COEFFICIENT MULTIPLIERS ON SPACES OF VECTOR-VALUED ENTIRE DIRICHLET SERIES

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Abstract. The spaces of entire functions represented by Dirichlet series have been studied by Hussein and Kamthan and others. In the present paper we consider the space X of all entire functions defined by vector-valued Dirichlet series and study the properties of a sequence space which is defined using the type of an entire function represented by vector-valued Dirichlet series. The main result concerns with obtaining the nature of the dual space of this sequence space and coefficient multipliers for some classes of vector-valued Dirichlet series.

Keywords: vector-valued Dirichlet series; analytic function; entire function; dual space; norm

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1. Introduction

Let

$$f(s)=\sum_{n=1}^{\infty}a_n\mathrm{e}^{s\lambda_n}, \tag{1.1}$$

where $s=\sigma+\mathrm{i}t$, σ and t are real variables, a_n's belong to a complex Banach algebra E with the unit element ω and $\{\lambda_n\}$ is an increasing sequence such that $0<\lambda_1<\lambda_2<\lambda_3<\ldots<\lambda_n\ldots$; $\lim\limits_{n\to\infty}\lambda_n=\infty$ and

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n}=D<\infty. \tag{1.2}$$

Let $\sigma_c(f)$ and $\sigma_a(f)$ be the abscissa of convergence and abscissa of absolute convergence, respectively, of the series in (1.1). Then under the condition (1.2), we have

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(see [3], page 59)

$$0 \leqslant \sigma_c(f) - \sigma_a(f) \leqslant D.$$

Further, if $D = 0$, then (see [3], page 62),

$$\sigma_c = \sigma_a = -\limsup_{n\to\infty} \frac{\ln \|a_n\|}{\lambda_n}. \tag{1.3}$$

Much earlier, Mandelbrojt (see [2], page 166) had obtained a result similar to (1.3) for the classical Dirichlet series $\sum_{n=1}^{\infty} a_n \mathrm{e}^{-s\lambda_n}$ under the condition

$$\limsup_{n\to\infty} \frac{n}{\lambda_n} < \infty. \tag{1.4}$$

It is evident that if (1.4) holds, then $D = 0$.

Suppose that the sequence $\{\lambda_n\}$ in the vector-valued Dirichlet series (1.1) given above satisfies the condition (1.4) so that (1.3) holds. If $\sigma_c(f) = \sigma_a(f) = \infty$, then $f(s)$ is a vector-valued entire function represented by the Dirichlet series in (1.1). We define its maximum modulus as

$$M(\sigma) = \sup_{-\infty<t<\infty} \|f(\sigma + \mathrm{i}t)\|.$$

The concepts of order and type of an entire function represented by vector-valued Dirichlet series of one complex variable were first introduced in [3] by Srivastava. Thus the order ϱ of the entire function $f(s)$ is defined as

$$\varrho = \limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma}, \quad 0 \leqslant \varrho \leqslant \infty. \tag{1.5}$$

When $0 < \varrho < \infty$, the type T of $f(s)$ is defined as

$$T = \limsup_{\sigma\to\infty} \frac{\log M(\sigma)}{\exp(\varrho\sigma)}, \quad 0 \leqslant T \leqslant \infty. \tag{1.6}$$

Srivastava in [3] also obtained the coefficient characterizations of order and type. Thus $f(s)$ is an entire function of order ϱ if and only if

$$\varrho = \limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log \|a_n\|^{-1}}. \tag{1.7}$$

Further, if $f(s)$ is an entire of order ϱ, then it is of type T if and only if

$$T = \limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}}. \tag{1.8}$$

Let X denote the linear space of all entire functions $f(s)$ defined by vector-valued Dirichlet series (1.1) over the complex field and satisfying

$$\limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} \leqslant T. \tag{1.9}$$

Lê Hai Khôi in [1] introduced various concepts of duality for sequence spaces which we state below.

Let A and B be two sequence spaces. We denote the sequence space of "multipliers" from A to B by (A, B) such that

$$(A, B) = \{u = (u_n)\colon\ (u_n a_n) \in B,\ \forall\,(a_n) \in A\}.$$

A sequence space A is said to be normal, if whenever A contains (a_n) it also contains the sequence (b_n) satisfying $\|b_n\| \leqslant \|a_n\|$ for $n = 1, 2, \ldots$ Equivalently, A is normal if $l^\infty \subset (A, A)$. If D is a fixed sequence space, then the D-dual of a sequence space A is defined to be (A, D), the space of multipliers from A to D, and denoted by A^D. Some duals are defined with some conditions such as Köthe dual or Abel dual. The Köthe dual is obtained when $D = l^1$, and will be denoted by A^α (it is also denoted by A^K). The Abel dual is obtained when D is the space of Abel-summable sequences, that is, the space of sequences (d_n) for which $\lim\limits_{r\to 1} \sum\limits_{n=1}^{\infty} d_n r^n$ exists. Note that when $d_n \geqslant 0$, the existence of this limit is equivalent to the condition $\sum d_n < \infty$. We denote the Abel dual of A by A^a. It is clear that $A^\alpha \subseteq A^a$. The reverse inclusion is true if space A is normal.

The aim of this paper is to introduce a new sequence space using the type of entire functions represented by vector-valued Dirichlet series (VVDS) and obtain some auxiliary conditions of convergence of VVDS given in (1.1). In what follows we shall always consider E to be a complex Banach algebra and assume that the sequence $\{\lambda_n\}$ satisfies the condition (1.4). Consequently, (1.2) also holds and $D = 0$.

2. Main results

We denote by E_T the sequence space

$$E_T = \{(a_n)\colon\ a_n \in E \text{ and } (a_n) \text{ satisfies (1.9)}\}.$$

In this section, we study some dual spaces of the space E_T. We note that if the sequence $\{\lambda_n\}$ satisfies condition (1.2), then

$$\sum_{n=1}^{\infty} r_n^{\lambda} < \infty \quad \forall\, r \in (0, 1). \tag{2.1}$$

The Köthe dual of the space E_T is defined as

$$E_T^\alpha = \Big\{(u_n)\colon \sum_{n=1}^{\infty} \|u_n a_n\| \text{ converges } \forall\, (a_n) \in E_T\Big\}.$$

Now we introduce another sequence space E_T^β defined as

$$E_T^\beta = \Big\{(u_n)\colon \sum_{n=1}^{\infty} u_n a_n \text{ converges } \forall\, (a_n) \in E_T\Big\}.$$

It can be easily verified that $E_T^\alpha \subseteq E_T^\beta$. We now find the criteria for the reverse inclusion relation to be true

We now prove the following statement.

Theorem 1. *If* $(u_n) \in E_T^\beta$, *then we have*

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} \geqslant T. \tag{2.2}$$

Conversely, if the sequence (u_n) *satisfies* (2.2), *then* $(u_n) \in E_T^\alpha$.

P r o o f. Let us assume that (2.2) does not hold, i.e.,

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} < T.$$

Then for a given $\varepsilon > 0$ there exists a sequence (n_k) of positive integers such that

$$\frac{\lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} < T + \varepsilon \quad \forall\, k \geqslant 1.$$

Let (a_n) be a sequence defined as

$$a_n = \begin{cases} \dfrac{\omega}{\|u_n\|}, & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} = \lim_{k\to\infty} \frac{\lambda_{n_k} \|a_{n_k}\|^{\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} = \lim_{k\to\infty} \frac{\lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \leqslant T.$$

It follows that $(a_n) \in E_T$. But $\|a_n u_n\| = 1$ for $n = n_k$, $k = 1, 2, \ldots$, that is, $\lim\limits_{n\to\infty} \|a_n u_n\| \neq 0$. So the series $\sum\limits_{n=1}^{\infty} \|u_n a_n\|$ does not converge. Hence if $(u_n) \in E_T^\beta$, then (2.2) will always be satisfied.

Conversely, suppose that (2.2) holds, that is,

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} = M \geqslant T.$$

Then for a given $\varepsilon > 0$, there exists N_1 such that for all $n \geqslant N_1$ we have

$$\frac{\lambda_n \|u_n\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \geqslant M - \varepsilon.$$

Also, for every sequence $(a_n) \in E_T$, using (1.9), we can find a positive integer N_2 such that for all $n \geqslant N_2$

$$\|a_n\|^{\varrho/\lambda_n} < \frac{(T+\varepsilon)\varrho \mathrm{e}}{\lambda_n} \quad \forall\, n \geqslant N_2.$$

Therefore for all $n \geqslant \max\{N_1, N_2\}$,

$$\|a_n u_n\|^{\varrho/\lambda_n} \leqslant \frac{T+\varepsilon}{M-\varepsilon}, \quad \text{i.e.,} \ \|a_n u_n\| \leqslant \Big(\frac{T+\varepsilon}{M-\varepsilon}\Big)^{\lambda_n/\varrho}.$$

For $M > T$, we choose any $\varepsilon > 0$ such that $M-\varepsilon > T+\varepsilon$. Then from (2.1) we can see that the series $\sum\limits_{n=1}^{\infty} \|a_n u_n\|$ converges. Hence $(u_n) \in E_T^{\beta}$. This proves Theorem 1. □

Theorem 2. *The space E_T is perfect, i.e., $E_T^{\alpha\alpha} = E_T$.*

Proof. Let the sequence $(a_n) \notin E_T$. Then we have

$$\limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} \geqslant T.$$

We denote by T' the left-hand side limit if it is finite and a number $> T$ if the limit is infinite. Then for a given arbitrarily small $\varepsilon > 0$, there exists a sequence (n_k) of positive integers such that

$$\|a_{n_k}\|^{\varrho/\lambda_{n_k}} \geqslant \frac{(T'-\varepsilon)\varrho \mathrm{e}}{\lambda_{n_k}}, \quad k = 1, 2, \ldots$$

Let us define a sequence

$$u_n = \begin{cases} \dfrac{\omega}{\|a_n\|} & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} = \lim_{k\to\infty} \frac{\lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} = \lim_{k\to\infty} \frac{\lambda_{n_k} \|a_{n_k}\|^{\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \geqslant T.$$

Hence from Theorem 1, $(u_n) \in E_T^{\alpha}$. But $\|a_n u_n\| = 1$ for $n = n_k$, i.e., $\sum a_n u_n$ does not converge. Therefore $(a_n) \notin E_T^{\alpha\alpha}$. Hence $E_T^{\alpha\alpha} \subseteq E_T$. The reverse inclusion always holds. Hence the space E_T is perfect. □

Theorem 3. *For the sequence space E_T defined as above, we have*

$$(E_T, l^p) = E_T^{\alpha} \quad \forall\, 0 < p \leqslant \infty.$$

P r o o f. Suppose that a sequence $(u_n) \notin E_T^{\alpha}$. Then from Theorem 1, we have

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} \leqslant T.$$

Then for an arbitrarily small $\varepsilon > 0$, there exists a sequence (n_k) of positive integers such that

$$\frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} \leqslant T + \varepsilon, \quad n = n_k \ \forall\, k \geqslant 1.$$

Let $0 < p < \infty$. We consider the sequence

$$a_n = \begin{cases} \dfrac{\omega}{\|u_{n_k}\|} & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} = \limsup_{k\to\infty} \frac{\lambda_{n_k} \|a_{n_k}\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} = \limsup_{k\to\infty} \frac{\lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \leqslant T.$$

Hence we get $(a_n) \in E_T$. By the definition of (E_T, l^p), $\sum\limits_{n=1}^{\infty} \|a_n u_n\|^p$ should be convergent. But $\|a_n u_n\| = 1$, $n = 1, 2, \ldots$ This implies $(a_n u_n) \notin l^p$.

For the case when $p = \infty$, consider a sequence

$$a_n = \begin{cases} \dfrac{\omega n_k}{\|u_{n_k}\|} & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} &= \limsup_{k\to\infty} \frac{\lambda_{n_k} \|a_{n_k}\|^{\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \\ &= \limsup_{k\to\infty} \frac{n_k^{\varrho/\lambda_{n_k}} \lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \leqslant T, \end{aligned}$$

since $\lim\limits_{k\to\infty} n_k^{1/n_k} = 1$. Hence $(a_n) \in E_T$. Since $\lim\limits_{k\leftrightarrow\infty} \|a_{n_k} u_{n_k}\| = \infty$, this implies that $(a_n u_n) \notin l^{\infty}$. Hence we conclude that for $0 < p \leqslant \infty$, $(u_n) \notin E_T^{\alpha} \Rightarrow (u_n) \notin (E_T, l^p)$ or equivalently, $(E_T, l^p) \subseteq E_T^{\alpha}$, $0 < p \leqslant \infty$.

Conversely, assume that $(u_n) \in E_T^{\alpha}$. Then for a given $M > T$, there exists N_1 such that

$$\|u_n\| \leqslant \Big(\frac{\varrho e M}{\lambda_n}\Big)^{-\lambda_n/\varrho} \qquad \forall\, n \geqslant N_1.$$

Suppose that $(a_n) \in E_T$, then for $\delta \in (0, M-T)$ there exists N_2 such that for all $n \geqslant N_2$

$$\|a_n\| \leqslant \Big(\frac{\varrho e(T+\delta)}{\lambda_n}\Big)^{\lambda_n/\varrho} \qquad \forall\, n \geqslant N_2.$$

Consequently, for all $n \geqslant N = \max\{N_1, N_2\}$, we have

$$\|a_n u_n\| \leqslant \|a_n\|\|u_n\| < \Big(\frac{T+\delta}{M}\Big)^{\lambda_n}.$$

If $0 < p < \infty$, then since $(T+\delta)/M < 1$, we have by condition (2.1),

$$\sum_{n=N}^{\infty} \|a_n u_n\|^p \leqslant \sum_{n=N}^{\infty} \Big(\frac{T+\delta}{M}\Big)^{p\lambda_n/\varrho} < \infty,$$

which implies that $(a_n u_n) \in l^p$.

Now let us take $p = \infty$, then we have $\|a_n u_n\| \leqslant ((T+\delta)/M)^{\lambda_n/\varrho} < 1$ for all $n \geqslant N$, which shows that $(a_n u_n) \in l^\infty$. Thus in both cases, $(u_n) \in (E_T, l^p)$ and consequently, $E_T^{\alpha} \subset (E_T, l^p)$, $0 < p \leqslant \infty$. This completes the proof of Theorem 3. □

In the next theorem we obtain the sequence space of multipliers from l^p to E_T.

Theorem 4. *A sequence (u_n) is a multiplier from l^p to E_T if*

$$(l^p, E_T) = E_T, \quad 0 < p \leqslant \infty.$$

P r o o f. Let $(u_n) \in (l^p, E_T)$, $0 < p \leqslant \infty$ and suppose that $(u_n) \notin E_T$. Then we have

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho e} = M < T.$$

Then for a given number δ, $0 < 2\delta < T - M$, there exists a sequence (n_k) of positive integers such that $\lambda_{n_k}\|u_{n_k}\|^{-\varrho/\lambda_{n_k}}\varrho^{-1}e^{-1} \leqslant M + \delta$ for all $k \geqslant 1$. This implies $\|u_{n_k}\|^{-1} \leqslant ((M+\delta)\varrho e \lambda_{n_k}^{-1})^{\lambda_{n_k}/\varrho}$ for all $k \geqslant 1$.

Define a new sequence (b_n) such that

$$b_n = \begin{cases} \dfrac{\omega((M+2\delta)\varrho e \lambda_{n_k}^{-1})^{-\lambda_{n_k}/\varrho}}{\|u_{n_k}\|} & \text{if } n = n_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have by (2.1),

$$\sum_{n=1}^{\infty}\|b_n\|^p=\sum_{k=1}^{\infty}\|b_{n_k}\|^p=\sum_{k=1}^{\infty}\|u_{n_k}\|^{-p}\Big\|\omega\Big(\frac{(M+2\delta)\varrho\mathrm{e}}{\lambda_{n_k}}\Big)\Big\|^{-\lambda_{n_k}p/\varrho}$$
$$\leqslant\sum_{k=1}^{\infty}\Big(\frac{M+\delta}{M+2\delta}\Big)^{\lambda_{n_k}p/\varrho}<\infty,$$

which shows that $(b_n)\in l^p$. Now consider

$$\liminf_{n\to\infty}\frac{\lambda_n\|b_nu_n\|^{-\varrho/\lambda_n}}{\varrho\mathrm{e}}=\liminf_{k\to\infty}\frac{\lambda_{n_k}\|b_{n_k}u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho\mathrm{e}}=(M+2\delta)<T.$$

In the second case, i.e., for $p=\infty$, we define a sequence (c_n) such that

$$c_n=\begin{cases}\dfrac{\omega((M+\delta)\varrho\mathrm{e}\lambda_{n_k}^{-1})^{-\lambda_{n_k}/\varrho}}{\|u_{n_k}\|} & \text{if } \ n=n_k,\\ 0 & \text{otherwise.}\end{cases}$$

We can see that $\|c_n\|\leqslant 1$ for all $n\geqslant 1$, which shows that $(c_n)\in l^\infty$. Then we have

$$\liminf_{n\to\infty}\frac{\lambda_n\|c_nu_n\|^{-\varrho\lambda_n}}{\varrho\mathrm{e}}=\liminf_{k\to\infty}\frac{\lambda_{n_k}\|c_{n_k}u_{n_k}\|^{-\varrho\lambda_{n_k}}}{\varrho\mathrm{e}}=M+\delta<T.$$

Hence we see that in both cases, the sequences (b_nu_n) and (c_nu_n) do not belong to E_T even though $(b_n)\in l^p$ and $(c_n)\in l^\infty$. This is a contradiction. Thus $(l^p,E_T)\subset E_T$, $0<p\leqslant\infty$.

To prove the converse, assume that $(u_n)\in E_T$. Then we have

$$\liminf_{n\to\infty}\frac{\lambda_n\|u_n\|^{-\varrho/\lambda_n}}{\varrho\mathrm{e}}\geqslant T.$$

Let (d_n) be an arbitrary sequence such that $(d_n)\in l^p$, $0<p\leqslant\infty$. In both cases, there exists a constant P such that $\|d_n\|\leqslant P$ for all $n\geqslant 1$. Hence we have

$$\liminf_{n\to\infty}\frac{\lambda_n\|d_nu_n\|^{-\varrho/\lambda_n}}{\varrho\mathrm{e}}=\liminf_{k\to\infty}\frac{\lambda_{n_k}\|d_{n_k}u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho\mathrm{e}}$$
$$=\liminf_{k\to\infty}\frac{\lambda_{n_k}P^{-\varrho/\lambda_{n_k}}\|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho\mathrm{e}}\leqslant T.$$

which shows that $(d_nu_n)\in E_T$. Thus $E_T\subset(l^p,E_T)$ for all $0<p\leqslant\infty$. Hence the result follows. □

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