A STUDY OF VARIOUS RESULTS FOR A CLASS OF ENTIRE DIRICHLET SERIES WITH COMPLEX FREQUENCIES

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Abstract. Let F be a class of entire functions represented by Dirichlet series with complex frequencies $\sum a_k \mathrm{e}^{\langle \lambda^k, z\rangle}$ for which $(|\lambda^k|/\mathrm{e})^{|\lambda^k|} k!|a_k|$ is bounded. Then F is proved to be a commutative Banach algebra with identity and it fails to become a division algebra. F is also proved to be a total set. Conditions for the existence of inverse, topological zero divisor and continuous linear functional for any element belonging to F have also been established.

Keywords: Dirichlet series; Banach algebra; topological zero divisor; division algebra; continuous linear functional; total set

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1. Introduction

Consider a series of the form

$$f(z) = \sum_{k=1}^{\infty} a_k \mathrm{e}^{\langle \lambda^k, z\rangle}, \quad z \in \mathbb{C}^n, \tag{1.1}$$

where $\{\lambda^k\}$, $k \in \mathbb{N}$ and $\lambda^k = (\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k)$, is a sequence of complex vectors in $\mathbb{C}^n$ and $\langle \lambda^k, z\rangle = \lambda_1^k z_1 + \lambda_2^k z_2 + \ldots + \lambda_n^k z_n$. If $a_k \in \mathbb{C}$ for $k \in \mathbb{N}$ and $\{\lambda^k\}$ satisfies the condition $|\lambda^k| \to \infty$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \frac{\log |a_k|}{|\lambda^k|} = -\infty \tag{1.2}$$

$$\limsup_{k\to\infty} \frac{\log k}{|\lambda^k|} = D < \infty, \tag{1.3}$$

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then from [1] the series (1.1) represents an entire function. Let F be the set of series (1.1) for which $(|\lambda^k|/\mathrm{e})^{|\lambda^k|}k!\,|a_k|$ is bounded. Then every element of F represents an entire function.

Various results have been proved for different classes of entire Dirichlet series $\sum\limits_{n=1}^{\infty} a_n \mathrm{e}^{\lambda_n s}$, where few of them may be found in [3], [2], [7], [8]. Very recently Kumar and Manocha in [4] established several results on a class of entire functions represented by Dirichlet series having complex frequencies (1.1). In the present article we establish certain results, namely Banach algebra, topological zero divisor, division algebra, continuous linear functional and total set for the Dirichlet series of the form (1.1). If

$$f(z) = \sum_{k=1}^{\infty} a_k \mathrm{e}^{\langle \lambda^k, z\rangle} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k \mathrm{e}^{\langle \lambda^k, z\rangle},$$

define binary operation, i.e. addition and scalar multiplication in F as

$$\begin{aligned} f(z)+g(z) &= \sum_{k=1}^{\infty} (a_k+b_k)\mathrm{e}^{\langle \lambda^k, z\rangle}, \\ \alpha f(z) &= \sum_{k=1}^{\infty} (\alpha a_k)\mathrm{e}^{\langle \lambda^k, z\rangle}, \\ f(z)g(z) &= \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, a_k b_k \mathrm{e}^{\langle \lambda^k, z\rangle}. \end{aligned}$$

The norm in F is defined as

$$\|f\| = \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\,|a_k|. \tag{1.4}$$

In the sequel, following definitions are required for proving the main results.

Definition 1. Let F be a normed algebra. Then $x \in F$ is said to be a left (right) topological zero divisor if there exists a sequence $\{y_k\} \subset F$ such that $\|y_k\| = 1$, $k \in \mathbb{N}$ and $\lim\limits_k \|xy_k\| = 0$ (or $\lim\limits_k \|y_k x\| = 0$). If the algebra F is commutative, then the notions of left and right topological zero divisors are identical and in this case we shall speak only of topological zero divisors.

Definition 2. Let F be an algebra with identity. If each $x \in F$ $(x \neq 0)$ is regular, then F is said to be a division algebra.

Definition 3. Let F be a locally convex topological vector space. A set $E' \subset F$ is said to be total if and only if for any $\psi \in F^*$ with $\psi(E') = 0$ we have $\psi = 0$.

2. Main Results

In this section, main results are proved. For the definitions of terms used refer to [5]–[6].

Theorem 1. *F is a commutative Banach algebra with identity.*

Proof. Let $\{f_{p_1}\}$ be a Cauchy sequence in F. For given $\varepsilon > 0$ find p such that

$$\|f_{p_1} - f_{p_2}\| < \varepsilon, \quad \text{where } p_1, p_2 \geqslant p.$$

This implies that

$$\sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\,|a_{p_{1k}} - a_{p_{2k}}| < \varepsilon, \quad \text{where } p_1, p_2 \geqslant p.$$

This shows that $\{a_{p_{1k}}\}$ forms a Cauchy sequence in the set of complex numbers for every value of $k \geqslant 1$ and hence it converges and say to a_k. Therefore $f_{p_1} \to f = \sum\limits_{k=1}^{\infty} a_k \mathrm{e}^{\langle \lambda^k, z\rangle}$. Also

$$\sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\,|a_k| \leqslant \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\,|a_{p_{1k}} - a_k| + \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\,|a_{p_{1k}}|.$$

Hence $f(z) \in F$. Thus, F is complete under the norm defined by (1.4).

If $f(z), g(z) \in F$, then

$$\begin{aligned} \|fg\| &= \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, \Big|a_k b_k \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\Big| \\ &\leqslant \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\,|a_k| \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\,|b_k| = \|f\|\|g\|. \end{aligned}$$

The identity element in F is

$$e(z) = \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} \mathrm{e}^{\langle \lambda^k, z\rangle}.$$

This completes the proof of the theorem. □

Theorem 2. *The function $f(z) = \sum\limits_{k=1}^{\infty} a_k \mathrm{e}^{\langle \lambda^k, z\rangle}$ is invertible in F if and only if*

$$\Big\{\Big|a_k^{-1} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!}\Big|\Big\}$$

is a bounded sequence.

P r o o f. Let $f(z)$ be invertible and $g(z) = \sum\limits_{k=1}^{\infty} b_k \mathrm{e}^{\langle \lambda^k, z\rangle}$ be its inverse. Then $f(z)g(z) = e(z)$ implies that

$$\Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, a_k b_k = \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!}.$$

Thus,

$$\Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, |b_k| = \Big| a_k^{-1} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!}\Big|.$$

Since $g(z) \in F$, $\{|a_k^{-1}(|\lambda^k|/\mathrm{e})^{-|\lambda^k|}(k!)^{-1}|\}$ is a bounded sequence.

Conversely, suppose $\{|a_k^{-1}(|\lambda^k|/\mathrm{e})^{-|\lambda^k|}(k!)^{-1}|\}$ is a bounded sequence. Define $g(z)$ such that

$$g(z) = \sum_{k=1}^{\infty} a_k^{-1} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-2|\lambda^k|} \frac{1}{(k!)^2} \mathrm{e}^{\langle \lambda^k, z\rangle}.$$

Obviously $g(z) \in F$. Moreover,

$$f(z)g(z) = \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k! a_k a_k^{-1} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-2|\lambda^k|} \frac{1}{(k!)^2} \mathrm{e}^{\langle \lambda^k, z\rangle} = e(z).$$

Hence the theorem. □

Theorem 3. *A necessary and a sufficient condition that an element $f(z)$ of F is a topological zero divisor is*

(2.1)
$$\lim_{k\to\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, |a_k| = 0.$$

P r o o f. Let (2.1) hold. Construct a sequence $\{g_k\}$ such that

$$g_k(z) = \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} \mathrm{e}^{\langle \lambda^k, z\rangle}.$$

Thus, for all $k \geqslant 1$, $g_k \in F$ and $\|g_k\| = 1$. Then

$$\begin{aligned} g_k(z)f(z) &= f(z)g_k(z) \\ &= \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, a_k \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} \mathrm{e}^{\langle \lambda^k, z\rangle} = \sum_{k=1}^{\infty} a_k \mathrm{e}^{\langle \lambda^k, z\rangle}. \end{aligned}$$

Therefore

$$\|g_k f\| = \|f g_k\| = \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, |a_k|.$$

As $k \to \infty$,

$$\|g_k f\| = \|f g_k\| \to 0.$$

Thus, $f(z)$ is a topological zero divisor by Definition 1.

Conversely, suppose if possible (2.1) is not true, that is

$$\lim_{k\to\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\,|a_k| = \mu > 0.$$

Then given ν with $0 < \nu < \mu$ find an integer $k_0 \geqslant 1$ such that for all $k \geqslant k_0$

$$\Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\,|a_k| > \mu - \nu$$

holds true. Also since $f(z)$ is a topological zero divisor, there exists an arbitrary sequence $\{g_t\}$ of elements in F with unit norm such that for all $k \geqslant 1$ one has

$$g_t(z) = \sum_{k=1}^{\infty} b_{k_t} \mathrm{e}^{\langle \lambda^{k_t}, z\rangle}.$$

This implies

$$\sum_{k=1}^{\infty} \Big(\frac{|\lambda^{k_t}|}{\mathrm{e}}\Big)^{|\lambda^{k_t}|} (k_t!)|b_{k_t}| = 1.$$

Next, for given ε satisfying $0 < \varepsilon < 1$ there exists an integer K_t and a subsequence $\{k_t\}$ of the sequence of indices $\{k\}$ such that

$$\Big(\frac{|\lambda^{k_t}|}{\mathrm{e}}\Big)^{|\lambda^{k_t}|} k_t!\,|b_{k_t}| > 1 - \varepsilon \quad \text{for all } k = k_t \geqslant K_t.$$

Hence,

$$\|f g_t\| \nrightarrow 0,$$

which is a contradiction. Thus, the proof of the theorem is completed. □

Theorem 4. *F is not a division algebra.*

Proof. Let

$$p(z) = \sum_{k=1}^{\infty} k^{-1} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} \mathrm{e}^{\langle \lambda^k, z\rangle}.$$

Then $p(z) \in F$ and it does not possess inverse in F. Let, if possible,

$$q(z) = \sum_{k=1}^{\infty} d_k \mathrm{e}^{\langle \lambda^k, z\rangle}$$

be its inverse. Hence,

$$p(z)q(z) = e(z).$$

This implies

$$\Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, k^{-1} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} d_k = \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!},$$

which further implies that

$$d_k = k\Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} \quad \text{does not belong to } F.$$

Clearly by Definition 2, F fails to become a division algebra. This proves the theorem.

□

Theorem 5. *Every continuous linear functional* $\varphi\colon F \to \mathbb{C}$ *is of the form*

$$\varphi(f) = \sum_{k=1}^{\infty} a_k p_k \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!,$$

where

$$f(z) = \sum_{k=1}^{\infty} a_k \mathrm{e}^{\langle \lambda^k, z\rangle}$$

and $\{p_k\}$ *is a bounded sequence in* $\mathbb{C}$.

P r o o f. Let $\varphi\colon F \to \mathbb{C}$ be a continuous linear functional. Since φ is continuous,

$$\varphi(f) = \varphi\Big(\lim_{N\to\infty} f^{(N)}\Big),$$

where

$$f^{(N)}(z) = \sum_{k=1}^{N} a_k \mathrm{e}^{\langle \lambda^k, z\rangle}.$$

This implies

$$\varphi(f) = \varphi\Bigg(\lim_{N\to\infty} \sum_{k=1}^{N} a_k \mathrm{e}^{\langle \lambda^k, z\rangle}\Bigg).$$

Define a sequence $\{f_k\} \subseteq F$ as

$$f_k = \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} \mathrm{e}^{\langle \lambda^k, z\rangle}.$$

Then

$$\varphi(f) = \varphi\Big(\lim_{N\to\infty}\sum_{k=1}^{N} a_k\Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, f_k\Big)$$
$$= \lim_{N\to\infty}\sum_{k=1}^{N} a_k\Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, \varphi(f_k).$$

Since φ is a linear functional,

$$\varphi(f_k) = p_k.$$

This implies

$$\varphi(f) = \sum_{k=1}^{\infty} a_k p_k \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!.$$

Now

$$|p_k| = |\varphi(f_k)| \leqslant M\|f_k\|$$

and $\|f_k\| = 1$. This implies $|p_k| \leqslant M$. Thus, $\{p_k\}$ is a bounded sequence in $\mathbb{C}$.

Conversely, let $\{p_k\}$ be a bounded sequence in $\mathbb{C}$ satisfying

$$\varphi(f) = \sum_{k=1}^{\infty} a_k p_k \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!.$$

Then φ is well defined and linear. Also

$$|\varphi(f)| = \sum_{k=1}^{\infty} |a_k p_k| \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k! = \sum_{k=1}^{\infty} |a_k||p_k| \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k! \leqslant M\|f\|.$$

Thus, φ is a continuous linear functional. This completes the proof of the theorem. □

Theorem 6. *Let* $f(z) = \sum_{k=1}^{\infty} a_k \mathrm{e}^{\langle \lambda^k, z\rangle} \in F$, *where* $a_k \neq 0$ *for every* $k \geqslant 1$. *Let* $K \in \mathbb{C}^n$ *be a set having at least one finite limit point. Define*

$$f_\tau(z) = \sum_{k=1}^{\infty} a_k \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} \mathrm{e}^{\langle \lambda^k, z+\tau\rangle}. \tag{2.2}$$

Then the set $A_f = \{f_\tau\colon\ \tau \in K\}$ *is a total set with respect to the family of continuous linear transformations* $\varphi\colon\ F \to \mathbb{C}$.

P r o o f. We have

$$f_\tau(z) = \sum_{k=1}^{\infty} a_k \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} \mathrm{e}^{\langle \lambda^k, z+\tau\rangle}.$$

Note that for all $\tau \in \mathbb{C}^n$

$$\|f_\tau\| = \sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k! \Big| a_k \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} \mathrm{e}^{\langle \lambda^k, \tau\rangle}\Big| = \sum_{k=1}^{\infty} |a_k| \mathrm{e}^{\mathrm{Re}\langle \lambda^k, \tau\rangle}.$$

Since $f(z)$ is an entire Dirichlet series which converges absolutely in the whole complex plane, for every $\tau \in K$ the series on the right-hand side of the above equation is clearly convergent. Hence, $f_\tau(z) \in F$ for every $\tau \in K$.

Let $\varphi \in F^*$ be a continuous linear transformation such that $\varphi(A_f) \equiv 0$, that is $\varphi(f_\tau) = 0$ for all $\tau \in K$. Then by Theorem 5,

$$\sum_{k=1}^{\infty} \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{|\lambda^k|} k!\, a_k \Big(\frac{|\lambda^k|}{\mathrm{e}}\Big)^{-|\lambda^k|} \frac{1}{k!} \mathrm{e}^{\langle \lambda^k, \tau\rangle} p_k = 0$$

$$\Rightarrow \sum_{k=1}^{\infty} a_k p_k \mathrm{e}^{\langle \lambda^k, \tau\rangle} = 0 \tag{2.3}$$

for all $\tau \in K$. Now define $h(z) = \sum\limits_{k=1}^{\infty} a_k p_k \mathrm{e}^{\langle \lambda^k, z\rangle}$. Since $\{p_k\}$ is a bounded sequence in $\mathbb{C}$ and $f(z) = \sum\limits_{k=1}^{\infty} a_k \mathrm{e}^{\langle \lambda^k, z\rangle} \in F$, $h(z)$ also belongs to F. But

$$h(\tau) = \sum_{k=1}^{\infty} a_k p_k \mathrm{e}^{\langle \lambda^k, \tau\rangle} = 0 \quad \text{for all } \tau \in K.$$

Since K has a finite limit point, this means that $h \equiv 0$. This however implies that

$$a_k p_k = 0 \quad \text{for all } k \geqslant 1$$

and as $a_k \neq 0$ for every k, $p_k = 0$ for all $k \geqslant 1$. Thus, $\varphi = 0$ and hence the theorem. $\square$

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