ON THE ORDER OF CONVOLUTION CONSISTENCE OF THE ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Received May 4, 2015. First published February 2, 2017.
Communicated by Dagmar Medková

Abstract. Making use of a modified Hadamard product, or convolution, of analytic functions with negative coefficients, combined with an integral operator, we study when a given analytic function is in a given class. Following an idea of U. Bednarz and J. Sokół, we define the order of convolution consistence of three classes of functions and determine a given analytic function for certain classes of analytic functions with negative coefficients.

Keywords: analytic function with negative coefficients; univalent function; extreme point; order of convolution consistence; starlikeness; convexity

MSC 2010: 30C45, 30C50

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of analytic functions in the unit disc $\mathcal{U} = \{z\colon |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$ and let $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Definition 1 ([4]). We define the operator $D^n\colon A \to A$, $n \in \mathbb{N}$ for $z \in U$ by:
a) $D^0 f(z) = f(z)$,
b) $D^1 f(z) = Df(z) = zf'(z)$,
c) $D^n f(z) = D(D^{n-1} f(z))$.

Definition 2 ([4]). Let $\alpha \in [0, 1)$ and let $n \in \mathbb{N}$. We define the class $\mathcal{S}_n(\alpha)$ of *n-starlike functions of order* α by

$$\mathcal{S}_n(\alpha) = \Big\{ f \in A\colon \operatorname{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha, \quad z \in U \Big\}. \tag{1.1}$$

Denote by $\mathcal{S}_n$ the class $\mathcal{S}_n(0)$. We note that $\mathcal{S}_0 = \mathcal{ST}$ is the class of starlike functions and $\mathcal{S}_1 = \mathcal{CV}$ is the class of convex functions.

DOI: 10.21136/MB.2017.0019-15

The convolution, or the Hadamard product, of two functions f and g in $\mathcal{A}$ of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad \text{and} \quad g(z) = z + \sum_{j=2}^{\infty} b_j z^j$$

is the function $(f * g)$ defined as

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j.$$

Let us consider the integral operator (see [2], [1], [4]) $\mathcal{I}^s\colon \mathcal{A} \to \mathcal{A}$, $s \in \mathbb{R}$, such that

$$\mathcal{I}^s f(z) = \mathcal{I}^s \left(z + \sum_{j=2}^{\infty} a_j z^j \right) = z + \sum_{j=2}^{\infty} \frac{a_j}{j^s} z^j. \tag{1.2}$$

Definition 3 ([2]). Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be subsets of $\mathcal{A}$. We say that the triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is S-closed under the convolution if there exists a number $S\ (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that

$$S(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = \min\{s \in \mathbb{R}\colon \mathcal{I}^s(f * g) \in \mathcal{Z},\ f \in \mathcal{X},\ g \in \mathcal{Y}\}. \tag{1.3}$$

The number $S(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is called the *order of convolution consistence* of the triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.

Bednarz and Sokòł in [2] obtained the order of convolution consistence for certain classes of univalent functions (starlike, convex, uniform-starlike or uniform-convex functions). For example they proved the following statement.

Theorem 1 ([2]). *We have the following orders of convolution consistence:*

(i) $S(\mathcal{ST}, \mathcal{ST}, \mathcal{ST}) = 1$,
(ii) $S(\mathcal{CV}, \mathcal{CV}, \mathcal{ST}) = -1$,
(iii) $S(\mathcal{CV}, \mathcal{ST}, \mathcal{ST}) = 0$,
(iv) $S(\mathcal{ST}, \mathcal{ST}, \mathcal{CV}) = 2$,
(v) $S(\mathcal{CV}, \mathcal{CV}, \mathcal{CV}) = 0$,
(vi) $S(\mathcal{CV}, \mathcal{ST}, \mathcal{CV}) = 1$.

Let $\mathcal{N}$ denote the subclass of $\mathcal{A}$ consisting of analytic functions of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \quad a_j \geqslant 0,\ j \in \{2, 3, 4, \ldots\}. \tag{1.4}$$

Then $\mathcal{T}_n(\alpha) = \mathcal{S}_n(\alpha) \cap \mathcal{N}$ is the class of n-starlike functions of order α with negative coefficients. In particular, $\mathcal{T}_0(\alpha)$ and $\mathcal{T}_1(\alpha)$ are the class of starlike functions of order α with negative coefficients and the class of convex functions of order α with negative coefficients, respectively, introduced by Silverman [8]. We denote $\mathcal{T}_n(0)$ by $\mathcal{T}_n$.

The modified Hadamard product, or $\circledast$-convolution, of two functions f and g in $\mathcal{N}$ of the form

$$(1.5)\qquad f(z) = z - \sum_{j=2}^{\infty} a_j z^j \quad \text{and} \quad g(z) = z - \sum_{j=2}^{\infty} b_j z^j, \quad a_j, b_j \geqslant 0,$$

is the function $(f \circledast g)$ defined as (see [7])

$$(f \circledast g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j.$$

As in Definition 3, we define the *order of $\circledast$-convolution consistence* of the triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, where $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are subsets of $\mathcal{N}$, denoted $S_\circledast$ by

$$(1.6)\qquad S_\circledast(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = \min\{s \in \mathbb{R} \colon \mathcal{I}^s(f \circledast g) \in \mathcal{Z},\ f \in \mathcal{X},\ g \in \mathcal{Y}\}.$$

In this paper we obtain similar results as in Theorem 1 but for the class $\mathcal{T}_n$ and for $\circledast$-convolution.

We need the following characterization of the class $\mathcal{T}_n$.

Theorem 2. *Let $n \in \mathbb{N}$ and let $f \in \mathcal{N}$ be a function of the form* (1.4). *Then f belongs to $\mathcal{T}_n$ if and only if*

$$\sum_{j=2}^{\infty} j^{n+1} a_j \leqslant 1.$$

The result is sharp and the extremal functions are

$$(1.7)\qquad f_j(z) = z - \frac{1}{j^{n+1}} z^j, \quad j \in \{2, 3, \ldots\}.$$

A proof of this theorem in the particular cases $n = 0$ and $n = 1$ is given by Silverman in [8] and by Gupta and Jain in [3]. In a more general form (for $\mathcal{T}_n(\alpha)$) it is given in [5] and [6].

2. Main results

Theorem 3. *If* $f \in \mathcal{T}_{n+p}$ *and* $g \in \mathcal{T}_{n+q}$, *then* $\mathcal{I}^s(f \circledast g) \in T_{n+r}$, *where* $p, q, r, n \in \mathbb{N}$ *and when*

$$s = r - p - q - n - 1. \tag{2.1}$$

The result is sharp.

P r o o f. Since $f \in \mathcal{T}_{n+p}$ and $g \in \mathcal{T}_{n+q}$, if f and g have the form (1.5), then from Theorem 1 we have

$$\sum_{j=2}^{\infty} j^{n+p+1} a_j \leqslant 1 \quad \text{and} \quad \sum_{j=2}^{\infty} j^{n+q+1} b_j \leqslant 1$$

and by the Cauchy-Schwarz inequality we deduce

$$\sum_{j=2}^{\infty} j^{n+(p+q)/2+1} \sqrt{a_j b_j} \leqslant 1. \tag{2.2}$$

We need to find conditions on s, r, p, q, n such that

$$\sum_{j=2}^{\infty} j^{n+r+1-s} a_j b_j \leqslant 1.$$

Thus it is sufficient to show that

$$j^{n+r+1-s} a_j b_j \leqslant j^{n+(p+q)/2+1} \sqrt{a_j b_j},$$

that is, that

$$\sqrt{a_j b_j} \leqslant j^{s-r+(p+q)/2}, \quad j \in \{2, 3, \ldots\}.$$

But from (2.2) we know that

$$\sqrt{a_j b_j} \leqslant j^{-n-(p+q)/2-1}, \quad j \in \{2, 3, \ldots\}.$$

Consequently, it is sufficient to show that

$$j^{-n-(p+q)/2-1} \leqslant j^{s-r+(p+q)/2}, \quad j \in \{2, 3, \ldots\},$$

or, equivalently, that

$$j^{r-s-n-p-q-1} \leqslant 1, \quad j \in \{2, 3, \ldots\}, \tag{2.3}$$

but the inequalities (2.3) hold for s, r, p, q, n satisfying (2.1).

Finally, by using the extremal functions (see (1.7)) $f_2(z) = z - z^2/2^{n+p+1} \in \mathcal{T}_{n+p}$ and $g_2(z) = z - z^2/2^{n+q+1} \in \mathcal{T}_{n+q}$ we can see that

$$\mathcal{I}^s(f_2 \circledast g_2) = z - \frac{z^2}{2^{2n+s+p+q+2}}.$$

But from (2.1) we deduce

$$\mathcal{I}^s(f_2 \circledast g_2) = z - \frac{z^2}{2^{n+r+1}} \in \mathcal{T}_{n+r}, \tag{2.4}$$

and this shows that the result in Theorem 3 is sharp. □

Theorem 4. *Let $p, q, r, n \in \mathbb{N}$ and let s be given by (2.1). Then the order of $\circledast$-convolution consistence*

$$S_{\circledast}(\mathcal{T}_{n+p}, \mathcal{T}_{n+q}, \mathcal{T}_{n+r}) = s = r - p - q - n - 1. \tag{2.5}$$

Proof. Theorem 3 shows that $S_{\circledast}(\mathcal{T}_{n+p}, \mathcal{T}_{n+q}, \mathcal{T}_{n+r}) \leqslant s$ and from (2.4) we have $S_{\circledast}(\mathcal{T}_{n+p}, \mathcal{T}_{n+q}, \mathcal{T}_{n+r}) \geqslant s$. □

Corollary 1. *We have the following orders of $\circledast$-convolution consistence:*
(a) $S_{\circledast}(\mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_0) = -1$,
(b) $S_{\circledast}(\mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_1) = 0$,
(c) $S_{\circledast}(\mathcal{T}_1, \mathcal{T}_0, \mathcal{T}_0) = -2$,
(d) $S_{\circledast}(\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_0) = -3$,
(e) $S_{\circledast}(\mathcal{T}_1, \mathcal{T}_0, \mathcal{T}_1) = -1$,
(f) $S_{\circledast}(\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1) = -2$.

We note that $\mathcal{T}_0 = \mathcal{ST} \cap \mathcal{N}$ and $\mathcal{T}_1 = \mathcal{CV} \cap \mathcal{N}$ and it is easy to compare the results of Theorem 1 to those of Corollary 1.

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