NONLINEAR FRACTIONAL DIFFERENTIAL INCLUSION WITH NONLOCAL FRACTIONAL INTEGRO-DIFFERENTIAL BOUNDARY CONDITIONS IN BANACH SPACES

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Abstract. We consider a nonlinear fractional differential inclusion with nonlocal fractional integro-differential boundary conditions in a Banach space. The existence of at least one solution is proved by using the set-valued analog of Mönch fixed point theorem associated with the technique of measures of noncompactness.

Keywords: differential inclusion; Caputo fractional derivative; nonlocal boundary conditions; Banach space; existence; fixed point; measure of noncompactness

MSC 2010: 26A33, 34A60, 34B15

1. Introduction

The literature on fractional calculus is now vast and a variety of new results have been found. In the past several years, fractional differential and integral equations and inclusions and applications have been addressed extensively by several researchers. For example, we refer the reader to [1], [6], [7], [8], [11], [24] and the references cited therein.

Boundary value problems constitute a very interesting and important class of problems since they have applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, biophysics, aerodynamics, viscoelasticity and damping, wave propagation, see [17], [18], [25], [26]. Many researchers have studied the existence theory for nonlinear fractional differential equations with a variety of boundary conditions, for instance, see the papers by Ahmad et al. [4], Cui et al. [13], Han et al. [15], Ntouyas et al. [21], [22], Xu et al. [28] and the references therein.

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In this paper, we consider the following problem of fractional differential inclusion with nonlocal fractional integro-differential boundary conditions of the form

$$(1.1)\qquad \begin{cases} {}^cD^r x(t) \in F(x, x(t)), \quad t \in J = [0,1],\ 1 < r \leqslant 2, \\ \alpha_1 x(0) + \beta_1({}^cD^r x(0)) = \gamma_1 \displaystyle\int_0^\eta \frac{(\eta - s)^{r-2}}{\Gamma(r-1)} x(s)\,\mathrm{d}s, \quad 0 < r < 1, \\ \alpha_2 x(1) + \beta_2({}^cD^r x(1)) = \gamma_2 \displaystyle\int_0^\sigma \frac{(\sigma - s)^{r-2}}{\Gamma(r-1)} x(s)\,\mathrm{d}s, \quad 0 < \eta,\ \sigma < 1, \end{cases}$$

where ${}^cD^r$ is the Caputo fractional derivative of order r, $F\colon J \times E \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{P}(E)$ is the family of all subsets of E, α_i, β_i, γ_i, $i = 1, 2$ are suitably chosen constants in $\mathbb{R}$, and E is a Banach space with norm $|\cdot|$.

The present work is motivated by a recent paper of Ahmad et al. [3] where problem (1.1) was considered for a single-valued case, and the existence of solutions was shown by means of a variety of fixed point theorems such as Banach's contraction principle, Krasnoselskii's fixed point theorem, and Leray-Schauder nonlinear alternative.

The existence of solutions for the given multivalued problem is established by using the set-valued analog of the fixed point theorem of Mönch associated with the technique of measure of noncompactness. This last has proved to be one of the most powerful tool in studying the existence of solutions for differential and integral equations and inclusions, see Banaś and Goebel [9], Agarwal et al. [2], Akhmerov et al. [5], Benchohra et al. [10], [12], Guo et al. [14], Mönch [20], and Szufla [27] and the references cited therein.

2. Preliminaries

We use the following notations: 2^E is the collection of all subsets of E and $\mathcal{P}(E) = 2^E \setminus \emptyset$.

$$\mathcal{P}_c(E) = \{A \subset E\colon\ A \text{ is nonempty, convex}\},$$
$$\mathcal{P}_{kc}(E) = \{A \subset E\colon\ A \text{ is nonempty, compact, convex}\}.$$

Let X, Y be two sets, $\mathfrak{G}\colon X \to 2^Y$ a set-valued map, and $A \subset Y$. We define

$$\operatorname{graph}(\mathfrak{G}) = \{(x, y)\colon\ x \in X,\ y \in \mathfrak{G}(X)\} \quad \text{(the graph of } \mathfrak{G}).$$

Let $R > 0$ and let

$$B = \{x \in E\colon\ |x| \leqslant R\}$$

and

$$U = \{u \in C(J, E) \colon \ \|u\|_C < R\}.$$

Clearly $\overline{U} = C(J, B)$.

Definition 2.1. A multivalued map $F \colon J \times E \to \mathcal{P}(E)$ is said to be Carathéodory if

(i) $t \to F(t, u)$ is measurable for each $u \in E$,

(ii) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in C(J, E)$ define the set of selections of F by

$$S_{F,y} = \{f \in L^1(J, E) \colon \ f(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

Theorem 2.2 ([16]). *Let E be a Banach space and $C \subset L^1(J, E)$ countable with $|u(t)| \leqslant h(t)$ for a.e. $t \in J$, and every $u \in C$ where $h \in L^1(J, \mathbb{R}_+)$. Then the function $\varphi(t) = \alpha(C(t))$ belongs to $L^1(J, \mathbb{R}_+)$ and satisfies*

$$\alpha\left(\left\{\int_0^1 u(s)\,\mathrm{d}s \colon \ u \in C\right\}\right) \leqslant 2\int_0^1 \alpha(C(s))\,\mathrm{d}s.$$

Recall the set-valued analog of Mönch fixed point theorem.

Theorem 2.3 ([23]). *Let K be a closed, convex subset of a Banach space E, U a relatively open subset of K, and $\mathfrak{G} \colon \overline{U} \to P_c(K)$. Assume that* graph$(\mathfrak{G})$ *is closed, $\mathfrak{G}$ maps compact sets into relatively compact sets, and that for some $x_0 \in U$ the following two conditions are satisfied:*

$$\begin{cases} M \subset \overline{U}, \ M \subset \operatorname{conv}(x_0 \cup \mathfrak{G}(M)) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \subset M \text{ countable} \end{cases} \implies \quad \overline{M} \text{ compact}, \tag{2.1}$$

$$x \notin (1 - \lambda)x_0 + \lambda\mathfrak{G}(x) \quad \forall\, x \in \overline{U} \setminus U, \ \lambda \in (0, 1).$$

Then there exists $x \in \overline{U}$ with $x \in \mathfrak{G}(x)$.

Lemma 2.4 ([14]). *Let $V \subset C(J, E)$ be bounded and the elements of V be equicontinuous on each J_k, $k = 1, \ldots, m$. Then the map $t \mapsto \alpha(V(t))$ is continuous on J_k, $k = 1, \ldots, m$ and*

$$\alpha\left(\int_J V(t)\,\mathrm{d}t\right) \leqslant \int_J \alpha(V(t))\,\mathrm{d}t,$$

where $V(t) = \{u(t) \colon \ u \in V\}$.

Lemma 2.5 ([3]). *For any $y \in C[0,1]$ the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^r x(t) = y(t), \quad t \in J = [0,1],\ 1 < r \leqslant 2, \\ \alpha_1 x(0) + \beta_1({}^cD^q x(0)) = \gamma_1 \displaystyle\int_0^\eta \frac{(\eta - s)^{r-2}}{\Gamma(r-1)} x(s)\,\mathrm{d}s, \quad 0 < q < 1, \\ \alpha_2 x(1) + \beta_2({}^cD^q x(1)) = \gamma_2 \displaystyle\int_0^\sigma \frac{(\sigma - s)^{r-2}}{\Gamma(r-1)} x(s)\,\mathrm{d}s, \quad 0 < \eta,\ \sigma < 1, \end{cases}$$

is

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} y(s)\,\mathrm{d}s + \mu_1(t) \int_0 \eta \frac{(\eta - s)^{2r-2}}{\Gamma(2r-1)} y(s)\,\mathrm{d}s \\ & + \mu_2 \biggl[\gamma_2 \int_0^\sigma \frac{(\sigma - s)^{2r-2}}{\Gamma(2r-1)} y(s)\,\mathrm{d}s - \beta_2 \int_0^1 \frac{(1-s)^{r-q-1}}{\Gamma(r-q)} y(s)\,\mathrm{d}s \\ & - \alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} y(s)\,\mathrm{d}s \biggr], \end{aligned}$$

where

$$\mu_1(t) = \gamma_1(\Delta_1 - \Delta_2 t) \in E, \quad \mu_2(t) = \gamma_1 \Delta_3 + \Delta_4 t \in E,$$

$$\Delta_1 = \frac{1}{\Delta}\Bigl(\alpha_2 - \frac{\gamma_2 \sigma^r}{\Gamma(r+1)} + \frac{\beta_2}{\Gamma(2-q)}\Bigr), \quad \Delta_2 = \frac{1}{\Delta}\Bigl(\alpha_2 - \frac{\gamma_2 \sigma^{r-1}}{\Gamma(r)}\Bigr),$$

$$\Delta_3 = \frac{\eta^r}{\Delta \Gamma(r+1)}, \quad \Delta_4 = \frac{1}{\Delta}\Bigl(\alpha_1 - \frac{\gamma_1 \eta^{r-1}}{\Gamma(r)}\Bigr),$$

$$\Delta = \Bigl(\alpha_1 - \frac{\gamma_1 \eta^{r-1}}{\Gamma(r)}\Bigr)\Bigl(\alpha_2 - \frac{\gamma_2 \sigma^r}{\Gamma(r+1)} + \frac{\beta_2}{\Gamma(2-q)}\Bigr) + \frac{\gamma_1 \eta^r}{\Gamma(r+1)}\Bigl(\alpha_2 - \frac{\gamma_2 \sigma^{r-1}}{\Gamma(r)}\Bigr) \neq 0.$$

Now we recall the definition of the Kuratowski measure of noncompactness. Let $M \subset E$ be bounded. Then

$$\alpha(M) = \inf\Bigl\{\varepsilon > 0\colon\ M \subseteq \bigcup_{i=1}^n M_i \text{ and } \operatorname{diam}(M_i) \leqslant \varepsilon\Bigr\}.$$

Properties: The Kuratowski measure of noncompactness satisfies some properties (for more details see [9]):

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact),
(b) $\alpha(B) = \alpha(\overline{B})$,
(c) $A \subset B \Rightarrow \alpha(A) \leqslant \alpha(B)$,
(d) $\alpha(A+B) \leqslant \alpha(A) + \alpha(B)$,
(e) $\alpha(cB) = |c|\alpha(B)$; $c \in \mathbb{R}$,
(f) $\alpha(\mathrm{conv}B) = \alpha(B)$.

3. Main results

Let us list the following hypotheses:

(H1) $F\colon J \times E \to \mathcal{P}_{kc}(E)$ is a Carathéodory multi-valued map.

(H2) For each $R > 0$ there exists a function $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\|_{\mathcal{P}} = \sup\{|v|\colon v(t) \in F(t, y)\} \leqslant p(t)$$

for each $(t, y) \in J \times E$ with $|y| \leqslant R$, and

$$\liminf_{R\to\infty} \frac{\omega}{R} \int_0^1 p(s)\,\mathrm{d}s < \infty,$$

where

$$\omega = \max_{t\in[0,1]} \Big\{\frac{t^r}{\Gamma(r+1)} + \frac{|\mu_1(t)|\eta^{2r-1}}{\Gamma(2r)} + |\mu_2(t)|\Big(\frac{|\gamma_2|\sigma^{2r-1}}{\Gamma(2r)} + \frac{|\beta_2|}{\Gamma(r-q+1)} + \frac{|\alpha_2|}{\Gamma(r+1)}\Big)\Big\}.$$

(H3) There exists a Carathéodory function $\psi\colon J \times [0, 2R] \to \mathbb{R}_+$ such that

$$\alpha(F(t, M(t))) \leqslant \psi(t, \alpha(M(t))), \quad \text{a.e. } t \in J, \text{ and each } M \subset B,$$

and the unique solution $\varphi \in C(J, [0, 2R])$ of the inequality

$$\begin{aligned}\varphi(t) \leqslant{}& \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \psi(s, \alpha(M(s)))\,\mathrm{d}s \\ &+ \frac{|\mu_1(t)|}{\Gamma(2r-1)} \int_0^\eta (\eta - s)^{2r-2} \psi(s, \alpha(M(s)))\,\mathrm{d}s \\ &+ |\mu_2(t)|\Big[\frac{|\gamma_2|}{\Gamma(2r-1)} \int_0^\sigma (\sigma - s)^{2r-2} \psi(s, \alpha(M(s)))\,\mathrm{d}s \\ &+ \frac{|\beta_2|}{\Gamma(r-q)} \int_0^1 (1-s)^{r-q-1} \psi(s, \alpha(M(s)))\,\mathrm{d}s \\ &+ \frac{|\alpha_2|}{\Gamma(r)} \int_0^1 (1-s)^{r-1} \psi(s, \alpha(M(s)))\,\mathrm{d}s\Big], \quad t \in J,\end{aligned}$$

is $\varphi \equiv 0$.

Lemma 3.1 ([19]). *Let J be a compact real interval. Let F be a multivalued map satisfying* (H1) *and let Θ be a linear continuous map from $L^1(J, E) \to C(J, E)$. Then the operator*

$$\Theta \circ S_{F,y}\colon C(J, E) \to \mathcal{P}_{kc}(C(J, E)),\ y \mapsto (\Theta \circ S_{F,y})(y) = \Theta(S_{F,y})$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

Theorem 3.2. *Suppose that* (H1)–(H3) *are satisfied. Then problem* (1.1) *has at least one solution on* $C(J,B)$, *provided that*

$$\frac{\omega}{R}\int_0^1 p(s)\,\mathrm{d}s \leqslant 1. \tag{3.1}$$

Proof. Consider the multi-valued map $\mathfrak{G}\colon C(J,E)\to\mathcal{P}(C(J,E))$ defined by

$$\begin{aligned}\mathfrak{G}(y)=\biggl\{h\in C(J,E)\colon\ h(t)=\biggl\{&\int_0^t\frac{(t-s)^{r-1}}{\Gamma(r)}f(s,x(s))\,\mathrm{d}s\\&+\mu_1(t)\int_0^\eta\frac{(\eta-s)^{2r-2}}{\Gamma(2r-1)}f(s,x(s))\,\mathrm{d}s\\&+\mu_2(t)\biggl[\gamma_2\int_0^\sigma\frac{(\sigma-s)^{2r-2}}{\Gamma(2r-1)}f(s,x(s))\,\mathrm{d}s\\&-\beta_2\int_0^1\frac{(1-s)^{r-q-1}}{\Gamma(r-q)}f(s,x(s))\,\mathrm{d}s\\&-\alpha_2\int_0^1\frac{(1-s)^{r-1}}{\Gamma(r)}f(s,x(s))\,\mathrm{d}s\biggr];\ f\in S_{F,y}\biggr\}\biggr\}\end{aligned}$$

The fixed points of $\mathfrak{G}$ are solutions to (1.1). We show that $\mathfrak{G}$ satisfies the assumptions of Theorem 2.3. This is achieved in several steps.

First we show that $\mathfrak{G}(y)$ *is convex for each* $y\in C(J,E)$. If h_1, h_2 belong to $\mathfrak{G}(y)$, then there exist f_1, $f_2\in S_{F,y}$ such that for a.e. $t\in J$ we have

$$\begin{aligned}h_i(t)=&\int_0^t\frac{(t-s)^{r-1}}{\Gamma(r)}f_i(s)\,\mathrm{d}s+\mu_1(t)\int_0^\eta\frac{(\eta-s)^{2r-2}}{\Gamma(2r-1)}f_i(s)\,\mathrm{d}s\\&+\mu_2(t)\biggl[\gamma_2\int_0^\sigma\frac{(\sigma-s)^{2r-2}}{\Gamma(2r-1)}f_i(s)\,\mathrm{d}s-\beta_2\int_0^1\frac{(1-s)^{r-q-1}}{\Gamma(r-q)}f_i(s)\,\mathrm{d}s\\&-\alpha_2\int_0^1\frac{(1-s)^{r-1}}{\Gamma(r)}f_i(s)\,\mathrm{d}s\biggr],\quad i=1,2.\end{aligned}$$

Let $0\leqslant\lambda\leqslant 1$, for each $t\in J$ we have

$$\begin{aligned}(\lambda h_1+(1-\lambda)h_2)(t)=&\int_0^t\frac{(t-s)^{r-1}}{\Gamma(r)}(\lambda f_1+(1-\lambda)f_2)(s)\,\mathrm{d}s\\&+\mu_1(t)\int_0^\eta\frac{(\eta-s)^{2r-2}}{\Gamma(2r-1)}(\lambda f_1+(1-\lambda)f_2)(s)\,\mathrm{d}s\\&+\mu_2(t)\biggl[\gamma_2\int_0^\sigma\frac{(\sigma-s)^{2r-2}}{\Gamma(2r-1)}(\lambda f_1+(1-\lambda)f_2)(s)\,\mathrm{d}s\\&-\beta_2\int_0^1\frac{(1-s)^{r-q-1}}{\Gamma(r-q)}(\lambda f_1+(1-\lambda)f_2)(s)\,\mathrm{d}s\\&-\alpha_2\int_0^1\frac{(1-s)^{r-1}}{\Gamma(r)}(\lambda f_1+(1-\lambda)f_2)(s)\,\mathrm{d}s\biggr].\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$\lambda h_1 + (1-\lambda) h_2 \in \mathfrak{G}(y).$$

As the second step we show that $\mathfrak{G}(M)$ *is relatively compact for each compact* $M \subset \overline{U}$. Let $M \subset \overline{U}$ be a compact set and let (h_n) be any sequence of elements of $\mathfrak{G}(M)$. We show that (h_n) has a convergent subsequence by using the Arzèla-Ascoli criterion of noncompactness in $C(J,E)$. Since $(h_n) \in \mathfrak{G}(M)$, there exist $(y_n) \in M$ and $(f_n) \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) = {} & \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} f_n(s)\,\mathrm{d}s + \mu_1(t) \int_0^\eta \frac{(\eta-s)^{2r-2}}{\Gamma(2r-1)} f_n(s)\,\mathrm{d}s \\ & + \mu_2(t) \biggl[\gamma_2 \int_0^\sigma \frac{(\sigma-s)^{2r-2}}{\Gamma(2r-1)} f_n(s)\,\mathrm{d}s - \beta_2 \int_0^1 \frac{(1-s)^{r-q-1}}{\Gamma(r-q)} f_n(s)\,\mathrm{d}s \\ & - \alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f_n(s)\,\mathrm{d}s \biggr]. \end{aligned}$$

Using Theorem 2.2 and the properties of the measure of Kuratowski α, we obtain that

$$\begin{aligned} \alpha(\{h_n(t)\}) \leqslant {} & \frac{1}{\Gamma(r)} \int_0^t \alpha(\{(t-s)^{r-1} f_n(s)\})\,\mathrm{d}s \\ & + \frac{|\mu_1(t)|}{\Gamma(2r-1)} \int_0^\eta \alpha(\{(\eta-s)^{2r-2} f_n(s)\})\,\mathrm{d}s \\ & + |\mu_2(t)| \biggl[\frac{|\gamma_2|}{\Gamma(2r-1)} \int_0^\sigma \alpha(\{(\sigma-s)^{2r-2} f_n(s)\})\,\mathrm{d}s \\ & + \frac{|\beta_2|}{\Gamma(r-q)} \int_0^1 \alpha(\{(1-s)^{r-q-1} f_n(s)\})\,\mathrm{d}s \\ & + \frac{|\alpha_2|}{\Gamma(r)} \int_0^1 \alpha(\{(1-s)^{r-1} f_n(s)\})\,\mathrm{d}s \biggr]. \end{aligned} \tag{3.2}$$

On the other hand, since $M(s)$ is compact in E, the set $\{f_n(s)\colon\ n \geqslant 1\}$ is compact. Consequently, $\alpha(\{f_n(s)\colon\ n \geqslant 1\}) = 0$ for a.e. $s \in J$. Furthermore,

$$\begin{aligned} \alpha(\{(t-s)^{r-1} f_n(s)\colon\ n \geqslant 1\}) &= (t-s)^{r-1} \alpha(\{f_n(s)\colon\ n \geqslant 1\}) = 0, \\ \alpha(\{(k-s)^{r-1} f_n(s)\colon\ n \geqslant 1\}) &= (k-s)^{r-1} \alpha(\{f_n(s)\colon\ n \geqslant 1\}) = 0; \quad k = \eta, \sigma, \\ \alpha(\{(1-s)^{r-j} f_n(s)\colon\ n \geqslant 1\}) &= (1-s)^{r-j} \alpha(\{f_n(s)\colon\ n \geqslant 1\}) = 0; \quad j = 1, q+1 \end{aligned}$$

for a.e. $t, s \in J$. Now (3.2) implies that $\{h_n(t)\colon\ n \geqslant 1\}$ is relatively compact in E for each $t \in J$.

In addition, for each t_1 and t_2 from J, $t_1 < t_2$ we have

$$
\begin{aligned}
(3.3)\quad &|h_n(t_2) - h_n(t_1)| \\
&= \bigg| \frac{1}{\Gamma(r)} \int_0^{t_1} [(t_2 - s)^{r-1} - (t_1 - s)^{r-1}] f_n(s)\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (t_2 - s)^{r-1} f_n(s)\,\mathrm{d}s \\
&\quad + (\mu_1(t_2) - \mu_1(t_1)) \int_0^{\eta} \frac{(\eta - s)^{2r-2}}{\Gamma(2r-1)} f_n(s)\,\mathrm{d}s \\
&\quad + (\mu_2(t_2) - \mu_2(t_1)) \bigg[\gamma_2 \int_0^{\sigma} \frac{(\sigma - s)^{2r-2}}{\Gamma(2r-1)} f_n(s)\,\mathrm{d}s \\
&\quad - \beta_2 \int_0^1 \frac{(1-s)^{r-q-1}}{\Gamma(r-q)} f_n(s)\,\mathrm{d}s - \alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f_n(s)\,\mathrm{d}s \bigg] \bigg| \\
&\leqslant \frac{1}{\Gamma(r)} \int_0^{t_1} [(t_2 - s)^{r-1} - (t_1 - s)^{r-1}] p(s)\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (t_2 - s)^{r-1} p(s)\,\mathrm{d}s \\
&\quad + |\gamma_1 \Delta_2| (t_2 - t_1) \int_0^{\eta} \frac{(\eta - s)^{2r-2}}{\Gamma(2r-1)}\,\mathrm{d}s \\
&\quad + |\Delta_4| (t_2 - t_1) \bigg[|\gamma_2| \int_0^{\sigma} \frac{(\sigma - s)^{2r-2}}{\Gamma(2r-1)} p(s)\,\mathrm{d}s \\
&\quad + |\beta_2| \int_0^1 \frac{(1-s)^{r-q-1}}{\Gamma(r-q)}\,\mathrm{d}s + |\alpha_2| \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} \bigg].
\end{aligned}
$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n \colon n \geqslant 1\}$ is equicontinuous. Consequently, $\{h_n \colon n \geqslant 1\}$ is relatively compact in $C(J, E)$. $\square$

Now we show that $\mathfrak{G}$ *has a closed graph.* Let $(y_n, h_n) \in \operatorname{graph}(\mathfrak{G})$, $n \geqslant 1$, with $\|y_n - y\|_C, \|h_n - h\|_C \to 0$ as $n \to \infty$. We must show that $(y, h) \in \operatorname{graph}(\mathfrak{G})$.

The fact that $(y_n, h_n) \in \operatorname{graph}(\mathfrak{G})$ means that $h_n \in \mathfrak{G}(y_n)$, which means that there exists $f_n \in S_{F, y_n}$, such that for each $t \in J$

$$
\begin{aligned}
h_n(t) = &\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} f_n(s)\,\mathrm{d}s + \mu_1(t) \int_0^{\eta} \frac{(\eta - s)^{2r-2}}{\Gamma(2r-1)} f_n(s)\,\mathrm{d}s \\
&+ \mu_2(t) \bigg[\gamma_2 \int_0^{\sigma} \frac{(\sigma - s)^{2r-2}}{\Gamma(2r-1)} f_n(s)\,\mathrm{d}s - \beta_2 \int_0^1 \frac{(1-s)^{r-q-1}}{\Gamma(r-q)} f_n(s)\,\mathrm{d}s \\
&- \alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f_n(s)\,\mathrm{d}s \bigg].
\end{aligned}
$$

Consider the continuous linear operator

$$\Theta\colon\ L^1(J,E) \to C(J,E)$$

$$f \mapsto \Theta(f)(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} f(s)\,\mathrm{d}s + \mu_1(t) \int_0^\eta \frac{(\eta-s)^{2r-2}}{\Gamma(2r-1)} f(s)\,\mathrm{d}s \\ \quad \displaystyle +\mu_2(t)\Bigg[\gamma_2 \int_0^\sigma \frac{(\sigma-s)^{2r-2}}{\Gamma(2r-1)} f(s)\,\mathrm{d}s \\ \quad \displaystyle -\beta_2 \int_0^1 \frac{(1-s)^{r-q-1}}{\Gamma(r-q)} f(s)\,\mathrm{d}s \\ \quad \displaystyle -\alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f(s)\,\mathrm{d}s\Bigg]. \end{cases}$$

Clearly

$$\begin{aligned} |h_n(t) - h(t)| = \Bigg| &\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} (f_n(s) - f(s))\,\mathrm{d}s \\ &+ \mu_1(t) \int_0^\eta \frac{(\eta-s)^{2r-2}}{\Gamma(2r-1)} (f_n(s) - f(s))\,\mathrm{d}s \\ &+ \mu_2(t)\Bigg[\gamma_2 \int_0^\sigma \frac{(\sigma-s)^{2r-2}}{\Gamma(2r-1)} (f_n(s) - f(s))\,\mathrm{d}s \\ &- \beta_2 \int_0^1 \frac{(1-s)^{r-q-1}}{\Gamma(r-q)} (f_n(s) - f(s))\,\mathrm{d}s \\ &- \alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} (f_n(s) - f(s))\Bigg]\,\mathrm{d}s\Bigg| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Consequently, from Lemma 3.1 it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover, we have

$$h_n(t) \in \Theta(S_{F,y_n}).$$

Since $y_n \to y$, we get

$$\begin{aligned} h(t) = &\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} f(s)\,\mathrm{d}s + \mu_1(t) \int_0^\eta \frac{(\eta-s)^{2r-2}}{\Gamma(2r-1)} f(s)\,\mathrm{d}s \\ &+ \mu_2(t)\Bigg[\gamma_2 \int_0^\sigma \frac{(\sigma-s)^{2r-2}}{\Gamma(2r-1)} f(s)\,\mathrm{d}s - \beta_2 \int_0^1 \frac{(1-s)^{r-q-1}}{\Gamma(r-q)} f(s)\,\mathrm{d}s \\ &- \alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f(s)\,\mathrm{d}s\Bigg] \end{aligned}$$

for some $f \in S_{F,y}$.

In our next step we show that (2.1) is satisfied. Suppose $M \subset \overline{U}$, $M \subset \operatorname{conv}(\{0\} \cup \mathfrak{G}(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Using an estimation of type (3.3), we see that $\mathfrak{G}(M)$ is equicontinuous. Then from $M \subset \operatorname{conv}(\{0\} \cup \mathfrak{G}(M))$ we deduce that M is equicontinuous too. In order to apply the Arzèla-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since

$$C \subset M \subset \operatorname{conv}(\{0\} \cup \mathfrak{G}(M)) \text{ and } C \text{ is countable,}$$

we can find a countable set $H = \{h_n \colon n \geqslant 1\} \subset \mathfrak{G}(M)$ with $C \subset \operatorname{conv}(\{0\} \cup H)$. Then there exist $y_n \in M$ and $f_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) = {} & \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} f_n(s)\,\mathrm{d}s + \mu_1(t) \int_0^\eta \frac{(\eta-s)^{2r-2}}{\Gamma(2r-1)} f_n(s)\,\mathrm{d}s \\ & + \mu_2(t) \Big[\gamma_2 \int_0^\sigma \frac{(\sigma-s)^{2r-2}}{\Gamma(2r-1)} f_n(s)\,\mathrm{d}s - \beta_2 \int_0^1 \frac{(1-s)^{r-q-1}}{\Gamma(r-q)} f_n(s)\,\mathrm{d}s \\ & - \alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f_n(s)\,\mathrm{d}s \Big]. \end{aligned}$$

From $M \subset \overline{C} \subset \overline{\operatorname{conv}}(\{0\} \cup H)$ and the properties of the measure of noncompactness we have

$$\alpha(M(t)) \leqslant (\alpha(\overline{C}(t)) \leqslant \alpha(H(t)) = \alpha(\{h_n(t) \colon n \geqslant 1\}).$$

Using Theorem 2.2 and inequality (3.2), we obtain

$$\begin{aligned} \alpha(M(t)) \leqslant {} & \frac{1}{\Gamma(r)} \int_0^t \alpha(\{(t-s)^{r-1} f_n(s) \colon n \geqslant 1\})\,\mathrm{d}s \\ & + \frac{|\mu_1(t)|}{\Gamma(2r-1)} \int_0^\eta \alpha(\{(\eta-s)^{2r-2} f_n(s) \colon n \geqslant 1\})\,\mathrm{d}s \\ & + |\mu_2(t)| \Big[\frac{|\gamma_2|}{\Gamma(2r-1)} \int_0^\sigma \alpha(\{(\sigma-s)^{2r-2} f_n(s) \colon n \geqslant 1\})\,\mathrm{d}s \\ & + \frac{|\beta_2|}{\Gamma(r-q)} \int_0^1 \alpha(\{(1-s)^{r-q-1} f_n(s) \colon n \geqslant 1\})\,\mathrm{d}s \\ & + \frac{|\alpha_2|}{\Gamma(r)} \int_0^1 \alpha(\{(1-s)^{r-1} f_n(s) \colon n \geqslant 1\})\,\mathrm{d}s \Big]. \end{aligned}$$

Now, since $f_n(s) \in F(s, y_n(s))$ and $y_n \in M$, (H3) guarantees

$$\begin{aligned} \alpha(\{(t-s)^{r-1} f_n(s) \colon n \geqslant 1\}) & \leqslant (t-s)^{r-1} \alpha(F(s, M(s))) \\ & \leqslant (t-s)^{r-1} \psi(s, \alpha(M(s))); \\ \alpha(\{(k-s)^{2r-2} f_n(s) \colon n \geqslant 1\}) & \leqslant (k-s)^{2r-2} \alpha(F(s, M(s))) \\ & \leqslant (k-s)^{2r-2} \psi(s, \alpha(M(s))); \quad k = \eta, \sigma \end{aligned}$$

and

$$\alpha(\{(1-s)^{r-j} f_n(s)\colon\ n \geqslant 1\}) \leqslant (1-s)^{j-1}\alpha(F(s, M(s)))$$
$$\leqslant (1-s)^{r-j}\psi(s, \alpha(M(s))); \quad j = 1, q+1.$$

It follows that

$$\begin{aligned}
\alpha(M(t)) \leqslant{}& \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\psi(s,\alpha(M(s)))\,\mathrm{d}s \\
&+ \frac{|\mu_1(t)|}{\Gamma(2r-1)}\int_0^\eta (\eta-s)^{2r-2}\psi(s,\alpha(M(s)))\,\mathrm{d}s \\
&+ |\mu_2(t)|\biggl[\frac{|\gamma_2|}{\Gamma(2r-1)}\int_0^\sigma (\sigma-s)^{2r-2}\psi(s,\alpha(M(s)))\,\mathrm{d}s \\
&+ \frac{|\beta_2|}{\Gamma(r-q)}\int_0^1 (1-s)^{r-q-1}\psi(s,\alpha(M(s)))\,\mathrm{d}s \\
&+ \frac{|\alpha_2|}{\Gamma(r)}\int_0^1 (1-s)^{r-1}\psi(s,\alpha(M(s)))\,\mathrm{d}s\biggr].
\end{aligned}$$

Also, the function φ given by $\varphi(t) = \alpha(M(t))$ belongs to $C(J, [0, 2R])$. Consequently, by (H3), $\varphi \equiv 0$, that is $\alpha(M(t)) = 0$ for all $t \in J$.

Now, by the Arzèla-Ascoli theorem, M is relatively compact in $C(J, E)$.

Finally, let $h \in \mathfrak{G}(y)$ with $y \in \overline{U}$. Since $|y(s)| \leqslant R$ and (H2) holds, we have $\mathfrak{G}(\overline{U}) \subseteq \overline{U}$. If it was not true, there would exist a function $y \in \overline{U}$, but $\|\mathfrak{G}(y)\|_{\mathcal{P}} > R$ and

$$\begin{aligned}
h(t) ={}& \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} f(s)\,\mathrm{d}s + \mu_1(t)\int_0^\eta \frac{(\eta-s)^{2r-2}}{\Gamma(2r-1)} f(s)\,\mathrm{d}s \\
&+ \mu_2(t)\biggl[\gamma_2\int_0^\sigma \frac{(\sigma-s)^{2r-2}}{\Gamma(2r-1)} f(s)\,\mathrm{d}s - \beta_2\int_0^1 \frac{(1-s)^{r-q-1}}{\Gamma(r-q)} f(s)\,\mathrm{d}s \\
&- \alpha_2\int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f(s)\,\mathrm{d}s\biggr]
\end{aligned}$$

for some $f \in S_{F,y}$. On the other hand we have

$$\begin{aligned}
R <{}& \|\mathfrak{G}(y)\|_{\mathcal{P}} \\
<{}& \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}|f(s)|\,\mathrm{d}s + \frac{|\mu_1(t)|}{\Gamma(2r-1)}\int_0^\eta (\eta-s)^{2r-2}|f(s)|\,\mathrm{d}s \\
&+ |\mu_2(t)|\biggl[\frac{|\gamma_2|}{\Gamma(2r-1)}\int_0^\sigma (\sigma-s)^{2r-2}|f(s)|\,\mathrm{d}s \\
&+ \frac{|\beta_2|}{\Gamma(r-q)}\int_0^1 (1-s)^{r-q-1}|f(s)|\,\mathrm{d}s + \frac{|\alpha_2|}{\Gamma(r)}\int_0^1 (1-s)^{r-1}|f(s)|\,\mathrm{d}s\biggr]
\end{aligned}$$

$$
\begin{aligned}
&< \frac{t^r}{\Gamma(r+1)} \int_0^1 p(s)\,\mathrm{d}s + \frac{|\mu_1(t)|\eta^{2r-1}}{\Gamma(2r)} \int_0^1 p(s)\,\mathrm{d}s + |\mu_2(t)| \biggl[\frac{|\gamma_2|\sigma^{2r-1}}{\Gamma(2r)} \int_0^1 p(s)\,\mathrm{d}s \\
&\quad + \frac{|\beta_2|}{\Gamma(r-q+1)} \int_0^1 p(s)\,\mathrm{d}s + \frac{|\alpha_2|}{\Gamma(r+1)} \int_0^1 p(s)\,\mathrm{d}s \biggr] \\
&< \omega \int_0^1 p(s)\,\mathrm{d}s.
\end{aligned}
$$

Dividing both sides by R and taking the lower limit as $R \to \infty$, we conclude that

$$
\liminf_{R\to\infty} \frac{\omega}{R} \int_0^1 p(s)\,\mathrm{d}s > 1
$$

which contradicts (3.1). Hence $\mathfrak{G}(\overline{U}) \subseteq \overline{U}$.

As a consequence of all steps above together with Theorem 2.3, we can conclude that $\mathfrak{G}$ has a fixed point $y \in C(J, B)$, which is a solution of problem (1.1). $\square$

References

[1] *R. P. Agarwal, B. Ahmad*: Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions. Comput. Math. Appl. *62* (2011), 1200–1214. zbl MR doi

[2] *R. P. Agarwal, M. Benchohra, D. Seba*: On the application of measure of noncompactness to the existence of solutions for fractional differential equations. Result. Math. *55* (2009), 221–230. zbl MR doi

[3] *B. Ahmad, A. Alsaedi*: Nonlinear fractional differential equations with nonlocal fractional integro-differential boundary conditions. Bound. Value Probl. (electronic only) (2012), Article ID 124, 10 pages. zbl MR doi

[4] *B. Ahmad, J. J. Nieto*: Existence and approximation of solutions for a class of nonlinear impulsive functional differential equations with anti-periodic boundary conditions. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *69* (2008), 3291–3298. zbl MR doi

[5] *R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina, B. N. Sadovskiĭ*: Measures of Noncompactness and Condensing Operators. Operator Theory: Advances and Applications 55. Birkhäuser, Basel, 1992. zbl MR doi

[6] *A. Alsaedi, S. K. Ntouyas, B. Ahmad*: Existence results for Langevin fractional differential inclusions involving two fractional orders with four-point multiterm fractional integral boundary conditions. Abstr. Appl. Anal. *2013* (2013), Article ID 869837, 17 pages. zbl MR doi

[7] *H. H. Alsulami*: Application of fixed point theorems for multivalued maps to anti-periodic problems of fractional differential inclusions. Filomat *28* (2014), 91–98. MR doi

[8] *K. Balachandran, J. Y. Park, J. J. Trujillo*: Controllability of nonlinear fractional dynamical systems. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *75* (2012), 1919–1926. zbl MR doi

[9] *J. Banaś, K. Goebel*: Measures of Noncompactness in Banach Spaces. Lecture Notes in Pure and Applied Mathematics 60. Marcel Dekker, New York, 1980. zbl MR

[10] *M. Benchohra, J. Henderson, D. Seba*: Measure of noncompactness and fractional differential equations in Banach spaces. Commun. Appl. Anal. *12* (2008), 419–427. zbl MR

[11] *M. Benchohra, J. Henderson, D. Seba*: Boundary value problems for fractional differential inclusions in Banach spaces. Fract. Differ. Calc. *2* (2012), 99–108. MR doi

[12] *M. Benchohra, G. M. N'Guérékata, D. Seba*: Measure of noncompactness and nondensely defined semilinear functional differential equations with fractional order. Cubo *12* (2010), 35–48. zbl MR doi

[13] *Y. Cui*: Uniqueness of solution for boundary value problems for fractional differential equations. Appl. Math. Lett. *51* (2016), 48–54. zbl MR doi

[14] *D. Guo, V. Lakshmikantham, X. Liu*: Nonlinear Integral Equations in Abstract Spaces. Mathematics and Its Applications 373. Kluwer Academic Publishers, Dordrecht, 1996. zbl MR doi

[15] *J. Han, Y. Liu, J. Zhao*: Integral boundary value problems for first order nonlinear impulsive functional integro-differential differential equations. Appl. Math. Comput. *218* (2012), 5002–5009. zbl MR doi

[16] *H.-P. Heinz*: On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions. Nonlinear Anal., Theory Methods Appl. *7* (1983), 1351–1371. zbl MR doi

[17] *A. A. Kilbas, H. M. Srivastava, J. J. Trujillo*: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies 204. Elsevier, Amsterdam, 2006. zbl MR doi

[18] *V. Lakshmikantham, S. Leela, J. Vasundhara Devi*: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge, 2009. zbl

[19] *A. Lasota, Z. Opial*: An application of the Kakutani—Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. *13* (1965), 781–786. zbl MR

[20] *H. Mönch*: Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. Nonlinear Anal., Theory Methods Appl. *4* (1980), 985–999. zbl MR doi

[21] *S. K. Ntouyas*: Existence results for nonlocal boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions. Discuss. Math., Differ. Incl. Control Optim. *33* (2013), 17–39. zbl MR doi

[22] *S. K. Ntouyas, J. Tariboon*: Nonlocal boundary value problems for Langevin fractional differential inclusions with Riemann-Liouville fractional integral boundary conditions. Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal. *22* (2015), 123–141. zbl MR

[23] *D. O'Regan, R. Precup*: Fixed point theorems for set-valued maps and existence principles for integral inclusions. J. Math. Anal. Appl. *245* (2000), 594–612. zbl MR doi

[24] *I. Podlubny*: Geometric and physical interpretation of fractional integration and fractional differentiation. Fract. Calc. Appl. Anal. *5* (2002), 367–386; Correction: "Geometric and physical interpretation of fractional integration and fractional differentiation", ibid. *6* (2003), 109–110. zbl MR

[25] *J. Sabatier, O. P. Agrawal, J. A. Tenreiro Machado* (eds.): Advances in Fractional Calculus—Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht, 2007. zbl MR doi

[26] *S. G. Samko, A. A. Kilbas, O. I. Marichev*: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, New York, 1993. zbl MR

[27] *S. Szufla*: On the application of measure of noncompactness to existence theorems. Rend. Sem. Mat. Univ. Padova *75* (1986), 1–14. zbl MR

[28] *J. Xu, Z. Wei, W. Dong*: Uniqueness of positive solutions for a class of fractional boundary value problems. Appl. Math. Lett. *25* (2012), 590–593. zbl MR doi

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