WHEN SPECTRA OF LATTICES OF z-IDEALS ARE STONE-ČECH COMPACTIFICATIONS

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Abstract. Let X be a completely regular Hausdorff space and, as usual, let $C(X)$ denote the ring of real-valued continuous functions on X. The lattice of z-ideals of $C(X)$ has been shown by Martínez and Zenk (2005) to be a frame. We show that the spectrum of this lattice is (homeomorphic to) βX precisely when X is a P-space. This we actually show to be true not only in spaces, but in locales as well. Recall that an ideal of a commutative ring is called a d-ideal if whenever two elements have the same annihilator and one of the elements belongs to the ideal, then so does the other. We characterize when the spectrum of the lattice of d-ideals of $C(X)$ is the Stone-Čech compactification of the largest dense sublocale of the locale determined by X. It is precisely when the closure of every open set of X is the closure of some cozero-set of X.

Keywords: completely regular frame; coherent frame; z-ideal; d-ideal; Stone-Čech compactification; booleanization

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1. Introduction

Although the summary of results in the abstract is written mainly in the language of topological spaces, we shall employ the techniques of pointfree topology, and, in fact, prove our results in that broader context. The topological results (which are also new, we hasten to add) will then be corollaries. Throughout, L denotes a completely regular frame, and $\mathcal{R}L$ the ring of real functions on L. The notion of z-ideal in $C(X)$, defined by the requirement that if two functions have the same zero-set and one of the functions belongs to the ideal then so does the other, extends easily to ideals

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of $\mathcal{R}L$. It is shown in [8] that the lattice $\mathsf{Z}(\mathcal{R}L)$ of z-ideals of $\mathcal{R}L$ is a frame. We have recalled in the abstract what a d-ideal of a ring is. The lattice $\mathsf{D}(\mathcal{R}L)$ of d-ideals of $\mathcal{R}L$ is also a frame. Some categorical connections between $\mathsf{Z}(\mathcal{R}L)$ and $\mathsf{D}(\mathcal{R}L)$ are studied in [9].

Our aim in this note is to study some properties of the lattices $\mathsf{Z}(\mathcal{R}L)$ and $\mathsf{D}(\mathcal{R}L)$, including determining when they are Stone-Čech compactifications of some frames associated with L. We start by constructing a frame homomorphism ϱ_L mapping βL to $\mathsf{Z}(\mathcal{R}L)$. We show that it is an injective $*$-dense proper map, and its right adjoint is a frame homomorphism such that $\varrho_L(\varrho_L)_*$ is the identity map (Theorem 3.1). This leads to characterizations of the L for which the frame of z-ideals of $\mathcal{R}L$ is the Stone-Čech compactification of L (Corollary 3.5). They are precisely the P-frames. Since the lattice of z-ideals of $C(X)$ is isomorphic to $\mathsf{Z}(\mathfrak{O}X)$, and since X is a P-space if and only if $\mathfrak{O}X$ is a P-frame, taking spectra yields that the spectrum of the lattice of z-ideals of $C(X)$ is βX precisely when X is a P-space.

Composing the map $\varrho_L\colon \beta L \to \mathsf{Z}(\mathcal{R}L)$ with the homomorphism $\delta_L\colon \mathsf{Z}(\mathcal{R}L) \to \mathsf{D}(\mathcal{R}L)$ induced by the d-nucleus (see below) gives a frame homomorphism $\varrho_L\colon \beta L \to \mathsf{D}(\mathcal{R}L)$ mapping exactly as ϱ_L. It is an isomorphism if and only if L is basically disconnected (Proposition 4.2), which then tells us that if L is basically disconnected, then $\mathsf{D}(\mathcal{R}L)$ is the Stone-Čech compactification of L (Corollary 4.3). Applied to spaces, we have that if X is basically disconnected, then the spectrum of the lattice of d-ideals of $C(X)$ is βX.

On the other hand, $\mathsf{D}(\mathcal{R}L)$ is the Stone-Čech compactification of the booleanization of L precisely when every regular element of L is the pseudocomplement of a cozero element (Theorem 4.6).

2. Preliminaries

2.1. Frames and their homomorphisms. Our general references for frames are [14] and [17], and our notation is fairly standard. We denote by βL the Stone-Čech compactification of a completely regular frame L, and we take it to be the frame of completely regular ideals of L. We write

$$j_L\colon \beta L \to L \quad \text{given by} \quad I \mapsto \bigvee I$$

for the coreflection map from compact completely regular frames to L. The right adjoint of j_L is denoted by r_L. Recall that, for any $a \in L$,

$$r_L(a) = \{x \in L\colon x \prec\!\!\prec a\}.$$

We write $\operatorname{Coz} L$ for the set of cozero elements of L. For any $c, d \in \operatorname{Coz} L$, $r_L(c \vee d) = r_L(c) \vee r_L(d)$.

The *booleanization* of L will be denoted by $\mathfrak{B}L$. Recall that the join in $\mathfrak{B}L$ of any $S \subseteq \mathfrak{B}L$ is given by $(\bigvee S)^{**}$, and meets are calculated as in L. We write $\mathsf{b}_L\colon L \to \mathfrak{B}L$ for the dense onto frame map $x \mapsto x^{**}$. A frame homomorphism $h\colon L \to M$ is *skeletal* if, for any $a \in L$, $h(a^{**}) \leqslant h(a)^{**}$. These are precisely the homomorphisms that induce a frame homomorphism $\mathfrak{B}h\colon \mathfrak{B}L \to \mathfrak{B}M$ such that the square

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "\mathsf{b}_L"'] & M \arrow[d, "\mathsf{b}_M"] \\ \mathfrak{B}L \arrow[r, "\mathfrak{B}h"] & \mathfrak{B}M \end{tikzcd}$$

commutes.

2.2. The ring $\mathcal{R}L$ and some of its ideals. Our approach to the ring $\mathcal{R}L$ follows that of [3]. An ideal Q of $\mathcal{R}L$ is a *z-ideal* if, for any $\alpha, \beta \in \mathcal{R}L$, $\operatorname{coz}\alpha = \operatorname{coz}\beta$ and $\alpha \in Q$ imply $\beta \in Q$. The equality can be replaced with the inequality $\leqslant$. It is shown in [10] that, exactly as in $C(X)$, the sum of z-ideals of $\mathcal{R}L$ is a z-ideal. This actually is shown for two z-ideals, but simple calculation shows that it holds for any collection of z-ideals.

An ideal Q of $\mathcal{R}L$ is a d-ideal if and only if, for any $\alpha, \beta \in \mathcal{R}L$, $(\operatorname{coz}\alpha)^{**} = (\operatorname{coz}\beta)^{**}$ and $\alpha \in Q$ imply $\beta \in Q$. For any $a \in L$, the ideal $\boldsymbol{M}_a$ is defined by

$$\boldsymbol{M}_a = \{\alpha \in \mathcal{R}L\colon\ \operatorname{coz}\alpha \leqslant a\}.$$

Clearly, $\boldsymbol{M}_a$ is a z-ideal. For any $I \in \beta L$, ideal $\boldsymbol{O}^I$ of $\mathcal{R}L$ is defined by

$$\boldsymbol{O}^I = \{\alpha \in \mathcal{R}L\colon\ r_L(\operatorname{coz}\alpha) \prec\!\!\prec I\} = \{\alpha \in \mathcal{R}L\colon\ \operatorname{coz}\alpha \in I\}.$$

An ideal I of a ring is called *pure* if for every $u \in I$ there exists $v \in I$ such that $u = uv$. The pure ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{O}^I$ for $I \in \beta L$ (see [5], Proposition 4.3).

A surjective frame homomorphism $h\colon M \to L$ is called a *C^*-quotient map* (see [1]) if for every bounded $f \in \mathcal{R}L$, there is a (necessarily unique) $f^\beta \in \mathcal{R}M$ such that the triangle below commutes.

$$\begin{tikzcd} & \mathfrak{L}(\mathbb{R}) \arrow[dl, "f^\beta"'] \arrow[dr, "f"] & \\ M \arrow[rr, "h"] & & L \end{tikzcd}$$

2.3. Algebraic frames. We write $\mathfrak{k}(A)$ for the set of compact elements of a frame A. If $\mathfrak{k}(A)$ generates A, in the sense that every element of A is the join of compact elements below it, then A is said to be *algebraic*. An algebraic frame A is said to have the *finite intersection property* (FIP) if $a \wedge b \in \mathfrak{k}(A)$ for all $a, b \in \mathfrak{k}(A)$. A compact algebraic frame with FIP is called *coherent*, as is a frame homomorphism $\varphi\colon A \to B$ between coherent frames that takes compact elements to compact elements.

When we are dealing with coherent frames we shall denote the pseudocomplement of an element a by $a^{\perp}$, and refer to $a^{\perp}$ as the *polar* of a. The *d-nucleus* (see [15]) on an algebraic frame A with FIP is defined by

$$d(a) = \bigvee\{c^{\perp}\colon\ c \in \mathfrak{k}(A),\, c \leqslant a\}.$$

We write dA for the frame $\mathrm{Fix}(d)$, and denote by $d_A\colon\ A \to dA$ the dense onto frame homomorphism it induces. It is shown in [15] that $d_A(c) = c^{\perp\perp}$ for any $c \in \mathfrak{k}(A)$, and that $\mathfrak{k}(dA) = \{c^{\perp\perp}\colon\ c \in \mathfrak{k}(A)\}$. Also, $x^{\perp} \in dA$ for any $x \in A$, an upshot of which is that the polar of any $a \in dA$, considered in dA, is precisely the polar $a^{\perp}$ of a as an element of A. Elements of dA are called *d-elements* of A.

2.4. Frames of d-ideals and z-ideals in $\mathcal{R}L$. We summarize some results from the articles [7], [8], and [9] that we shall need. The lattices $\mathsf{Z}(\mathcal{R}L)$ and $\mathsf{D}(\mathcal{R}L)$ are normal coherent frames. Their sublattices of compact elements are, respectively, given by

$$\mathfrak{k}(\mathsf{Z}(\mathcal{R}L)) = \{\boldsymbol{M}_c\colon\ c \in \mathrm{Coz}\, L\} \qquad \text{and} \qquad \mathfrak{k}(\mathsf{D}(\mathcal{R}L)) = \{\boldsymbol{M}_{c^{**}}\colon\ c \in \mathrm{Coz}\, L\}.$$

It is shown in Proposition 4.1 of [7] that $\mathsf{D}(\mathcal{R}L) = d(\mathsf{Z}(\mathcal{R}L))$, where d denotes the d-nucleus on $\mathsf{Z}(\mathcal{R}L)$. We shall frequently denote the bottom of $\mathsf{Z}(\mathcal{R}L)$ and $\mathsf{D}(\mathcal{R}L)$ by $\perp$, and the top by $\top$.

For any frame L we have dense onto frame homomorphisms $\sigma_L\colon\ \mathsf{Z}(\mathcal{R}L) \to L$ and $\tau_L\colon\ \mathsf{D}(\mathcal{R}L) \to \mathfrak{B}L$ given by

$$\sigma_L(Q) = \bigvee\{\mathrm{coz}\,\alpha\colon\ \alpha \in Q\} \qquad \text{and} \qquad \tau_L(Q) = \Big(\bigvee\{\mathrm{coz}\,\alpha\colon\ \alpha \in Q\}\Big)^{**}.$$

A frame homomorphism $h\colon\ L \to M$ induces coherent maps $\mathsf{Z}(h)\colon\ \mathsf{Z}(\mathcal{R}L) \to \mathsf{Z}(\mathcal{R}M)$ and $\mathsf{D}(h)\colon\ \mathsf{D}(\mathcal{R}L) \to \mathsf{D}(\mathcal{R}M)$ such that the squares

$$\begin{tikzcd} \mathsf{Z}(\mathcal{R}L) \arrow[r, "\mathsf{Z}(h)"] \arrow[d, "\sigma_L"'] & \mathsf{Z}(\mathcal{R}M) \arrow[d, "\sigma_M"] \\ L \arrow[r, "h"] & M \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathsf{D}(\mathcal{R}L) \arrow[r, "\mathsf{D}(h)"] \arrow[d, "\tau_L"'] & \mathsf{D}(\mathcal{R}M) \arrow[d, "\tau_M"] \\ \mathfrak{B}L \arrow[r, "\mathfrak{B}h"] & \mathfrak{B}M \end{tikzcd}$$

commute; with the proviso of course that for the one on the right h must be a skeletal map. Explicitly, for any $Q \in \mathsf{Z}(\mathcal{R}L)$,

$$\mathsf{Z}(h)(Q) = \bigvee_{\mathsf{Z}(\mathcal{R}M)} \{\boldsymbol{M}_{h(\operatorname{coz}\alpha)} \colon \alpha \in Q\} = \bigcup \{\boldsymbol{M}_{h(\operatorname{coz}\alpha)} \colon \alpha \in Q\},$$

and, for any $Q \in \mathsf{D}(\mathcal{R}L)$,

$$\mathsf{D}(h)(Q) = \bigvee_{\mathsf{D}(\mathcal{R}M)} \{\boldsymbol{M}_{(h(\operatorname{coz}\alpha))^{**}} \colon \alpha \in Q\} = \bigcup \{\boldsymbol{M}_{(h(\operatorname{coz}\alpha))^{**}} \colon \alpha \in Q\}.$$

In each case the join is a union since the collection in question is up-directed.

We write $\delta_L \colon \mathsf{Z}(\mathcal{R}L) \to \mathsf{D}(\mathcal{R}L)$ for the coherent map induced by the d-nucleus on $\mathsf{Z}(\mathcal{R}L)$. To spell it out, for any $Q \in \mathsf{Z}(\mathcal{R}L)$,

$$\delta_L(Q) = \bigvee \{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \colon \alpha \in Q\} = \bigcup \{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \colon \alpha \in Q\}.$$

To avoid ambiguity, we shall denote the binary join in $\mathsf{D}(\mathcal{R}L)$ by $\sqcup$. It is shown in [9], Lemma 3.2 that if $c, d \in \operatorname{Coz} L$, then $\boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{d^{**}} = \boldsymbol{M}_{(c \vee d)^{**}}$.

3. Some properties of $\mathsf{Z}(\mathcal{R}L)$

Recall that a frame L is a *P-frame* if every cozero element of L is complemented. Less restricted than P-frames are *cozero complemented* frames, which are the L such that for every $c \in \operatorname{Coz} L$ there exists some $d \in \operatorname{Coz} L$ with $c \wedge d = 0$ and $c \vee d$ dense. Clearly, every P-frame is cozero complemented. It is shown in [8], Proposition 3.10 that $\mathsf{Z}(\mathcal{R}L)$ is regular if and only if L is a P-frame. On the other hand, $\mathsf{D}(\mathcal{R}L)$ is regular if and only if L is cozero-complemented, see [9], Proposition 3.6.

Recall from [12] that a frame homomorphism $h \colon L \to M$ is called *$*$-dense* if, for any $m \in M$, $h_*(m) = 0$ implies $m = 0$. These homomorphisms generalize the property of a continuous map sending no proper closed subset of its domain onto its codomain; that is, the irreducible maps. A frame homomorphism $h \colon L \to M$ is called *closed* precisely when, for every $a, b \in L$ and any $u \in M$,

$$h(a) \leqslant h(b) \vee u \implies a \leqslant b \vee h_*(u).$$

A closed frame homomorphism whose right adjoint preserves directed joins is called a *proper map*.

Theorem 3.1. *Let L be completely regular, and define a mapping $\varrho_L \colon \beta L \to \mathsf{Z}(\mathcal{R}L)$ by $\varrho_L(I) = \boldsymbol{O}^I$. The following statements hold:*

(a) *ϱ_L is an injective frame homomorphism making the triangle below commute,*

$$\begin{tikzcd}
\beta L \arrow[rr, "\varrho_L"] \arrow[dr, "j_L"'] & & \mathsf{Z}(\mathcal{R}L) \arrow[dl, "\sigma_L"] \\
& L &
\end{tikzcd}$$

(b) *ϱ_L is $*$-dense,*
(c) *$(\varrho_L)_*$ is a frame homomorphism, and is left inverse to ϱ_L,*
(d) *ϱ_L is a proper map.*

P r o o f. (a) We show first that ϱ_L is a frame homomorphism. It is immediate that ϱ_L maps $0_{\beta L}$ to $\perp$ and $1_{\beta L}$ to $\top$. Also clear is that ϱ_L is order-preserving. It is easy to check that, for any $I, J \in \beta L$, $\boldsymbol{O}^{I\wedge J} = \boldsymbol{O}^I \cap \boldsymbol{O}^J$, whence ϱ_L preserves finite meets. Let $\{I_\lambda\colon\ \lambda \in \Lambda\} \subseteq \beta L$. Since the sum of z-ideals is a z-ideal in $\mathcal{R}L$,

$$\bigvee_{\mathsf{Z}(\mathcal{R}L)} \varrho_L(I_\lambda) = \bigvee_{\mathsf{Z}(\mathcal{R}L)} \boldsymbol{O}^{I_\lambda} = \sum_\lambda \boldsymbol{O}^{I_\lambda},$$

so if we can show that $\boldsymbol{O}^{\bigvee_\lambda I_\lambda} \subseteq \sum\limits_\lambda \boldsymbol{O}^{I_\lambda}$, it will follow that ϱ_L preserves arbitrary joins. Let f be in the set on the left; which says $\operatorname{coz} f \in \bigvee\limits_\lambda I_\lambda$. We can therefore find finitely many indices $\lambda_1, \ldots, \lambda_n$ and positive functions $f_{\lambda_1}, \ldots, f_{\lambda_n}$ such that each $\operatorname{coz}(f_{\lambda_i}) \in I_{\lambda_i}$, and

$$\operatorname{coz} f = \operatorname{coz}(f_{\lambda_1}) \vee \ldots \vee \operatorname{coz}(f_{\lambda_n}) = \operatorname{coz}(f_{\lambda_1} + \ldots + f_{\lambda_n}).$$

Observe that

$$f_{\lambda_1} + \ldots + f_{\lambda_n} \in \boldsymbol{O}^{I_{\lambda_1}} + \ldots + \boldsymbol{O}^{I_{\lambda_1}} \subseteq \sum_\lambda \boldsymbol{O}^{I_\lambda},$$

which, in light of $\sum\limits_\lambda \boldsymbol{O}^{I_\lambda}$ being a z-ideal, implies $f \in \sum\limits_\lambda \boldsymbol{O}^{I_\lambda}$. Therefore $\boldsymbol{O}^{\bigvee_\lambda I_\lambda} \subseteq \sum\limits_\lambda \boldsymbol{O}^{I_\lambda}$, and so ϱ_L preserves joins, and is thus a frame homomorphism.

Next, we show that ϱ_L is injective. Let $I, J \in \beta L$ be such that $\boldsymbol{O}^I = \boldsymbol{O}^J$. To show that $I = J$, it suffices (by symmetry) to show that every cozero element in I is in J because, as ideals, I and J are generated by their cozero members. Let c be a cozero element in I. Pick $f \in \mathcal{R}L$ with $c = \operatorname{coz} f$. Then $f \in \boldsymbol{O}^I = \boldsymbol{O}^J$, which implies $c \in J$.

We now show that the triangle commutes. Let $I \in \beta L$. Note that since I is a completely regular ideal, $\bigvee I = \bigvee\{c \in \operatorname{Coz} L\colon\ c \in I\}$. Consequently,

$$\begin{aligned}\sigma_L \varrho_L(I) = \sigma_L(\boldsymbol{O}^I) &= \bigvee\{\operatorname{coz}\alpha\colon\ \alpha \in \boldsymbol{O}^I\} \\ &= \bigvee\{\operatorname{coz}\alpha\colon\ \operatorname{coz}\alpha \in I\} = \bigvee I = j_L(I),\end{aligned}$$

which proves commutativity of the triangle.

(b) We start by calculating the right adjoint of ϱ_L. We claim that, for any $Q \in \mathsf{Z}(\mathcal{R}L)$,

$$(\varrho_L)_*(Q) = \bigvee\{r_L(\operatorname{coz}\alpha)\colon\ \alpha \in Q\}.$$

For any $\alpha \in Q$, it is clear that $\boldsymbol{O}^{r_L(\operatorname{coz}\alpha)} \subseteq Q$ since Q is a z-ideal. Therefore

$$\begin{aligned}\varrho_L\Big(\bigvee\{r_L(\operatorname{coz}\alpha)\colon\ \alpha \in Q\}\Big) &= \bigvee\{\varrho_L(r_L(\operatorname{coz}\alpha))\colon\ \alpha \in Q\}\\ &= \bigvee\{\boldsymbol{O}^{r_L(\operatorname{coz}\alpha)}\colon\ \alpha \in Q\} \subseteq Q.\end{aligned}$$

On the other hand, if $\varrho_L(I) \subseteq Q$, then $\boldsymbol{O}^I \subseteq Q$. For any $t \in I$ we can find $\gamma \in \mathcal{R}L$ such that $t \prec\!\!\prec \operatorname{coz}\gamma \in I$. Then $\gamma \in Q$, and hence $t \in \bigvee\{r_L(\operatorname{coz}\alpha)\colon\ \alpha \in Q\}$, showing that $I \subseteq \bigvee\{r_L(\operatorname{coz}\alpha)\colon\ \alpha \in Q\}$. Thus, $(\varrho_L)_*(Q) = \bigvee\{r_L(\operatorname{coz}\alpha)\colon\ \alpha \in Q\}$, as claimed. Consequently, if $(\varrho_L)_*(Q) = 0_{\beta L}$, then, for any $\alpha \in Q$, $r_L(\operatorname{coz}\alpha) = 0_{\beta L}$, which implies $\operatorname{coz}\alpha = 0$, and hence $\alpha = \mathbf{0}$, so that Q is the zero ideal, which is the bottom element of $\mathsf{Z}(\mathcal{R}L)$. Therefore ϱ_L is $*$-dense.

(c) Since $(\varrho_L)_*$ preserves the top and bottom elements (the latter in view of ϱ_L being dense as it is injective), we need to show that it preserves joins. So let $\{Q_i\colon i \in I\} \subseteq \mathsf{Z}(\mathcal{R}L)$. Then

$$(\varrho_L)_*\Big(\bigvee_i Q_i\Big) = (\varrho_L)_*\Big(\sum_i Q_i\Big) = \bigvee\Big\{r_L(\operatorname{coz}\xi)\colon\ \xi \in \sum_i Q_i\Big\}.$$

Given $\xi \in \sum_i Q_i$, we can find finitely many indices $i_1, \dots, i_n$ in I and elements $\xi_{i_k} \in Q_{i_k}$ such that $\xi = \xi_{i_1} + \ldots + \xi_{i_n}$. Then $\operatorname{coz}\xi \leqslant \operatorname{coz}(\xi_{i_1}) \vee \ldots \vee \operatorname{coz}(\xi_{i_n})$, which implies

$$\begin{aligned}r_L(\operatorname{coz}\xi) &\leqslant r_L(\operatorname{coz}(\xi_{i_1})) \vee \ldots \vee r_L(\operatorname{coz}(\xi_{i_n}))\\ &\leqslant (\varrho_L)_*(Q_{i_1}) \vee \ldots \vee (\varrho_L)_*(Q_{i_n})\\ &\leqslant \bigvee\{(\varrho_L)_*(Q_i)\colon\ i \in I\},\end{aligned}$$

so that $(\varrho_L)_*\big(\bigvee_i Q_i\big) \leqslant \bigvee_i (\varrho_L)_*(Q_i)$, and hence equality.

To see that $(\varrho_L)_*$ is left inverse to ϱ_L, we need only show that $(\varrho_L)_*\varrho_L \leqslant \mathrm{id}_{\beta L}$. For any $I \in \beta L$ we have

$$\begin{aligned}(\varrho_L)_*\varrho_L(I) &= \bigvee\{r_L(\operatorname{coz}\alpha)\colon\ \alpha \in \varrho_L(I)\} = \bigvee\{r_L(\operatorname{coz}\alpha)\colon\ \alpha \in \boldsymbol{O}^I\}\\ &= \bigvee\{r_L(\operatorname{coz}\alpha)\colon\ \operatorname{coz}\alpha \in I\} \leqslant I,\end{aligned}$$

as desired.

(d) Let $I, J \in \beta L$ and $Q \in \mathsf{Z}(\mathcal{R}L)$ be such that $\varrho_L(I) \leqslant \varrho_L(J) \vee Q$. Deciphering, and using the fact that the join in $\mathsf{Z}(\mathcal{R}L)$ is the sum, this implies $\boldsymbol{O}^I \subseteq \boldsymbol{O}^J + Q$. Let $a \in I$, and pick $\gamma \in \mathcal{R}L$ such that $a \prec\!\!\prec \operatorname{coz}\gamma$ and $\operatorname{coz}\gamma \in I$. Then $\gamma \in \boldsymbol{O}^I$, hence there exist $\delta \in \boldsymbol{O}^J$ and $\xi \in Q$ such that $\gamma = \delta + \xi$. Therefore $\operatorname{coz}\gamma \leqslant \operatorname{coz}\delta \vee \operatorname{coz}\xi$, which implies

$$r_L(\operatorname{coz}\gamma) \subseteq r_L(\operatorname{coz}\delta \vee \operatorname{coz}\xi) = r_L(\operatorname{coz}\delta) \vee r_L(\operatorname{coz}\xi).$$

Since $a \in r_L(\operatorname{coz}\gamma)$, there exists $c \in r_L(\operatorname{coz}\delta)$ and $v \in r_L(\operatorname{coz}\xi)$ such that $a = c \vee v$. But $\operatorname{coz}\delta \in J$ and $v \in (\varrho_L)_*(Q)$; so $a \in J \vee (\varrho_L)_*(Q)$. Therefore ϱ_L is a closed map. Hence it is a proper map because its right adjoint preserves directed joins as it is a frame homomorphism. □

R e m a r k 3.2. That ϱ_L is a frame homomorphism can also be deduced from [4], Proposition 3.11 by noting that the frame of pure ideals on $\mathcal{R}L$ is a subframe of $\mathsf{Z}(\mathcal{R}L)$.

R e m a r k 3.3. In [18] Plewe defines triquotient maps in Loc, and obverses that they include retractions in Loc. It follows therefore that, for any frame L, the localic map $(\varrho_L)_*\colon \mathsf{Z}(\mathcal{R}L) \to \beta L$ is a triquotient map.

For the following result let us recall how the functor $\beta\colon \mathsf{CRFrm} \to \mathsf{KRFrm}$ acts on morphisms. For any $h\colon L \to M$ in CRFrm, $\beta h\colon \beta L \to \beta M$ is the unique frame homomorphism making the square

$$\begin{tikzcd} \beta L \arrow[r, "\beta h"] \arrow[d, "j_L"'] & \beta M \arrow[d, "j_M"] \\ L \arrow[r, "h"] & M \end{tikzcd}$$

commute. Explicitly, for any $I \in \beta L$,

$$\beta h(I) = \{v \in M\colon\ v \leqslant h(u) \text{ for some } u \in I\}.$$

Corollary 3.4. *For any frame homomorphism $h\colon L \to M$, the diagram below commutes.*

$$\begin{tikzcd} \beta L \arrow[rrr, "\beta h"] \arrow[dr, "\varrho_L"] \arrow[dd, "j_L"'] & & & \beta M \arrow[dl, "\varrho_M"'] \arrow[dd, "j_M"] \\ & \mathsf{Z}(\mathcal{R}L) \arrow[r, "\mathsf{Z}(h)"] \arrow[dl, "\sigma_L"'] & \mathsf{Z}(M) \arrow[dr, "\sigma_M"] & \\ L \arrow[rrr, "h"] & & & M \end{tikzcd}$$

P r o o f. In Subsection 2.4 we recalled (from [7]) that the lower trapezoid commutes, and we already know from the foregoing theorem that the triangles commute. So we are left with showing that the upper trapezoid commutes. Let $I \in \beta L$. Then, in view of the fact that $\alpha \in \boldsymbol{O}^I$ if and only if $\operatorname{coz}\alpha \in I$,

$$\mathsf{Z}(h)\varrho_L(I) = \mathsf{Z}(h)(\boldsymbol{O}^I) = \bigcup\bigl\{\boldsymbol{M}_{h(\operatorname{coz}\alpha)}\colon \operatorname{coz}\alpha \in I\bigr\}.$$

On the other hand, $\varrho_M(\beta h)(I) = \boldsymbol{O}^{\beta h(I)}$, so we need to show that

$$\bigcup\bigl\{\boldsymbol{M}_{h(\operatorname{coz}\alpha)}\colon \operatorname{coz}\alpha \in I\bigr\} = \boldsymbol{O}^{\beta h(I)}.$$

Now, for any $\gamma \in \mathcal{R}L$,

$$\begin{aligned}
\gamma \in \bigcup\bigl\{\boldsymbol{M}_{h(\operatorname{coz}\alpha)}\colon \operatorname{coz}\alpha \in I\bigr\} &\iff \gamma \in \boldsymbol{M}_{h(\operatorname{coz}\alpha)}, \text{ for some } \operatorname{coz}\alpha \in I \\
&\iff \operatorname{coz}\gamma \leqslant h(\operatorname{coz}\alpha), \text{ for some } \operatorname{coz}\alpha \in I \\
&\iff \operatorname{coz}\gamma \in \beta h(I) \\
&\iff \gamma \in \boldsymbol{O}^{\beta h(I)},
\end{aligned}$$

which establishes the equality. The upper trapezoid therefore also commutes. □

Corollary 3.5. *The following are equivalent for a completely regular frame L:*

(1) $\sigma_L\colon \mathsf{Z}(\mathcal{R}L) \to L$ *is the Stone-Čech compactification of* L,
(2) $\mathsf{Z}(\mathcal{R}L)$ *is regular,*
(3) ϱ_L *is surjective,*
(4) L *is a* P*-frame.*

Since a completely regular Hausdorff space is a P-space if and only if the frame of its open sets is a P-frame, we have the following topological interpretation. Let $\mathscr{C}_z(X)$ denote the lattice of z-ideals of a completely regular Hausdorff space X.

Corollary 3.6. *The spectrum of* $\mathscr{C}_z(X)$ *is* βX *if and only if* X *is a* P*-space.*

Let us note that, for any $I \in \beta L$, the ideal $\boldsymbol{O}^I$ is a d-ideal of $\mathcal{R}L$. For, if $(\operatorname{coz}\alpha)^{**} = (\operatorname{coz}\gamma)^{**}$ and $\gamma \in \boldsymbol{O}^I$, then $\operatorname{coz}\gamma \in I$, and so we can find $\eta \in \mathcal{R}L$ such that $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\eta$ and $\operatorname{coz}\eta \in I$. Thus,

$$\operatorname{coz}\alpha \leqslant (\operatorname{coz}\alpha)^{**} = (\operatorname{coz}\gamma)^{**} \leqslant \operatorname{coz}\eta,$$

so that $\operatorname{coz}\alpha \in I$, and hence $\alpha \in \boldsymbol{O}^I$. Consequently, for the map $\delta_L\colon \mathsf{Z}(\mathcal{R}L) \to \mathsf{D}(\mathcal{R}L)$ recalled in Subsection 2.4, we have $\delta_L(\boldsymbol{O}^I) = \boldsymbol{O}^I$. Thus, the map $\varrho_L\colon \beta L \to \mathsf{D}(\mathcal{R}L)$ given by $I \mapsto \boldsymbol{O}^I$ is exactly the composite $\beta L \xrightarrow{\varrho_L} \mathsf{Z}(\mathcal{R}L) \xrightarrow{\delta_L} \mathsf{D}(\mathcal{R}L)$, and is therefore a frame homomorphism. We show that it also makes the square below commute.

$$\begin{tikzcd}
\beta L \arrow[r, "\varrho_L"] \arrow[d, "j_L"'] & \mathsf{D}(\mathcal{R}L) \arrow[d, "\tau_L"] \\
L \arrow[r, "\mathsf{b}_L"] & \mathfrak{B}L
\end{tikzcd}$$

Indeed, for any $I \in \beta L$,

$$\begin{aligned}
\tau_L\varrho_L(I) = \tau_L(\boldsymbol{O}^I) &= \Big(\bigvee\{\operatorname{coz}\alpha\colon\ \alpha \in \boldsymbol{O}^I\}\Big)^{**} = (\{\operatorname{coz}\alpha\colon\ \operatorname{coz}\alpha \in I\})^{**} \\
&= \Big(\bigvee I\Big)^{**} = \mathsf{b}_L j_L(I).
\end{aligned}$$

In the case of $\mathsf{D}(\mathcal{R}L)$ we also have a result analogous to Corollary 3.4. Below we write $\mu_L = \mathsf{b}_L j_L$, and similarly for M.

Corollary 4.1. *For any skeletal frame homomorphism $h\colon L \to M$, the diagram below commutes.*

$$\begin{tikzcd}
\beta L \arrow[rrr, "\beta h"] \arrow[dr, "\varrho_L"] \arrow[dd, "\mu_L"'] & & & \beta M \arrow[dl, "\varrho_M"'] \arrow[dd, "\mu_M"] \\
& \mathsf{D}(\mathcal{R}L) \arrow[r, "\mathsf{D}(h)"] \arrow[dl, "\tau_L"'] & \mathsf{D}(\mathcal{R}M) \arrow[dr, "\tau_M"] & \\
\mathfrak{B}L \arrow[rrr, "\mathfrak{B}h"] & & & \mathfrak{B}M
\end{tikzcd}$$

P r o o f. We know that the lower trapezoid commutes. The triangles commute in light of the square immediately above. So we must show that the upper trapezoid commutes. The proof for that is essentially the same as in Corollary 3.4. Simply note that if $c \in I$, where $I \in \beta L$, then there is a $d \in I$ such that $c \prec\!\!\prec d$, and for this d we have $(h(c))^{**} \leqslant h(d)$. $\square$

We shall now determine when ϱ_L is onto (in which case it is an isomorphism). We need some background. Recall that a frame L is *basically disconnected* if $c^* \vee c^{**} = 1$ for every $c \in \operatorname{Coz} L$. A convenient characterization for present purposes is [5], Proposition 3.3, which states that

a frame L is basically disconnected if and only if every d-ideal of $\mathcal{R}L$ is generated by idempotents.

Let us also recall from [6], Corollary 3.3 that an ideal Q of $\mathcal{R}L$ is pure if and only if for every $\alpha \in Q$ there is a $\beta \in Q$ such that $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\beta$. The idempotents of $\mathcal{R}L$ are precisely the $\eta \in \mathcal{R}L$ for which $\operatorname{coz}\eta$ is complemented, see [5], Proposition 2.2.

Theorem 4.2. *The following are equivalent for a completely regular frame L:*

(1) *ϱ_L is surjective (in which case it is an isomorphism),*
(2) *every d-ideal of $\mathcal{R}L$ is pure,*
(3) *L is basically disconnected.*

P r o o f. The equivalence of (1) and (2) is immediate.

(2) $\Rightarrow$ (3): Let $c \in \operatorname{Coz} L$, and consider the d-ideal $\boldsymbol{M}_{c^{**}}$. Pick $\gamma \in \mathcal{R}L$ with $\operatorname{coz}\gamma = c$. Then $\gamma \in \boldsymbol{M}_{c^{**}}$. By hypothesis, the purity of $\boldsymbol{M}_{c^{**}}$ implies that there is a $\delta \in \boldsymbol{M}_{c^{**}}$ such that $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\delta$. Then $c \prec\!\!\prec \operatorname{coz}\delta \leqslant c^{**}$, which implies $c^* \vee c^{**} = 1$. Therefore L is basically disconnected.

(3) $\Rightarrow$ (2): Assume L is basically disconnected, and let Q be a d-ideal in $\mathcal{R}L$. By the result cited above, Q is generated by some set $\{\eta_i \colon i \in I\}$, say, consisting of idempotents. Let $\alpha \in Q$. Then there are finitely many indices $i_1, \ldots, i_n$ in I and elements $\alpha_{i_1}, \ldots, \alpha_{i_n}$ in $\mathcal{R}L$ such that $\alpha = \alpha_{i_1}\eta_{i_1} + \ldots + \alpha_{i_n}\eta_{i_n}$. Now,

$$\begin{aligned}
\operatorname{coz}\alpha &\leqslant \operatorname{coz}(\alpha_{i_1}\eta_{i_1}) \vee \ldots \vee \operatorname{coz}(\alpha_{i_n}\eta_{i_n}) \\
&\leqslant \operatorname{coz}(\eta_{i_1}) \vee \ldots \vee \operatorname{coz}(\eta_{i_n}) \\
&\prec\!\!\prec \operatorname{coz}(\eta_{i_1}) \vee \ldots \vee \operatorname{coz}(\eta_{i_n}) \qquad \text{since each } \operatorname{coz}(\eta_{i_k}) \text{ is complemented} \\
&= \operatorname{coz}(\eta_{1_1} + \ldots + \eta_{i_n}) \qquad \text{since idempotents are positive.}
\end{aligned}$$

Since $\eta_{1_1} + \ldots + \eta_{i_n} \in Q$, it follows from the characterization of purity cited above that Q is a pure ideal. $\square$

Corollary 4.3. *If L is basically disconnected, then the map $\mathsf{D}(\mathcal{R}L) \to L$ given by $Q \mapsto \bigvee \operatorname{coz}[Q]$ is the Stone-Čech compactification of L.*

P r o o f. By the preceding proposition, if L is basically disconnected then ϱ_L is an isomorphism. So it suffices to show that the composite $j_L\varrho_L^{-1} \colon \mathsf{D}(\mathcal{R}L) \to L$ sends any d-ideal Q to $\bigvee \operatorname{coz}[Q]$. Let $Q \in \mathsf{D}(\mathcal{R}L)$. Then $Q = \boldsymbol{O}^I$ for some (uniquely determined) $I \in \beta L$. Then $j_L \varrho_L^{-1}(Q) = \bigvee I$. If $\alpha \in Q$, then $\operatorname{coz}\alpha \in I$, and therefore $\bigvee \operatorname{coz}[Q] \leqslant \bigvee I$. But any $x \in I$ is below some $\operatorname{coz}\gamma \in I$, so $\bigvee I \leqslant \bigvee \operatorname{coz}[Q]$, and hence equality. $\square$

Write $\mathscr{C}_d(X)$ for the lattice of d-ideals of $C(X)$. Since a completely regular Hausdorff space X is basically disconnected if and only if the frame $\mathfrak{O}X$ is basically disconnected, we have the following result.

Corollary 4.4. *If X is basically disconnected, then the spectrum of $\mathscr{C}_d(X)$ is βX.*

We have already mentioned that $\mathsf{D}(\mathcal{R}L)$ is regular precisely when L is cozero complemented. Since τ_L is always dense onto, we therefore have that $\tau_L\colon \mathsf{D}(\mathcal{R}L) \to \mathfrak{B}L$ is a compactification of $\mathfrak{B}L$ precisely when L is cozero complemented. Call a frame L *cozero approximated* if for every $a \in L$ there exists some $c \in \operatorname{Coz} L$ such that $a^* = c^*$. These frames extend Gruenhage's cozero approximated spaces (see [11]), which are the Tychonoff spaces X such that for every open set U there is a cozero-set V such that $\overline{U} = \overline{V}$. These spaces are called *fraction dense* in [13].

Observation 4.5. Every cozero approximated frame L is cozero complemented. For, if $c \in \operatorname{Coz} L$, then there exists $d \in \operatorname{Coz} L$ such that $c^{**} = d^*$. Thus, $c \wedge d \leqslant c^{**} \wedge d = 0$, and $(c \vee d)^* = c^* \wedge d^* = c^* \wedge c^{**} = 0$, showing that $c \vee d$ is dense.

For use in the following proof, we recall from [15], Theorem 2.4 (a) that an algebraic frame is regular if and only if its compact elements are complemented. For normal coherent frames this was first proved by Banaschewski in [2]. In the proof that follows we shall use the following characterization of C^*-quotient map, which appears as part of [1], Theorem 7.1.1:

A quotient map $h\colon M \to L$ is C^*-quotient map if and only if for every $c, d \in \operatorname{Coz} L$ with $c \vee d = 1$, there exist $u, v \in \operatorname{Coz} M$ such that $u \vee v = 1$, $h(u) \leqslant c$, and $h(v) \leqslant d$.

We shall write the join in $\mathfrak{B}L$ as $\sqcup$, similarly to the join in $\mathsf{D}(\mathcal{R}L)$. This is to avoid confusion since $\mathfrak{B}L \subseteq L$.

Theorem 4.6. *A necessary and sufficient condition that $\tau_L\colon \mathsf{D}(\mathcal{R}L) \to \mathfrak{B}L$ be the Stone-Čech compactification of $\mathfrak{B}L$ is that L be cozero approximated.*

Proof. To show that the condition is sufficient, assume that L is cozero approximated. Then L is cozero complemented, hence $\tau_L\colon \mathsf{D}(\mathcal{R}L) \to \mathfrak{B}L$ is a compactification of $\mathfrak{B}L$. By [1], Corollary 8.2.7, it suffices to show that τ_L is a C^*-quotient map, and for that we use the characterization cited above. Since $\operatorname{Coz}(\mathfrak{B}L) = \mathfrak{B}L$, consider any $a, b \in L$ such that $a^{**} \sqcup b^{**} = 1$, that is, $(a \vee b)^{**} = 1$. We must produce $I, J \in \operatorname{Coz}(\mathsf{D}(\mathcal{R}L))$ such that $I \sqcup J = \top$, $\tau_L(I) \leqslant a^{**}$, and $\tau_L(J) \leqslant b^{**}$. Since L

is cozero approximated, there are cozero elements $c, d \in L$ such that $a^* = c^*$ and $b^* = d^*$. Observe that

$$(c \vee d)^{**} = (c^* \wedge d^*)^* = (a^* \wedge b^*)^* = (a \vee b)^{**},$$

so that

$$\boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{d^{**}} = \boldsymbol{M}_{(c\vee d)^{**}} = \boldsymbol{M}_{(a\vee b)^{**}} = \boldsymbol{M}_1 = \top.$$

Since $\mathsf{D}(\mathcal{R}L)$ is regular under the current hypothesis, and since $\mathsf{D}(\mathcal{R}L)$ is an algebraic frame, its compact elements are complemented, and are therefore cozero elements. Thus, $\boldsymbol{M}_{c^{**}}$ and $\boldsymbol{M}_{d^{**}}$ are cozero elements of $\mathsf{D}(\mathcal{R}L)$. Now,

$$\tau_L(\boldsymbol{M}_{c^{**}}) = c^{**} = a^{**} \quad \text{and} \quad \tau_L(\boldsymbol{M}_{d^{**}}) = d^{**} = b^{**};$$

so it follows that $\tau_L\colon \mathsf{D}(\mathcal{R}L) \to \mathfrak{B}L$ is the Stone-Čech compactification of $\mathfrak{B}L$.

Conversely, suppose $\tau_L\colon \mathsf{D}(\mathcal{R}L) \to \mathfrak{B}L$ is the Stone-Čech compactification of $\mathfrak{B}L$. Let $a \in L$. Then a^* and a^{**} are cozero elements of $\mathfrak{B}L$ with $a^* \sqcup a^{**} = 1$. By [1], Corollary 8.2.7, $(\tau_L)_*(a^*) \sqcup (\tau_L)_*(a^{**}) = \top$. Since $\mathsf{D}(\mathcal{R}L)$ is coherent, there are two compact elements of $\mathsf{D}(\mathcal{R}L)$, one below $(\tau_L)_*(a^*)$ and the other below $(\tau_L)_*(b^*)$, and such that they join at the top. So, there exist $c, d \in \operatorname{Coz} L$ such that $\boldsymbol{M}_{c^{**}} \leqslant (\tau_L)_*(a^*)$, $\boldsymbol{M}_{d^{**}} \leqslant (\tau_L)_*(a^{**})$, and $\boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{d^{**}} = \top$. From all this we deduce (on applying the map τ_L) that $c^{**} \leqslant a^*$, $d^{**} \leqslant a^{**}$, and $(c \vee d)^{**} = 1$. The latter implies $c^* \wedge d^* = 0$, so that $d^* \leqslant c^{**}$. Consequently, $a^* \leqslant d^* \leqslant c^{**} \leqslant a^*$, whence $a^* = d^*$. Therefore L is cozero approximated. $\square$

Recall that a frame is called *perfectly normal* if it is normal and every element in it is a join of countably many elements each rather below it. Metrizable frames are perfectly normal. Among completely regular frames, the perfectly normal ones are exactly those in which every element is a cozero element.

Corollary 4.7. *The Stone-Čech compactification of the booleanization of any perfectly normal frame L is $\mathsf{D}(\mathcal{R}L)$.*

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