EPIMORPHISMS BETWEEN FINITE MV-ALGEBRAS

Aldo V. Figallo, San Juan, Marina B. Lattanzi, La Pampa

Received December 5, 2014. First published February 1, 2017.
Communicated by Radomír Halaš

Abstract. MV-algebras were introduced by Chang to prove the completeness of the infinite-valued Łukasiewicz propositional calculus. Recently, algebraic theory of MV-algebras has been intensively studied. Wajsberg algebras are just a reformulation of Chang MV-algebras where implication is used instead of disjunction. Using these equivalence, in this paper we provide conditions for the existence of an epimorphism between two finite MV-algebras A and B. Specifically, we define the mv-functions with domain in the ordered set of prime elements of B and with range in the ordered set of prime elements of A, and prove that every epimorphism from A to B can be uniquely constructed from an mv-function.

Keywords: MV-algebras; mv-function; epimorphism

MSC 2010: 06D35, 08A35

1. Preliminaries and necessary properties

MV-algebras were originally defined by Chang [4], [5] as algebraic models of Łukasiewicz infinite-valued (also finite-valued) propositional calculi. However, let us recall that Łukasiewicz [13], [14] considered as the main propositional connectives *implication* $\rightarrow$ and *negation* $\sim$. Algebras introduced by Chang, instead, contain other operations which do not correspond, for example, to logical connectives MV-conjunction or MV-disjunction, to mention some.

Algebraic counterparts of Łukasiewicz propositional calculi (infinite or finite-valued), all of them polynomially equivalent, were originally defined by Komori [12], [11] under the name CN-algebras, and by Rodriguez [17] under the name Wajsberg algebras (see [10], [18] too). In this paper we will adopt the language of Wajsberg algebras (or W-algebras) to describe MV-algebras.

In [16] Luiz Monteiro determined the number of epimorphisms between finite Łukasiewicz algebras (see [3]). It is known that every finite W-algebra is a direct

DOI: 10.21136/MB.2017.0077-14

product of finite chains. In this work, we use this fact to find the number of epimorphisms between finite W-algebras. This representation for finite W-algebras is also used in [2] to find the structure and cardinality of finitely generated algebras in varieties of k-potent hoop residuation algebras.

In this section we review some definitions and properties necessary for what follows (see, for example, [6], [9], [10], [17]). In Section 2 we define the mv-functions between the ordered sets of prime elements of finite MV-algebras and prove that every epimorphism can be uniquely constructed from an mv-function. This results can be also obtained from the duality given by Martínez in [15]. More details on MV-algebras can be found in two very interesting papers [7] and [8].

Let us recall that a W-algebra $\mathcal{A} = \langle A, \to, \sim, 1\rangle$ is an algebra of type $(2, 1, 0)$ such that the following identities are satisfied:

(W1) $1 \to x = x$,
(W2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(W3) $(x \to y) \to y = (y \to x) \to x$,
(W4) $(\sim y \to \sim x) \to (x \to y) = 1$.

The unit real interval $[0, 1]$ endowed with the operations $x \to y := \min\{1, 1 - x + y\}$ and $\sim x := 1 - x$ is a Wajsberg algebra. For each integer $n \geqslant 1$, we denote by L_{n+1} the subalgebra of $[0, 1]$ with the universe $\{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$.

In every W-algebra $\mathcal{A} = \langle A, \to, \sim, 1\rangle$ the following terms can be defined:

(i) $0 := \sim 1$,
(ii) $a \vee b := (a \to b) \to b$,
(iii) $a \wedge b := \sim(\sim a \vee \sim b)$,
(iv) $a \oplus b := \sim b \to a$,
(v) $0 \cdot a := 0$, $(n + 1) \cdot a := n \cdot a \oplus a$, for every nonnegative integer n.

Then $\langle A, \oplus, \sim, 0\rangle$ is an MV-algebra and $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ is a Kleene algebra. The following properties hold in every W-algebra $\mathcal{A}$, for all nonnegative integers n, m (see [10], [17]):

(W5) $x \leqslant y$ if and only if $x \to y = 1$,
(W6) $x \to 0 = \sim x$,
(W7) $x \oplus 0 = x$,
(W8) $x \oplus y = y \oplus x$,
(W9) $x \vee y \leqslant x \oplus y$,
(W10) $x \leqslant y$ implies $x \oplus z \leqslant y \oplus z$,
(W11) $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z)$,
(W12) $\bigvee_{i=1}^{n} x_i \oplus \bigvee_{h=1}^{m} y_h = \bigvee_{i=1}^{n} \bigvee_{h=1}^{m} (x_i \oplus y_h)$,
(W13) $(n + m) \cdot x = n \cdot x \oplus m \cdot x$,

(W14) $(nm) \cdot x = n \cdot (m \cdot x)$,

(W15) $x \leqslant y$ implies $n \cdot x \leqslant n \cdot y$,

(W16) $n \leqslant m$ implies $n \cdot x \leqslant m \cdot x$.

Let $\mathcal{A}$ be a W-algebra. The set $B(\mathcal{A}) = \{x \in A\colon \sim x \to x = x\}$ is a Boolean algebra. Indeed, $B(\mathcal{A})$ is the Boolean algebra of the complemented elements of the bounded distributive lattice reduct of A. The elements of $B(\mathcal{A})$ are called the boolean elements of $\mathcal{A}$. For each $a \in B(\mathcal{A})$ the set $[0, a] = \{x \in A\colon x \leqslant a\}$ is a W-algebra with the operations $(x \to y) \wedge a$ and $\sim x \wedge a$, for all $x, y \in [0, a]$.

We will denote by $\mathrm{At}(\mathcal{A})$, $\mathcal{X}(\mathcal{A})$ and $\Pi(\mathcal{A})$ the set of atoms of $\mathcal{A}$ and the ordered sets of all prime filters and prime elements with respect to the lattice structure of $\mathcal{A}$, respectively. The function $\varphi\colon \mathcal{X}(\mathcal{A}) \to \mathcal{X}(\mathcal{A})$, defined by $\varphi(P) = \mathcal{X}(\mathcal{A}) \setminus \{\sim x\colon x \in P\}$ for each $P \in \mathcal{X}(\mathcal{A})$, is an involution and a dual isomorphism.

In what follows A is a finite W-algebra.

Then, it is isomorphic to a direct product of intervals determined by atoms of $B(\mathcal{A})$, i.e.,

$$\mathcal{A} \simeq \prod_{a \in \mathrm{At}(B(\mathcal{A}))} [0, a].$$

Moreover, if $a \in \mathrm{At}(B(\mathcal{A}))$ then $[0, a]$ is isomorphic to L_{r+1}, for some integer $r \geqslant 1$.

Let $\Psi\colon \Pi(\mathcal{A}) \to \Pi(\mathcal{A})$ be the function $\Psi = \mu^{-1} \circ \varphi \circ \mu$, where μ is the order-isomorphism from $\Pi(\mathcal{A})$ onto the dual of $\mathcal{X}(\mathcal{A})$, which exists because $\mathcal{A}$ is finite.

As an immediate consequence of this representation, we have the following result (see, for example, [17]).

Corollary 1.1. *Let $\mathcal{A}$ be a finite W-algebra and let $n = \max\limits_{a \in \mathrm{At}(B(\mathcal{A}))} \{r\colon [0, a] \simeq L_{r+1}\}$. Then the following statements hold:*

(i) *The ordered set $\Pi(\mathcal{A})$ is the disjoint union of t_r chains with r elements, where $t_n > 0$ and $t_r \geqslant 0$ for all r, $1 \leqslant r \leqslant n$.*

(ii) *Each element $p \in \Pi(\mathcal{A})$ can be identified with $j \cdot 1/r$ for some integers j, r, $j \geqslant 1$ and $1 \leqslant r \leqslant n$.*

(iii) *The atoms of $B(\mathcal{A})$ are the last elements of the chains and the cardinal number of $\mathrm{At}(B(\mathcal{A}))$ is $t_1 + t_2 + \ldots + t_n$.*

(iv) *If $p_j \in \Pi(\mathcal{A})$ for all $1 \leqslant j \leqslant r$ and $p_1 < p_2 < \ldots < p_r$, then $\Psi(p_j) = p_{r-j+1}$ for all j, $1 \leqslant j \leqslant r$.*

(v) *If $p \in \Pi(\mathcal{A})$, then $k \cdot p \in \Pi(\mathcal{A})$ for every $k \geqslant 1$.*

(vi) *If $p \in \Pi(\mathcal{A})$ then $m \cdot p \in \mathrm{At}(B(\mathcal{A}))$ for every integer $m \geqslant n$ and $n \cdot p$ is the last element in the chain which contains p.*

(vii) *Let p, $q \in \Pi(\mathcal{A})$. If p and q are comparable, then $p \oplus q \in \Pi(\mathcal{A})$.*

(viii) *If p, $q \in \Pi(\mathcal{A})$ are incomparable, then $p \oplus q = p \vee q$.*

Therefore, for every finite W-algebra A we will write $A = A_{t_1 t_2 \dots t_n}$ to identify the ordered set $\Pi(\mathcal{A})$.

E x a m p l e 1.1. Let $A = L_{1+1} \times L_{2+1} = \{0, a, b, c, d, 1\}$, where $0 = (0,0)$, $a = (1,0)$, $b = (0, \frac{1}{2})$, $c = (0,1)$, $d = (1, \frac{1}{2})$ and $1 = (1,1)$. It is clear that $\Pi(\mathcal{A}) = \{a, b, c\}$, with $b \leqslant c$; $\mathcal{X}(\mathcal{A}) = \{F(a), F(b), F(c)\}$ with $F(c) \subseteq F(b)$, where $F(t)$ is the lattice filter generated by $t \in \Pi(\mathcal{A})$ and $\mu(t) = F(t)$. Moreover, $\varphi(F(a)) = F(a)$, $\varphi(F(b)) = F(c)$ and $\varphi(F(c)) = F(b)$; $\Psi(a) = a$, $\Psi(b) = c$ and $\Psi(c) = b$. In this case $n = 2$, $r \in \{1, 2\}$, $t_1 = 1$, $t_2 = 1$ and we write $A = A_{11}$.

2. MV-FUNCTIONS AND EPIMORPHISMS

Let $\mathcal{A}$ be a finite W-algebra. From Corollary 1.1 (i), the ordered set $\Pi(\mathcal{A})$ is a disjoint union of finite chains; each connected component will be denoted by C. Then, if $C \subseteq \Pi(\mathcal{A})$ is a chain with first element p_0, we will write $C = C(p_0)$.

Definition 2.1. An mv-function is a map $f\colon \Pi(\mathcal{A}') \to \Pi(\mathcal{A})$ which satisfies the following conditions for all $p' \in \Pi(\mathcal{A}')$:

(F1) f is injective,
(F2) $f(k \cdot p') = k \cdot f(p')$ for all $k \geqslant 1$,
(F3) $f(\Psi'(p')) = \Psi(f(p'))$.

Properties (F1), (F2) and (F3) are independent. Indeed, let us consider the functions $f_1\colon \Pi(L_{2+1}) \to \Pi(L_{3+1})$, $f_2\colon \Pi(L_{2+1}) \to \Pi(L_{3+1})$ and $f_3\colon \Pi(L^2_{1+1}) \to \Pi(L_{1+1})$, defined by

$$f_1(x) = \begin{cases} \frac{2}{3} & \text{if } x = \frac{1}{2}, \\ 1 & \text{if } x = 1, \end{cases} \quad f_2(x) = \begin{cases} \frac{1}{3} & \text{if } x = \frac{1}{2}, \\ 1 & \text{if } x = 1, \end{cases} \quad f_3(x) = \begin{cases} 1 & \text{if } x = (0,1), \\ 1 & \text{if } x = (1,0). \end{cases}$$

It is easy to see that f_1 satisfies (F1) and (F2) but not (F3), f_2 satisfies (F1) and (F3) but not (F2) and f_3 satisfies (F2) and (F3) but not (F1).

Lemma 2.1. *Let $f\colon \Pi(\mathcal{A}') \to \Pi(\mathcal{A})$ be an* mv*-function. Then, for all $p', q' \in \Pi(\mathcal{A}')$, the following properties hold:*

(F4) *$p' \leqslant q'$ implies $f(p') \leqslant f(q')$.*
(F5) *Let $C' \subseteq \Pi(\mathcal{A}')$. If f' is the restriction of f to C' and $f'(C') \subseteq C \subseteq \Pi(\mathcal{A})$, then $f'(C') = C$.*
(F6) *$f(p') \leqslant f(q')$ implies $p' \leqslant q'$.*
(F7) *If p' and q' are comparable then $f(p' \oplus q') = f(p') \oplus f(q')$.*

P r o o f. Let $p', q' \in \Pi(\mathcal{A}')$.

(F4) Suppose that $p' \leqslant q'$. Let $C'(p'_0) \subseteq \Pi(\mathcal{A}')$ be a chain which contains p' and q'. By Corollary 1.1 (ii) there exist integers $j, t \geqslant 1$ such that $p' = j \cdot p'_0$ and $q' = t \cdot p'_0$. If $j > t$, then from (W16) it is clear that $j \cdot p'_0 \geqslant t \cdot p'_0$, i.e., $p' \geqslant q'$. Thus $p' = q'$ and then $f(p') = f(q')$. Let us suppose now that $j \leqslant t$. Then from (F2) we have that $f(p') = f(j \cdot p'_0) = j \cdot f(p'_0)$ and $f(q') = f(t \cdot p'_0) = t \cdot f(p'_0)$. Hence, from (W16) we obtain $f(p') \leqslant f(q')$.

(F5) Let $C'(p'_0) \subseteq \Pi(\mathcal{A}')$. Then from (F4) we have that $f(C'(p'_0)) \subseteq C(p_0)$ for some chain $C(p_0) \subseteq \Pi(\mathcal{A})$. Let f' be the restriction of f to $C'(p'_0)$.

Let p'_1 and p_1 be the last elements of $C'(p'_0)$ and $C(p_0)$, respectively. From Corollary 1.1 (vi) and (F2) we have that $p_1 = n \cdot f(p'_1) = f(n \cdot p'_1) = f(p'_1)$. So, by (F3) we obtain $f(p'_0) = p_0$.

Let $p \in C(p_0)$. So, $p = j \cdot p_0$ for some integer $j \geqslant 1$. Thus, $p = j \cdot p_0 = j \cdot f(p'_0) = f(j \cdot p'_0) = f(p')$, with $p' \in C'(p'_0)$. Hence, $C(p_0) \subseteq f(C'(p'_0))$.

(F6) Suppose that $f(p') \leqslant f(q')$. Let $C(p_0) \subseteq \Pi(\mathcal{A})$ be such that $f(p'), f(q') \in C(p_0)$. If we suppose that p' and q' belong to different connected components, let us say $p' \in C'_1(p'_0)$, $q' \in C'_2(q'_0)$, then by applying (F5) we obtain $f(p'_0) = f(q'_0) = p_0$ which is a contradiction because f is injective. Thus, let $C'(p'_0) \subseteq \Pi(\mathcal{A}')$ be such that $p', q' \in C'(p'_0)$. Then there exist integers $j, t \geqslant 1$ such that $f(p') = j \cdot p_0$ and $f(q') = t \cdot p_0$. If $j > t$ then $f(p') \geqslant f(q')$, so $f(p') = f(q')$ and we have $p' = q'$ because f is injective. Let us suppose now $j \leqslant t$. From (F2) and (F5) we have $f(p') = j \cdot p_0 = j \cdot f(p'_0) = f(j \cdot p'_0)$ and $f(q') = t \cdot p_0 = t \cdot f(p'_0) = f(t \cdot p'_0)$. Then $p' = j \cdot p'_0$ and $q' = t \cdot p'_0$ because f is injective. Hence, $p' \leqslant q'$ follows from (W16).

(F7) Suppose that p' and q' are comparable. Let $C'(p'_0) \subseteq \Pi(\mathcal{A}')$ be such that $p', q' \in C'(p'_0)$. From Corollary 1.1 (vii) and (ii) it is clear that $p' \oplus q' \in C'(p'_0)$, $p' = j \cdot p'_0$ and $q' = t \cdot p'_0$, for some integers $j, t \geqslant 1$. Then by applying (W13) and (F2) we get $f(p' \oplus q') = f(j \cdot p'_0 \oplus t \cdot p'_0) = f((j+t) \cdot p'_0) = (j+t) \cdot f(p'_0) = j \cdot f(p'_0) \oplus t \cdot f(p'_0) = f(p') \oplus f(q')$. □

Notice that by (F5) in Lemma 2.1 there exists an mv-function between $\Pi(L_{n+1})$ and $\Pi(L_{m+1})$ (the identity function) if and only if $m = n$.

Theorem 2.1. *Let $f\colon \Pi(\mathcal{A}') \to \Pi(\mathcal{A})$ be an* mv-*function. For each $x \in A$ let $A'_x = \{p' \in \Pi(\mathcal{A}')\colon\ f(p') \leqslant x\}$. If we define the function $h\colon\ A \to A'$ by*

$$h(x) = \begin{cases} 0 & \text{if } A'_x = \emptyset, \\ \bigvee\limits_{p' \in A'_x} p' & \text{otherwise} \end{cases}$$

then h is an epimorphism. We will say that h is the epimorphism induced by the mv-*function f.*

P r o o f. Let $x, y \in A$. To prove that h is a homomorphism it is enough to show that $h(\sim x) = \sim h(x)$ and $h(x \oplus y) = h(x) \oplus h(y)$, because $x \to y = \sim x \oplus y$. The proof of the first equality is an exact analogue of that given in [1]. In order to prove the second equality, let us suppose that $x \neq 0$ and $y \neq 0$ (the cases $x = 0$ or $y = 0$ are trivial). Let us consider the sets

$$\begin{aligned} A'_x &= \{p' \in \Pi(\mathcal{A}')\colon\ f(p') \leqslant x\},\\ A'_y &= \{q' \in \Pi(\mathcal{A}')\colon\ f(q') \leqslant y\},\\ A'_{x\oplus y} &= \{r' \in \Pi(\mathcal{A}') : f(r') \leqslant x \oplus y\}. \end{aligned}$$

Then by applying (W12) we have

$$h(x) \oplus h(y) = \bigvee_{p' \in A'_x} p' \oplus \bigvee_{q' \in A'_y} q' = \bigvee_{p' \in A'_x} \bigvee_{q' \in A'_y} (p' \oplus q'). \tag{2.1}$$

Let $B_{x,y}$ be the set $\{s' \in \Pi(\mathcal{A}')\colon\ s' \leqslant p' \oplus q',\ p' \in A'_x,\ q' \in A'_y,\ p', q' \text{ comparable}\}$. We claim that

$$A'_{x\oplus y} = A'_x \cup A'_y \cup B_{x,y}. \tag{2.2}$$

Indeed, since

$$x = \bigvee\{p \in \Pi(\mathcal{A})\colon\ p \leqslant x\} \quad \text{and} \quad y = \bigvee\{q \in \Pi(\mathcal{A})\colon\ q \leqslant y\},$$

by applying (W12) we obtain $x \oplus y = \bigvee_{p \leqslant x} \bigvee_{q \leqslant y} (p \oplus q)$.

Hence, if $r' \in A'_{x\oplus y}$, then $f(r') \leqslant x \oplus y$, so there exist $p, q \in \Pi(\mathcal{A})$, $p \leqslant x$, $q \leqslant y$ such that

$$f(r') \leqslant p \oplus q. \tag{2.3}$$

There are two cases to consider:

(i) If p and q are comparable, then from Corollary 1.1 (vii), (2.3), (F5) and (F6) we have that $p \oplus q \in \Pi(\mathcal{A})$ and there exists $s' \in \Pi(\mathcal{A}')$, $r' \leqslant s'$, such that $p \oplus q = f(s')$, $s' \in \Pi(\mathcal{A}')$.

Similarly, since $p, q \leqslant p \oplus q$, there exist $p', q' \in \Pi(\mathcal{A}')$ such that $p = f(p')$ and $q = f(q')$, where p' and q' are comparable. Then $f(s') = p \oplus q = f(p') \oplus f(q') = f(p' \oplus q')$. Thus, $s' = p' \oplus q'$ because f is injective. Hence, $r' \in B_{x,y}$. Then in this case we conclude $A'_{x\oplus y} \subseteq B_{x,y}$.

(ii) If p and q are incomparable then $p \oplus q = p \vee q$ by Corollary 1.1 (viii). So, from (2.3) we obtain $f(r') \leqslant p \leqslant x$ or $f(r') \leqslant q \leqslant y$; i.e., $r' \in A'_x$ or $r' \in A'_y$.

From (i) and (ii) we have proved $A'_{x\oplus y} \subseteq A'_x \cup A'_y \cup B_{x,y}$.

Conversely, let $r' \in A'_x \cup A'_y \cup B_{x,y}$. If $r' \in A'_x$ then $f(r') \leqslant x \leqslant x \oplus y$; so $r' \in A'_{x\oplus y}$. Analogously if $r' \in A'_y$. If $r' \in B_{x,y}$ then there exist $p' \in A'_x$ and $q' \in A'_y$, where p' and q' are comparable, such that $r' \leqslant p' \oplus q'$. Hence, by applying Corollary 1.1 (vii), (F7), (W10) and (F4) we have that

$$f(r') \leqslant f(p' \oplus q') = f(p') \oplus f(q') \leqslant x \oplus y,$$

i.e., $r' \in A'_{x\oplus y}$.

Therefore, $A'_x \cup A'_y \cup B_{x,y} \subseteq A'_{x\oplus y}$.

Now we claim that

$$\bigvee_{r' \in A'_{x\oplus y}} r' = \bigvee_{p' \in A'_x} \bigvee_{q' \in A'_y} (p' \oplus q'). \tag{2.4}$$

Indeed, let $r' \in A'_x \cup A'_y \cup B_{x,y}$.

If $r' \in A'_x$ then $r' \leqslant r' \oplus q' \leqslant \bigvee_{q' \in A'_y} (r' \oplus q') \leqslant \bigvee_{r' \in A'_x} \bigvee_{q' \in A'_y} (r' \oplus q')$ for every $q' \in A'_y$, so

$$\bigvee_{r' \in A'_x} r' \leqslant \bigvee_{r' \in A'_x} \bigvee_{q' \in A'_y} (r' \oplus q'). \tag{2.5}$$

In a similar way, if $r' \in A'_y$ then

$$\bigvee_{r' \in A'_y} r' \leqslant \bigvee_{p' \in A'_x} \bigvee_{r' \in A'_y} (p' \oplus r'). \tag{2.6}$$

Finally, if $r' \in B_{x,y}$ then $r' \leqslant p' \oplus q'$, $p' \in A'_x$, $q' \in A'_y$, hence $r' \leqslant p' \oplus q' \leqslant \bigvee_{p' \in A'_x} \bigvee_{q' \in A'_y} (p' \oplus q')$, so

$$\bigvee_{r' \in B_{x,y}} r' \leqslant \bigvee_{p' \in A'_x} \bigvee_{q' \in A'_y} (p' \oplus q'). \tag{2.7}$$

Then from (2.2), (2.5), (2.6) and (2.7) we obtain

$$\bigvee_{r' \in A'_{x\oplus y}} r' \leqslant \bigvee_{p' \in A'_x} \bigvee_{q' \in A'_y} (p' \oplus q').$$

In order to prove the other inequality, let $p' \in A'_x$ and $q' \in A'_y$. If p' and q' are comparable then $p' \oplus q' \in B_{x,y}$, thus

$$p' \oplus q' \leqslant \bigvee_{r' \in B_{x,y}} r' \leqslant \bigvee_{r' \in A'_x \cup A'_y \cup B_{x,y}} r'. \tag{2.8}$$

If p' and q' are incomparable then $p' \oplus q' = p' \vee q'$, hence

$$(2.9) \qquad p' \oplus q' = p' \vee q' \leqslant \bigvee_{p' \in A'_x} p' \vee \bigvee_{q' \in A'_y} q' = \bigvee_{r' \in A'_x \cup A'_y} r' \leqslant \bigvee_{r' \in A'_x \cup A'_y \cup B_{x,y}} r'.$$

From (2.8), (2.9) and (2.2) we have

$$\bigvee_{p' \in A'_x} \bigvee_{q' \in A'_y} (p' \oplus q') \leqslant \bigvee_{r' \in A'_{x \oplus y}} r'.$$

Therefore, equality $h(x \oplus y) = h(x) \oplus h(y)$ follows from (2.1) and (2.4).

So it remains to prove that h is surjective. Let $y \in A'$, $y \neq 0$. Let $x \in A$ be the element defined by

$$(2.10) \qquad x = \bigvee \{f(p') \colon\ p' \leqslant y,\ p' \in \Pi(\mathcal{A}')\}.$$

We claim that

$$(2.11) \qquad A'_x = \{p' \in \Pi(\mathcal{A}') \colon\ p' \leqslant y\}.$$

Indeed, if $q' \in A'_x$ then $f(q') \leqslant x$. From (2.10) we have that there exists $p' \in \Pi(\mathcal{A}')$, $p' \leqslant y$ such that $f(q') \leqslant f(p')$, because $f(q')$ is a prime element of A. Hence, by (F6) we have $q' \leqslant p' \leqslant y$, i.e., $q' \in \{p' \in \Pi(\mathcal{A}') \colon\ p' \leqslant y\}$.

Conversely, let $q' \in \{p' \in \Pi(\mathcal{A}') \colon\ p' \leqslant y\}$. Then $f(q') \leqslant x$ which implies $q' \in A'_x$. From (2.11) we conclude $h(x) = y$. □

Lemma 2.2. *If f is an* mv*-function and h is the epimorphism induced by f, then for each $p \in \Pi(\mathcal{A})$ we have either $h(p) = 0$ or $h(p) = r' \in \Pi(\mathcal{A}')$, with $f(r') = p$.*

P r o o f. Let $p \in C \subseteq \Pi(\mathcal{A})$. If $h(p) \neq 0$ then $A'_p \neq \emptyset$. Let $p' \in A'_p$. Suppose that $p' \in C' \subseteq \Pi(\mathcal{A}')$. Since $f(p') \leqslant p$ we have that $f(p') \in C$. Then, by Corollary 1.1 and (F5), $f(C') = C$. If there exists $q' \in A'_p \setminus C'$ then $f(q') \in C$, which is a contradiction because f is injective. Thus, $A'_p \subseteq C'$. Let $r' = \bigvee_{p' \in A'_p} p'$. Therefore, $r' \in C' \subseteq A'_p$ and $h(p) = r'$. To complete the proof we must show that $f(r') = p$. Indeed, there exists $t' \in C'$ such that $f(t') = p$. If $f(r') < p$ then $r' < t'$, which is a contradiction because $t' \in A'_p$ implies $t' \leqslant r'$. Therefore $f(r') = p$. □

Let us denote by $\mathrm{Epi}(\mathcal{A}, \mathcal{A}')$ the set of all epimorphisms from $\mathcal{A}$ to $\mathcal{A}'$.

Lemma 2.3. *Let $h \in \mathrm{Epi}(\mathcal{A}, \mathcal{A}')$. Then for each $p' \in \Pi(\mathcal{A}')$ there exists a unique element $p \in \Pi(\mathcal{A})$ such that $h(p) = p'$.*

P r o o f. Let $p' \in \Pi(\mathcal{A}')$. Suppose that $h^{-1}(\{p'\}) = \{x_1, x_2, \ldots, x_t\}$ and let $q = \bigwedge_{i=1}^{t} x_i$. It is easy to see that $q \in h^{-1}(\{p'\})$ and $q \neq 0$. Besides, $q \in \Pi(\mathcal{A})$ and $h^{-1}(\{p'\}) \cap \Pi(\mathcal{A}) = \{q\}$. Indeed, suppose that $q = a \vee b$ for some $a, b \in A$. Then $h(q) = h(a) \vee h(b) = p'$. Since p' is join-irreducible we have $h(a) = p'$ or $h(b) = p'$. Hence, $a \in h^{-1}(p')$ or $b \in h^{-1}(p')$, i.e., $q = a$ or $q = b$, which proves $q \in \Pi(\mathcal{A})$. On the other hand, let $p \in h^{-1}(p') \cap \Pi(\mathcal{A})$. Then $q \leqslant p$. Let $C \subseteq \Pi(\mathcal{A})$ be the chain which contains q and p and suppose that $C \simeq \Pi(L_{r+1})$ for some integer $r \geqslant 1$. If $q < p$ then we can write $q = j \cdot 1/r$ and $p = k \cdot 1/r$, for some integers j, k such that $1 \leqslant j < k \leqslant r$. Let $z = \sim(p \to q) = (k-j) \cdot 1/r$. Then $h(z) = (k-j) \cdot h(1/r)$. If $h(z) = 0$ then $h(1/r) = 0$ wherefrom we have $p' = h(q) = h(j/r) = j \cdot h(1/r) = 0$, which is a contradiction. Hence, $h(z) \neq 0$, which contradicts $h(z) = \sim(h(p) \to h(q)) = \sim(p' \to p') = 0$. Therefore $q = p$. □

The above result allows us to define, for each $h \in \mathrm{Epi}(\mathcal{A}, \mathcal{A}')$, a function $f\colon \Pi(\mathcal{A}') \to \Pi(\mathcal{A})$ by $f(p') = p$ if only if $h^{-1}(\{p'\}) = \{p\}$. We will say that f is the function induced by the epimorphism h.

Lemma 2.4. *Let $h \in \mathrm{Epi}(\mathcal{A}, \mathcal{A}')$. Then the function induced by the epimorphism h is an* mv*-function.*

P r o o f. Let $h \in \mathrm{Epi}(\mathcal{A}, \mathcal{A}')$. Let $f\colon \Pi(\mathcal{A}') \to \Pi(\mathcal{A})$ be defined by $f(p') = p$ if and only if $h^{-1}(\{p'\}) = \{p\}$, for each $p' \in A'$. We must show that conditions (F1), (F2) and (F3) in Definition 2.1 hold. Condition (F1) follows by definition. Let $p' \in \Pi(\mathcal{A}')$ and let k be an integer, $k \geqslant 1$. Let us consider the elements $p_1 = f(k \cdot p')$ and $p_2 = f(p')$. Since $h(k \cdot p_2) = k \cdot h(p_2) = k \cdot p' = h(p_1) \in \Pi(\mathcal{A})$, we conclude $p_1 = k \cdot p_2$ by applying Corollary 1.1 (v) and Lemma 2.3. This proves (F2). In order to prove (F3), note that the following properties hold:

(P1) f preserves the order (it is a consequence of (F2)).

(P2) If $f(C'(p'_0)) \subseteq C(p_0)$ then $f(p'_0) = p_0$.

Indeed, let us consider the elements $q = f(p'_0)$ and $p \in C(p_0) \subseteq \Pi(\mathcal{A})$ such that $p \leqslant q$. Then $h(p) \leqslant h(q) = p'_0$ which implies $p'_0 = h(p)$. Hence, by Lemma 2.3, we have that $p = q$.

(P3) If f' is the restriction of f to $C' \subseteq \Pi(\mathcal{A}')$ and $f'(C') \subseteq C \subseteq \Pi(\mathcal{A})$, then $f'(C') = C$ (this is a consequence of (P1), (F2) and (P2)).

We prove now (F3). Let $p' \in C'(p'_0) \subseteq \Pi(\mathcal{A}')$. Suppose that $C' \simeq \Pi(L_{r+1})$ for some integer $r \geqslant 1$. Thus, there exists an integer i, $1 \leqslant i \leqslant r$, such that $p' = i \cdot p'_0$. Then $\Psi'(p') = \Psi'(i \cdot p'_0) = (r-i+1) \cdot p'_0$ which implies $f(\Psi'(p')) = (r-i+1) \cdot f(p'_0)$. Moreover, $f(p') = i \cdot f(p'_0)$ and then by applying (P2) we have that $f(p'_0)$ is the first element in the chain. Then $\Psi(f(p')) = (r-i+1) \cdot f(p'_0)$, which completes the proof. □

Let $F_{\mathrm{mv}}(\mathcal{A}', \mathcal{A})$ denote the set of all mv-functions from $\Pi(\mathcal{A}')$ to $\Pi(\mathcal{A})$.

Theorem 2.2. *The sets* $F_{\mathrm{mv}}(\mathcal{A}', \mathcal{A})$ *and* $\mathrm{Epi}(\mathcal{A}, \mathcal{A}')$ *have the same cardinal number.*

P r o o f. Let $\varphi\colon F_{\mathrm{mv}}(\mathcal{A}', \mathcal{A}) \to \mathrm{Epi}(\mathcal{A}, \mathcal{A}')$ be the map defined by $\varphi(f) = h_f$ where h_f is the epimorphism induced by f, for each $f \in F_{\mathrm{mv}}(\mathcal{A}', \mathcal{A})$. Let $f, g \in F_{\mathrm{mv}}(\mathcal{A}', \mathcal{A})$. Suppose that $f \neq g$. Then there exists $p' \in \Pi(\mathcal{A}')$ such that $f(p') \neq g(p')$. Let $p, q \in \Pi(\mathcal{A})$ be such that $f(p') = p$ and $g(p') = q$. Then $g(p') \neq p$ and $h_f(p) = p'$. Therefore $h_f \neq h_g$, so φ is injective.

Let $h \in \mathrm{Epi}(\mathcal{A}, \mathcal{A}')$. Let f be the function induced by h, that is $f(p') = p$ if and only if $h(p) = p'$. From Lemma 2.4 we have that $f \in F_{\mathrm{mv}}(\mathcal{A}', \mathcal{A})$. We claim $h_f = h$, which proves that φ is surjective. Indeed, let $p \in \Pi(\mathcal{A})$. If $h_f(p) \neq 0$ then from Lemma 2.2 we have $h_f(p) = p' \in \Pi(\mathcal{A}')$ and $f(p') = p$. Thus $h(p) = p'$ wherefrom we conclude $h_f(p) = h(p)$. Suppose now that $h_f(p) = 0$, that is, $A'_p = \{p' \in \Pi(\mathcal{A}')\colon\ f(p') \leqslant p\} = \emptyset$. If $h(p) \neq 0$ then there exists an element $q' \in \Pi(\mathcal{A}')$ which satisfies $q' \leqslant h(p)$. By Lemma 2.3 there exists a unique $q \in \Pi(\mathcal{A})$ such that $h(q) = q'$. Then $h(q) = q' = q' \wedge h(p) = h(q) \wedge h(p) = h(q \wedge p)$ which implies $q = q \wedge p$. Hence, we get $q = f(q') \leqslant p$, that is, $q' \in A'_p$, which contradicts $A'_p = \emptyset$. □

Suppose that $\mathcal{A} = \mathcal{A}_{t_1 t_2 \ldots t_n}$ and $\mathcal{A}' = \mathcal{A}'_{r_1 r_2 \ldots r_m}$, with $n \geqslant m$.

If $n > m$ then, taking $r_j = 0$ for all $m+1 \leqslant j \leqslant n$, we can write $\mathcal{A}' = \mathcal{A}'_{r_1 r_2 \ldots r_m} = \mathcal{A}'_{r_1 r_2 \ldots r_n}$. Thus, it is clear that $F_{\mathrm{mv}}(\mathcal{A}', \mathcal{A}) \neq \emptyset$ if and only if $r_i \leqslant t_i$ for all i, $1 \leqslant i \leqslant n$. In this case, the cardinal number of $F_{\mathrm{mv}}(\mathcal{A}', \mathcal{A})$ is $V_{t_1}^{r_1} \cdot V_{t_2}^{r_2} \cdot \ldots \cdot V_{t_n}^{r_n}$, where

$$V_{t_i}^{r_i} = \begin{cases} \dfrac{t_i!}{(t_i - r_i)!} & \text{if } r_i > 0,\ t_i > 0, \\ 1 & \text{if } r_i = 0,\ t_i \geqslant 0. \end{cases}$$

It is clear that the function induced by $h \in \mathrm{Epi}(\mathcal{A}, \mathcal{A}')$ is surjective whenever h is injective. Conversely, if $f\colon \Pi(\mathcal{A}') \to \Pi(\mathcal{A})$ is a surjective mv-function then the epimorphism induced by f is injective. Let $F^*_{\mathrm{mv}}(\mathcal{A}', \mathcal{A})$ denote the set of all mv-functions from $\Pi(\mathcal{A}')$ onto $\Pi(\mathcal{A})$. Then $F^*_{\mathrm{mv}}(\mathcal{A}', \mathcal{A}) \neq \emptyset$ if and only if $n = m$ and $t_i = r_i$ for all i, $1 \leqslant i \leqslant n$. In this case, the cardinal number of F^*_{mv} is $t_1! \cdot t_2! \cdot \ldots \cdot t_n!$.

Corollary 2.1. *If* $\mathcal{A}$ *is a finite* W*-algebra and* $\mathcal{A} = \mathcal{A}_{t_1 t_2 \ldots t_n}$*, then the number of automorphisms of* $\mathcal{A}$ *is* $t_1! \cdot t_2! \cdot \ldots \cdot t_n!$.

References

[1] *M. Abad, A. Figallo*: On Łukasiewicz Homomorphisms. Facultad de Filosofía, Humanidades y Artes, Universidad Nacional de San Juan, 1992.

[2] *J. Berman, W. J. Blok*: Free Łukasiewicz and hoop residuation algebras. Stud. Log. *77* (2004), 153–180. zbl MR doi

[3] *V. Boicescu, A. Filipoiu, G. Georgescu, S. Rudeanu*: Łukasiewicz-Moisil Algebras. Annals of Discrete Mathematics 49. North-Holland, Amsterdam, 1991. zbl MR doi

[4] *C. C. Chang*: Algebraic analysis of many valued logics. Trans. Am. Math. Soc. *88* (1958), 467–490. zbl MR doi

[5] *C. C. Chang,*: A new proof of the completeness of Łukasiewicz axioms. Trans. Am. Math. Soc. *93* (1959), 74–80. zbl MR doi

[6] *R. Cignoli, I. M. L. D'Ottaviano, D. Mundici*: Algebraic Foundations of Many-Valued Reasoning. Trends in Logic-Studia Logica Library 7. Kluwer Academic Publishers, Dordrecht, 2000. zbl MR doi

[7] *R. Cignoli, E. J. Dubuc, D. Mundici*: Extending Stone duality to multisets and locally finite MV-algebras. J. Pure Appl. Algebra *189* (2004), 37–59. zbl MR doi

[8] *R. Cignoli, V. Marra*: Stone duality for real-valued multisets. Forum Math. *24* (2012), 1317–1331. zbl MR doi

[9] *A. V. Figallo*: Algebras implicativas de Łukasiewicz $(n+1)$-valuadas con diversas operaciones adicionales. Tesis Doctoral. Univ. Nac. del Sur, 1990.

[10] *J. M. Font, A. J. Rodríguez, A. Torrens*: Wajsberg algebras. Stochastica *8* (1984), 5–31. zbl MR

[11] *Y. Komori*: Super-Łukasiewicz implicational logics. Nagoya Math. J. *72* (1978), 127–133. zbl MR

[12] *Y. Komori*: Super Łukasiewicz propositional logics. Nagoya Math. J. *84* (1981), 119–133. zbl MR

[13] *J. Łukasiewicz*: On three-valued logics. Ruch filozoficzny *5* (1920), 169–171. (In Polish.)

[14] *J. Łukasiewicz, A. Tarski*: Untersuchungen über den Aussagenkalkül. C. R. Soc. Sc. Varsovie *23* (1930), 30–50. zbl

[15] *N. G. Martínez*: The Priestley duality for Wajsberg algebras. Stud. Log. *49* (1990), 31–46. zbl MR doi

[16] *L. F. Monteiro*: Number of epimorphisms between finite Łukasiewicz algebras. Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér. *49(97)* (2006), 177–187. zbl MR

[17] *A. J. Rodríguez*: Un studio algebraico de los cálculos proposicionales de Łukasiewicz. Ph. Doc. Diss. Universitat de Barcelona (1980).

[18] *A. J. Rodríguez, A. Torrens, V. Verdú*: Łukasiewicz logic and Wajsberg algebras. Bull. Sect. Log., Pol. Acad. Sci. *19* (1990), 51–55. zbl MR

Authors' addresses: *Aldo V. Figallo*, Instituto de Ciencias Básicas, Universidad Nacional de San Juan, Av. Ignacio de la Roza 230 Oeste, 5400 San Juan, Argentina, e-mail: `avfigallo@gmail.com`; *Marina B. Lattanzi*, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de La Pampa, Av. Uruguay 151, 6300 Santa Rosa, Argentina, e-mail: `mblatt@exactas.unlpam.edu.ar`.