APPROXIMATE TRI-QUADRATIC FUNCTIONAL EQUATIONS VIA LIPSCHITZ CONDITIONS

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Received February 17, 2016. First published January 26, 2017.
Communicated by Leonid Berezansky

Abstract. In this paper, we consider Lipschitz conditions for tri-quadratic functional equations. We introduce a new notion similar to that of the left invariant mean and prove that a family of functions with this property can be approximated by tri-quadratic functions via a Lipschitz norm.

Keywords: tri-quadratic functional equation; Lipschitz space; stability

MSC 2010: 39B82, 39B52

1. Introduction

A generalized stability problem for the quadratic functional equation

$$\mathcal{Q}(x+y)+\mathcal{Q}(x-y)=2\mathcal{Q}(x)+2\mathcal{Q}(y)$$

was proved by Skof in [11] for mappings from a normed space to a Banach space. Czerwik et al. in [1] verified the stability of the quadratic functional equations in Lipschitz spaces. The Lipschitz stability type problems for some functional equations were also studied by Tabor, see [12], [13]. In Lipschitz spaces we investigated the stability of cubic functional equations in [2] and the stability of quartic functional equations in [7] (see also [6], [5]). The stability problem for the quadratic and bi-quadratic functional equation has been studied by many mathematicians under various degrees of generality imposed on the equation or on the underlying space; see, for example, [3], [4], [10], [9] and the references therein. We obtained Lipschitz criteria for bi-quadratic functional equations in Lipschitz spaces in [8].

The algebra of Lipschitz functions on a complete metric space plays a role in noncommutative metric theory similar to that played by the algebra of continuous

DOI: 10.21136/MB.2017.0028-16

functions on a compact space in noncommutative topology. Let $\mathcal{H}$ be an abelian group and $\mathcal{W}$ a real vector space. A function $\mathcal{Q}\colon \mathcal{H}^3 \to \mathcal{W}$ is called tri-quadratic if $\mathcal{Q}$ satisfies the system of equations

$$\begin{aligned}\mathcal{Q}(x+y,z,w)+\mathcal{Q}(x-y,z,w)&=2\mathcal{Q}(x,z,w)+2\mathcal{Q}(y,z,w),\\ \mathcal{Q}(x,y+z,w)+\mathcal{Q}(x,y-z,w)&=2\mathcal{Q}(x,y,w)+2\mathcal{Q}(x,z,w),\\ \mathcal{Q}(x,y,z+w)+\mathcal{Q}(x,y,z-w)&=2\mathcal{Q}(x,y,z)+2\mathcal{Q}(x,y,w)\end{aligned}$$

for all $x,y,z \in \mathcal{H}$, that is, $\mathcal{Q}$ is quadratic in each variable. In this paper, we consider Lipschitz conditions for tri-quadratic functional equations. We prove that a family of functions satisfying tri-symmetric left invariant mean property can be approximated by tri-quadratic functions via a Lipschitz norm.

2. Lipschitz conditions for tri-quadratic functional equations

In this section, we introduce the notion of tri-symmetric left invariant mean (TSLIM in brief) and prove that a family of functions with TSLIM property can be approximated by tri-quadratic functions via a Lipschitz norm.

A family S of subsets of $\mathcal{W}$ is called linearly invariant if $A+\alpha B \in S$ for $A, B \in S$, $\alpha \in \mathbb{R}$ and $x + A \in S$ for $A \in S$, $x \in \mathcal{W}$. For example, the family of all closed balls with center at zero is a linearly invariant family in a normed vector space. We denote this family by $CB(\mathcal{W})$. Let $\mathcal{L}(\mathcal{W})$ be a linearly invariant family of subsets of $\mathcal{W}$. By $\mathcal{M}(\mathcal{H},\mathcal{L}(\mathcal{W}))$ we denote the family of all functions $\mathcal{Q}\colon \mathcal{H} \to \mathcal{W}$ such that $\operatorname{Im}\mathcal{Q} \subset B$ for some $B \in \mathcal{L}(\mathcal{W})$.

Definition 2.1. The function $\mathcal{Q}$ is called tri-symmetric if

$$\mathcal{Q}(x,y,z)=\mathcal{Q}(y,z,x)=\mathcal{Q}(z,x,y)=\mathcal{Q}(z,y,x)=\mathcal{Q}(x,z,y)=\mathcal{Q}(y,x,z)$$

for all $x,y,z \in \mathcal{H}$.

Definition 2.2. We say that $\mathcal{M}(\mathcal{H},\mathcal{L}(\mathcal{W}))$ admits a tri-symmetric left invariant mean (briefly TSLIM), if the family $\mathcal{L}(\mathcal{W})$ is linearly invariant and there exists a linear operator $\Gamma\colon \mathcal{M}(\mathcal{H},\mathcal{L}(\mathcal{W})) \to \mathcal{W}$ such that

(i) if $\mathcal{Q}_{x,y,z} \in \mathcal{M}(\mathcal{H},\mathcal{L}(\mathcal{W}))$ and $x,y,z \in \mathcal{H}$, then

$$\Gamma[\mathcal{Q}_{x,y,z}]=\Gamma[\mathcal{Q}_{y,z,x}]=\Gamma[\mathcal{Q}_{z,x,y}]=\Gamma[\mathcal{Q}_{z,y,x}]=\Gamma[\mathcal{Q}_{x,z,y}]=\Gamma[\mathcal{Q}_{y,x,z}],$$

(ii) if $\operatorname{Im}\mathcal{Q} \subset B$ for some $B \in \mathcal{L}(\mathcal{W})$, then $\Gamma[\mathcal{Q}] \in B$,

(iii) if $\mathcal{Q} \in \mathcal{M}(\mathcal{H},\mathcal{L}(\mathcal{W}))$ and $a \in \mathcal{H}$, then $\Gamma[\mathcal{Q}^a]=\Gamma[\mathcal{Q}]$, where $\mathcal{Q}^a(x)=\mathcal{Q}(x+a)$.

Definition 2.3. Let $\Delta\colon \mathcal{H}^3 \times \mathcal{H}^3 \to \mathcal{L}(\mathcal{W})$ be a set-valued function such that

$$\begin{aligned}&\Delta((x+a, y+b, z+c), (u+a, v+b, w+c))\\ &\quad= \Delta((a+x, b+y, c+z), (a+u, b+v, c+w)) = \Delta((x, y, z), (u, v, w))\end{aligned}$$

for all $(a, b, c), (x, y, z), (u, v, w) \in \mathcal{H}^3$. A function $\mathcal{Q}\colon \mathcal{H}^3 \to \mathcal{W}$ is said to be Δ-Lipschitz if

$$\mathcal{Q}(x, y, z) - \mathcal{Q}(u, v, w) \in \Delta((x, y, z), (u, v, w))$$

for all $(x, y, z), (u, v, w) \in \mathcal{H}^3$.

Let $\mathcal{Q}\colon \mathcal{H}^3 \to \mathcal{W}$ be a function. We consider its tri-quadratic difference as follows:

$$T\mathcal{Q}(x, y, z, w) := 2\mathcal{Q}(x, z, w) + 2\mathcal{Q}(y, z, w) - \mathcal{Q}(x+y, z, w) - \mathcal{Q}(x-y, z, w)$$

for all $x, y, z, w \in \mathcal{H}$.

Theorem 2.4. *Let $\mathcal{H}$ be an abelian group and let $\mathcal{W}$ be a vector space. Assume that the family $\mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ admits* TSLIM. *If $\mathcal{F}\colon \mathcal{H}^3 \to \mathcal{W}$ is a function and $T\mathcal{F}(t, \cdot, \cdot, \cdot)\colon \mathcal{H}^3 \to \mathcal{W}$ is Δ-Lipschitz for every $t \in \mathcal{H}$, then there exists a tri-quadratic function $\mathcal{Q}\colon \mathcal{H}^3 \to \mathcal{W}$ such that $\mathcal{F} - \mathcal{Q}$ is $\frac{1}{2}\Delta$-Lipschitz. Moreover, if $\operatorname{Im} T\mathcal{F} \subset A$ for some $A \in \mathcal{L}(\mathcal{W})$, then $\operatorname{Im}(\mathcal{F} - \mathcal{Q}) \subset \frac{1}{2}A$.*

P r o o f. For every $(x, y, z) \in \mathcal{H}^3$ we define $\varphi_x(\cdot, y, z)\colon \mathcal{H} \to \mathcal{W}$ by

$$\varphi_x(\cdot, y, z) := \frac{1}{2}\mathcal{F}(\cdot + x, y, z) + \frac{1}{2}\mathcal{F}(\cdot - x, y, z) - \mathcal{F}(\cdot, y, z).$$

We prove that $\operatorname{Im} \varphi_x(\cdot, y, z) \subseteq A$ for some $A \in \mathcal{L}(\mathcal{W})$. We have for $(x, y, z) \in \mathcal{H}^3$,

$$\begin{aligned}\varphi_x(\cdot, y, z) &= \frac{1}{2}\mathcal{F}(\cdot + x, y, z) + \frac{1}{2}\mathcal{F}(\cdot - x, y, z) - \mathcal{F}(\cdot, y, z) - \mathcal{F}(x, y, z)\\ &\quad - \frac{1}{2}\mathcal{F}(\cdot, y, z) - \frac{1}{2}\mathcal{F}(\cdot, y, z) + \mathcal{F}(\cdot, y, z) + \mathcal{F}(0, y, z)\\ &\quad + \mathcal{F}(x, y, z) - \mathcal{F}(0, y, z)\\ &= \frac{1}{2}T\mathcal{F}(\cdot, 0, y, z) - \frac{1}{2}T\mathcal{F}(\cdot, x, y, z) + \mathcal{F}(x, y, z) - \mathcal{F}(0, y, z).\end{aligned}$$

By assumption, since $T\mathcal{F}(t, \cdot, \cdot, \cdot)$ is Δ-Lipschitz for every $t \in \mathcal{H}$, $\operatorname{Im} \varphi_x(\cdot, y, z) \subseteq A$, where $A := \frac{1}{2}\Delta((0, y, z), (x, y, z)) + \mathcal{F}(x, y, z) - \mathcal{F}(0, y, z)$. The family $\mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ admits TSLIM, so there exists a linear operator $\Gamma\colon \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \to \mathcal{W}$ such that

(i) $\Gamma[\varphi_x(\cdot, y, z)] = \Gamma[\varphi_y(\cdot, z, x)] = \Gamma[\varphi_z(\cdot, x, y)] = \Gamma[\varphi_y(\cdot, x, z)] = \Gamma[\varphi_x(\cdot, z, y)] = \Gamma[\varphi_z(\cdot, y, x)]$ for every $(x, y, z) \in \mathcal{H}^3$

(ii) $\Gamma[\varphi_x(\cdot,y,z)] \in A$ for some $A \in \mathcal{L}(\mathcal{W})$ and every $(x,y,z) \in \mathcal{H}^3$,

(iii) if $u \in \mathcal{H}$ and $\varphi_x^u(\cdot,y,z)\colon \mathcal{H} \to \mathcal{W}$ defined by $\varphi_x^u(\cdot,y,z) := \varphi_x(\cdot+u,y,z)$ for every $(x,y,z) \in \mathcal{H}^3$, then $\varphi_x^u(\cdot,y,z) \in \mathcal{M}(\mathcal{H},\mathcal{L}(\mathcal{W}))$ and $\Gamma[\varphi_x^u(\cdot,y,z)] = \Gamma[\varphi_x(\cdot,y,z)]$.

Define the function $\mathcal{Q}\colon \mathcal{H}^3 \to \mathcal{W}$ by $\mathcal{Q}(x,y,z) := \Gamma[\varphi_x(\cdot,y,z)]$. In view of property (i) of Γ, $\mathcal{Q}$ is tri-symmetric. We prove that $\mathcal{F}-\mathcal{Q}$ is $\frac{1}{2}\Delta$-Lipschitz. Since $T\mathcal{F}(t,\cdot,\cdot,\cdot)$ is Δ-Lipschitz for $t \in \mathcal{H}$,

$$T\mathcal{F}(t,x,y,z) - T\mathcal{F}(t,u,v,w) \in \Delta((x,y,z),(u,v,w)) \tag{2.1}$$

for all $(x,y,z),(u,v,w) \in \mathcal{H}^3$ and so

$$\operatorname{Im}\Big(\frac{1}{2}T\mathcal{F}(\cdot,x,y,z) - \frac{1}{2}T\mathcal{F}(\cdot,u,v,w)\Big) \subseteq \frac{1}{2}\Delta((x,y,z),(u,v,w)).$$

In view of property (ii) of Γ, we find that

$$\Gamma\Big[\frac{1}{2}T\mathcal{F}(\cdot,x,y,z) - \frac{1}{2}T\mathcal{F}(\cdot,u,v,w)\Big] \in \frac{1}{2}\Delta((x,y,z),(u,v,w))$$

for all $(x,y,z),(u,v,w) \in \mathcal{H}^3$. Note that $\mathcal{M}(\mathcal{H},\mathcal{L}(\mathcal{W}))$ contains constant functions. Property (ii) of Γ entails that for a constant function $C\colon \mathcal{H} \to \mathcal{W}$, $\Gamma[C] = C$. For every $(x,y,z) \in \mathcal{H}^3$ we define the constant function $C_{x,y,z}\colon \mathcal{H} \to \mathcal{W}$ by $C_{x,y,z}(\cdot) := \mathcal{F}(x,y,z)$. We see that

$$\begin{aligned}
(\mathcal{F}(x,y,z) - \mathcal{Q}(x,y,z)) &- (\mathcal{F}(u,v,w) - \mathcal{Q}(u,v,w)) \\
&= (\Gamma[C_{x,y,z}(\cdot)] - \Gamma[\varphi_x(\cdot,y,z)]) - (\Gamma[C_{u,v,w}(\cdot)] - \Gamma[\varphi_u(\cdot,v,w)]) \\
&= \Gamma[C_{x,y,z}(\cdot) - \varphi_x(\cdot,y,z)] - \Gamma[C_{u,v,w}(\cdot) - \varphi_u(\cdot,v,w)] \\
&= \Gamma\Big[\frac{1}{2}T\mathcal{F}(\cdot,x,y,z) - \frac{1}{2}T\mathcal{F}(\cdot,u,v,w)\Big]
\end{aligned}$$

for all $(x,y,z),(u,v,w) \in \mathcal{H}^3$. This shows that

$$(\mathcal{F}(x,y,z) - \mathcal{Q}(x,y,z)) - (\mathcal{F}(u,v,w) - \mathcal{Q}(u,v,w)) \in \frac{1}{2}\Delta((x,y,z),(u,v,w))$$

for all $(x,y,z),(u,v,w) \in \mathcal{H}^3$, i.e., $\mathcal{F}-\mathcal{Q}$ is a $\frac{1}{2}\Delta$-Lipschitz function. Applying property (iii) of Γ and the definition of Γ, we find that

$$\begin{aligned}
2\mathcal{Q}(x,z,w) + 2\mathcal{Q}(y,z,w) &= 2\Gamma[\varphi_x(\cdot,z,w)] + 2\Gamma[\varphi_y(\cdot,z,w)] \\
&= \Gamma[\varphi_x^y(\cdot,z,w)] + \Gamma[\varphi_x^{-y}(\cdot,z,w)] + 2\Gamma[\varphi_y(\cdot,z,w)].
\end{aligned} \tag{2.2}$$

On the other hand, we have

$$
\begin{aligned}
(2.3)\quad & \Gamma[\varphi_x^y(\cdot,z,w)]+\Gamma[\varphi_x^{-y}(\cdot,z,w)]+2\Gamma[\varphi_y(\cdot,z,w)]\\
&=\Gamma\Big[\frac{1}{2}\mathcal{F}(\cdot+x+y,z,w)+\frac{1}{2}\mathcal{F}(\cdot-x+y,z,w)-\mathcal{F}(\cdot+y,z,w)\Big]\\
&\quad+\Gamma\Big[\frac{1}{2}\mathcal{F}(\cdot+x-y,z,w)+\frac{1}{2}\mathcal{F}(\cdot-x-y,z,w)-\mathcal{F}(\cdot-y,z,w)\Big]\\
&\quad+\Gamma[\mathcal{F}(\cdot+y,z,w)+\mathcal{F}(\cdot-y,z,w)-2\mathcal{F}(\cdot,z,w)]\\
&=\Gamma[\varphi_{x+y}(\cdot,z,w)]+\Gamma[\varphi_{x-y}(\cdot,z,w)]\\
&=\mathcal{Q}(x+y,z,w)+\mathcal{Q}(x-y,z,w).
\end{aligned}
$$

From (2.2) and (2.3) it follows that $\mathcal{Q}$ is quadratic in its first variable. Since $\mathcal{Q}$ is tri-symmetric, $\mathcal{Q}$ is quadratic in its second and third variables and hence $\mathcal{Q}$ is tri-quadratic. Moreover, if $\operatorname{Im} T\mathcal{F}\subset A$, then

$$\operatorname{Im}\Big(\frac{1}{2}T\mathcal{F}(\cdot,x,y,z)\Big)\subset\operatorname{Im}\Big(\frac{1}{2}T\mathcal{F}\Big)\subset\frac{1}{2}A.$$

In other words, $\frac{1}{2}T\mathcal{F}(\cdot,x,y,z)\in\mathcal{M}(\mathcal{H},\mathcal{L}(\mathcal{W}))$ for all $(x,y,z)\in\mathcal{H}^3$. Thus, property (ii) of Γ implies

$$\mathcal{F}(x,y,z)-\mathcal{Q}(x,y,z)=\Gamma\Big[\frac{1}{2}T\mathcal{F}(\cdot,x,y,z)\Big]\in\frac{1}{2}A$$

for all $(x,y,z)\in\mathcal{H}^3$. Therefore, $\operatorname{Im}(\mathcal{F}-\mathcal{Q})\subset\frac{1}{2}A$. □

Definition 2.5. Let $(\mathcal{H}^3,\varrho)$ be a metric group and $\mathcal{W}$ a normed space. A function $m_{\mathcal{F}}\colon\mathbb{R}^+\to\mathbb{R}^+$ is a module of continuity of $\mathcal{F}\colon\mathcal{H}^3\to\mathcal{W}$ if $\varrho((x,y,z),(u,v,w))\leqslant\delta$ implies $\|\mathcal{F}(x,y,z)-\mathcal{F}(u,v,w)\|\leqslant m_{\mathcal{F}}(\delta)$ for every $\delta>0$ and $(x,y,z),(u,v,w)\in\mathcal{H}^3$.

Definition 2.6. A function $\mathcal{F}\colon\mathcal{H}^3\to\mathcal{W}$ is called a Lipschitz function of order $\alpha>0$ if there exists a constant $L>0$ such that

$$\|\mathcal{F}(x,y,z)-\mathcal{F}(u,v,w)\|\leqslant L\varrho^\alpha((x,y,z),(u,v,w)) \tag{2.4}$$

for every $(x,y,z),(u,v,w)\in\mathcal{H}^3$.

For a metric group $(\mathcal{H}^3,\varrho)$, a normed space $\mathcal{W}$, and $\alpha\in(0,1]$, let $\operatorname{Lip}_\alpha(\mathcal{H}^3,\mathcal{W})$ be the Lipschitz space consisting of all bounded Lipschitz functions of order $\alpha>0$ with the norm

$$\|\mathcal{F}\|_\alpha:=\|\mathcal{F}\|_{\sup}+P_\alpha(\mathcal{F}),$$

where $\|\cdot\|_{\sup}$ is the supremum norm and

$$P_\alpha(\mathcal{F}) = \sup\Big\{\frac{\|\mathcal{F}(x,y,z)-\mathcal{F}(u,v,w)\|}{\varrho^\alpha((x,y,z),(u,v,w))}\colon (x,y,z),(u,v,w)\in\mathcal{H}^3,$$
$$(x,y,z)\neq(u,v,w)\Big\}.$$

Definition 2.7. Consider an abelian group $(\mathcal{H}^3,+)$ with a metric ϱ invariant under translation, i.e., satisfying the condition

$$\begin{aligned}&\varrho((x+a,y+b,z+c),(u+a,v+b,w+c))\\&\quad=\varrho((a+x,b+y,c+z),(a+u,b+v,c+w))=\varrho((x,y,z),(u,v,w))\end{aligned}$$

for all $(a,b,c),(x,y,z),(u,v,w)\in\mathcal{H}^3$. A metric D on $\mathcal{H}^3\times\mathcal{H}$ is called a metric pair if it is invariant under translation and the following condition holds:

$$D((x,y,z,a),(u,v,w,a))=D((a,x,y,z),(a,u,v,w))=\varrho((x,y,z),(u,v,w))$$

for all $a\in\mathcal{H}$, $(x,y,z),(u,v,w)\in\mathcal{H}^3$.

Theorem 2.8. *Let $(\mathcal{H}^3,+,\varrho,D)$ be a metric pair, $\mathcal{W}$ a normed space such that $\mathcal{M}(\mathcal{H},CB(\mathcal{W}))$ admits* TSLIM, *and $\mathcal{F}\colon\mathcal{H}^3\to\mathcal{W}$ a function. If $T\mathcal{F}\in\operatorname{Lip}_\alpha(\mathcal{H}\times\mathcal{H}^3,\mathcal{W})$, then there exists a tri-quadratic function $\mathcal{Q}$ such that*

$$\|\mathcal{F}-\mathcal{Q}\|_\alpha\leqslant\frac{1}{2}\|T\mathcal{F}\|_\alpha.$$

Proof. Assume that $m_{T\mathcal{F}}\colon\mathbb{R}^+\to\mathbb{R}^+$ is the module of continuity of $T\mathcal{F}$ with the metric pair D. Define the set-valued function $\Delta\colon\mathcal{H}^3\times\mathcal{H}^3\to CB(\mathcal{W})$ by

$$\Delta((x,y,z),(u,v,w)):=\inf_{\varrho((x,y,z),(u,v,w))\leqslant\delta}m_{T\mathcal{F}}(\delta)B(0,1),$$

where $B(0,1)$ is the closed unit ball with center at zero. The following inequality shows that $T\mathcal{F}(t,\cdot,\cdot,\cdot)$ is Δ-Lipschitz:

$$\begin{aligned}\|T\mathcal{F}(t,x,y,z)-T\mathcal{F}(t,u,v,w)\|&\leqslant\inf_{D((t,x,y,z),(t,u,v,w))\leqslant\delta}m_{T\mathcal{F}}(\delta)\\&=\inf_{\varrho((x,y,z),(u,v,w))\leqslant\delta}m_{T\mathcal{F}}(\delta)\end{aligned}$$

for all $t\in\mathcal{H}$, $(x,y,z),(u,v,w)\in\mathcal{H}^3$. Thus, there exists a tri-quadratic function $\mathcal{Q}$ such that $\mathcal{F}-\mathcal{Q}$ is $\frac{1}{2}\Delta$-Lipschitz by Theorem 2.4. Hence,

$$\|(\mathcal{F}-\mathcal{Q})(x,y,z)-(\mathcal{F}-\mathcal{Q})(u,v,w)\|\leqslant\inf_{\varrho((x,y,z),(u,v,w))\leqslant\delta}\frac{1}{2}m_{T\mathcal{F}}(\delta),$$

which entails that $m_{\mathcal{F}-\mathcal{Q}} = \frac{1}{2} m_{T\mathcal{F}}$. Moreover, $\|T\mathcal{F}\|_{\sup} < \infty$ and clearly $\operatorname{Im} T\mathcal{F} \subset \|T\mathcal{F}\|_{\sup} B(0,1)$. Using the last part of Theorem 2.4 we get

$$\|\mathcal{F} - \mathcal{Q}\|_{\sup} \leqslant \frac{1}{2}\|T\mathcal{F}\|_{\sup}. \tag{2.5}$$

Define the function $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ by $\omega(t) := P_\alpha(T\mathcal{F})t^\alpha$. In view of $T\mathcal{F} \in \operatorname{Lip}_\alpha(\mathcal{H} \times \mathcal{H}^3, \mathcal{W})$, we have

$$\|T\mathcal{F}(t,x,y,z) - T\mathcal{F}(t,u,v,w)\| \leqslant \omega(D((t,x,y,z),(t,u,v,w)),$$

which ensures that ω is the module of continuity of the function $T\mathcal{F}$ and consequently $m_{\mathcal{F}-\mathcal{Q}} = \frac{1}{2}\omega$. Then,

$$\begin{aligned}\|(\mathcal{F}-\mathcal{Q})(x,y,z) - (\mathcal{F}-\mathcal{Q})(u,v,w)\| &\leqslant \frac{1}{2}\omega(\varrho((x,y,z),(u,v,w))) \\ &= \frac{1}{2}P_\alpha(T\mathcal{F})\varrho^\alpha((x,y,z),(u,v,w)).\end{aligned}$$

This inequality implies that $\mathcal{F}-\mathcal{Q}$ is a Lipschitz function of order α and $P_\alpha(\mathcal{F}-\mathcal{Q}) \leqslant \frac{1}{2}P_\alpha(T\mathcal{F})$. From inequality (2.5) it follows that

$$\begin{aligned}\|\mathcal{F}-\mathcal{Q}\|_\alpha &= \|\mathcal{F}-\mathcal{Q}\|_{\sup} + P_\alpha(\mathcal{F}-\mathcal{Q}) \\ &\leqslant \frac{1}{2}\|T\mathcal{F}\|_{\sup} + \frac{1}{2}P_\alpha(T\mathcal{F}) = \frac{1}{2}\|T\mathcal{F}\|_\alpha.\end{aligned}$$

□

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