RELATIVELY PSEUDOCOMPLEMENTED POSETS

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Abstract. We extend the notion of a relatively pseudocomplemented meet-semilattice to arbitrary posets. We show some properties of the binary operation of relative pseudocomplementation and provide some corresponding characterizations. We show that relatively pseudocomplemented posets satisfying a certain simple identity in two variables are join-semilattices. Finally, we show that every relatively pseudocomplemented poset is distributive and that the converse holds for posets satisfying the ascending chain condition and one more natural condition. Suitable examples are provided.

Keywords: relatively pseudocomplemented poset; join-semilattice; distributive poset

MSC 2010: 06A11, 06A06, 06D15

1. Introduction

The study of pseudocomplementation and relative pseudocomplementation has a relatively long history. For semilattices it was originated by Balbes (cf. [1], see also [10] and [12]). The usefulness of these concepts in algebra and the applications in logic were pointed out by Köhler (see [10]) and Nemitz (see [12]). For the readers' convenience, these results are collected in the monograph [5]. It is known that relatively pseudocomplemented lattices are in fact distributive, see [1] or [5]. The first attempt to modify the concept of relative pseudocomplementation for non-distributive lattices was settled by the first author in [2]. The definition of a^* on page 12 of [13] coincides with the definition of $a * 0$ in our paper (cf. Definition 2.1) in the case when the considered poset has a least element 0. The definition of $x * y$ in formula (5) of [9] is equivalent to our definition in the case $y < x$. In [9] these

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elements are called "sectional" pseudo-complements and are extensions of Venkatanarasimhan's (see [13]) notion from $[0)$ to arbitrary $[y)$. In the present paper, we are going to consider a more general situation.

It turns out that a generalization of relative pseudocomplementation for posets can be useful in certain studies of non-classical logical systems whose underlying posets need not be necessarily semilattices. There are two independent ways how to treat it. The first one is to organize the poset in question into an algebra called directoid, see the monograph [7] for this approach. Then pseudocomplements and relative pseudocomplements can be studied similarly as for semilattices. The essential difference is that the binary operation of a directoid is not assumed to be associative. This approach has been used in the papers [3], [4] and [6]. Another possibility is to investigate pseudocomplements and relative pseudocomplements directly in the given posets provided the definition is slightly modified in order not to use the concept of infimum. This approach is used in the present paper. Analogously as for relatively pseudocomplemented lattices, also relatively pseudocomplemented posets have an additional property which has been called distributivity in [11] and was studied separately in [8].

We hope that this paper encourages other researchers to go on with the topic in order to obtain a full analogy with relatively pseudocomplemented semilattices.

2. Properties of relative pseudocomplementation

For the convenience of the reader we recall some useful concepts.

Let $(P, \leqslant)$ be a poset, $a \in P$ and $M, N \subseteq P$. We define $L(M) := \{x \in P \colon x \leqslant y$ for all $y \in M\}$, $U(M) := \{x \in P \colon x \geqslant y$ for all $y \in M\}$, $L(M, N) := L(M \cup N)$ and $U(M, N) := U(M \cup N)$. Instead of $L(\{a\})$ we write $L(a)$, instead of $L(\{a\}, N)$ we write $L(a, N)$ and so on. Clearly, $L(a, b) = L(b, a)$, $U(a, b) = U(b, a)$ and $M \subseteq N$ imply $L(M) \supseteq L(N)$ and $U(M) \supseteq U(N)$.

Recall that a meet-semilattice $\mathbf{S} = (S, \wedge)$ is called *relatively pseudocomplemented* if for every two elements a, b of S there exists a greatest element c in S such that $a \wedge c \leqslant b$; this element c is called the *relative pseudocomplement of a with respect to b* and it is denoted by $a * b$.

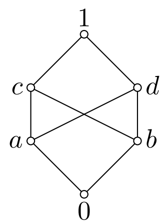
In what follows, we are going to modify this concept for posets which need not be meet-semilattices.

Definition 2.1. Let $(P, \leqslant)$ be a poset and $a, b, c \in P$. The element c is called the *relative pseudocomplement* of a with respect to b, in symbols $c = a * b$, if c is the greatest element of $\{x \in P \colon L(a, x) \subseteq L(b)\}$. If for all $x, y \in P$ the element

$x * y$ exists, then $(P, \leqslant)$ or $(P, \leqslant, *)$ is called a *relatively pseudocomplemented poset*. Let $\mathcal{P}$ denote the class of relatively pseudocomplemented posets.

R e m a r k 2.2. If $(P, \leqslant)$ is a meet-semilattice, then for all $a, b, x \in P$ the following are equivalent: $L(a, x) \subseteq L(b)$, $L(a \wedge x) \subseteq L(b)$, $a \wedge x \leqslant b$. Thus in this case we get the usual notion of relative pseudocomplement.

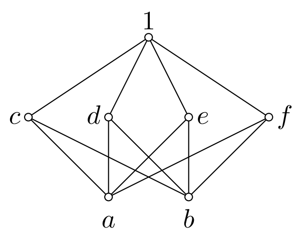
E x a m p l e 2.3. The poset with the Hasse diagram



is not a semilattice, but it belongs to $\mathcal{P}$ and the operation table of $*$ looks as follows:

$*$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	d	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

E x a m p l e 2.4. The poset $(P, \leqslant) = (\{a, b, c, d, e, f, 1\}, \leqslant)$ with the Hasse diagram



is neither a semilattice nor a member of $\mathcal{P}$ since the set

$$\{x \in P\colon\ L(c, x) \subseteq L(d)\} = \{a, b, d, e, f\}$$

has no greatest element.

Lemma 2.5. *Every nonempty member of $\mathcal{P}$ has a greatest element.*

P r o o f. If $(P, \leqslant, *) \in \mathcal{P}$ and $a, b \in P$, then $L(b, a) \subseteq L(b)$ and hence $a \leqslant b * b$ showing that $b * b$ is the greatest element of $(P, \leqslant)$. $\square$

In the following we denote nonempty members of $\mathcal{P}$ also in the form $(P, \leqslant, *, 1)$, thus indicating that 1 is the greatest element of $(P, \leqslant)$.

Lemma 2.6. *If* $(P, \leqslant, *, 1) \in \mathcal{P}$ *and* $a, b, c \in P$, *then*

(i) $a \leqslant b$ *if and only if* $a * b = 1$,
(ii) $a * a = a * 1 = 1$ *and* $1 * a = a$,
(iii) $a \leqslant b$ *implies* $c * a \leqslant c * b$ *and* $b * c \leqslant a * c$,
(iv) $a \leqslant b * a$,
(v) $a \leqslant (a * b) * b$,
(vi) $((a * b) * b) * b = a * b$,
(vii) $L(a, b) = L(a, a * b)$,
(viii) $a \leqslant b * c$ *if and only if* $b \leqslant a * c$,
(ix) $(a * b) * a \leqslant (a * b) * b$ *and*
(x) $((a * b) * a) * b = a * b$.

P r o o f. (i) The following assertions are equivalent: $a * b = 1$, $1 \leqslant a * b$, $L(a, 1) \subseteq L(b)$, $a \leqslant b$.

(ii) The relation $a \leqslant a$ implies $a * a = 1$ according to (i), and $a \leqslant 1$ implies $a * 1 = 1$ according to (i). Since the assertions $b \leqslant 1 * a$, $L(1, b) \subseteq L(a)$, $b \leqslant a$ are equivalent, we have $1 * a = a$.

(iii) Assume $a \leqslant b$. Then we have $L(c, c * a) \subseteq L(a)$, $L(c, c * a) \subseteq L(b)$, $c * a \leqslant c * b$ and $L(b, b * c) \subseteq L(c)$, $L(a, b * c) \subseteq L(c)$, $b * c \leqslant a * c$.

(iv) Since $b \leqslant 1$, we have $a = 1 * a \leqslant b * a$ according to (ii) and (iii).

(v) We have $L(a, a * b) \subseteq L(b)$, $L(a * b, a) \subseteq L(b)$, $a \leqslant (a * b) * b$.

(vi) Since $a \leqslant (a * b) * b$ according to (v), we have $((a * b) * b) * b \leqslant a * b$ according to (iii). Conversely, substituting a by $a * b$ in (v) yields $a * b \leqslant ((a * b) * b) * b$.

(vii) The following assertions are equivalent: $c \in L(a, b)$, $c \leqslant a$ and $L(a, c) \subseteq L(b)$, $c \in L(a, a * b)$.

(viii) The following assertions are equivalent: $a \leqslant b * c$, $L(b, a) \subseteq L(c)$, $L(a, b) \subseteq L(c)$, $b \leqslant a * c$.

(ix) We have

$$L(a * b, (a * b) * a) = L(a * b, a) = L(a, a * b) \subseteq L(b)$$

according to (vii) and hence $(a * b) * a \leqslant (a * b) * b$.

(x) Applying (iii) to $a * b \leqslant 1$ and using (ii) yields $((a * b) * a) * b \leqslant (1 * a) * b = a * b$. On the other hand, $(a * b) * a \leqslant (a * b) * b$ which holds according to (ix) implies

$a * b \leqslant ((a * b) * a) * b$ according to (viii). Together, we obtain $((a * b) * a) * b = a * b$. $\square$

The operation of relative pseudocomplementation can be characterized as follows.

Theorem 2.7. *Let $(P, \leqslant)$ be a poset and $*$ a binary operation on P. Then the following conditions are equivalent:*

(i) $(P, \leqslant, *) \in \mathcal{P}$;
(ii) *for all $x, y, z \in P$, $x \leqslant y * z$ is equivalent to $L(y, x) \subseteq L(z)$;*
(iii) *for all $x, y, z \in P$, $L(y, x) \subseteq L(z)$ implies $x \leqslant y * z$ and, moreover, we have $L(x, y) = L(x, x * y)$.*

Proof. Let $a, b, c \in P$.

(i) $\Rightarrow$ (ii): If $a \leqslant b * c$, then $L(b, a) \subseteq L(b, b * c) \subseteq L(c)$. If, conversely, $L(b, a) \subseteq L(c)$, then $a \leqslant b * c$.

(ii) $\Rightarrow$ (i): We have $L(b, b * c) \subseteq L(c)$. Moreover, $L(b, a) \subseteq L(c)$ implies $a \leqslant b * c$. This shows that $b * c$ is the relative pseudocomplement of b with respect to c.

(i) $\Rightarrow$ (iii): This follows from Definition 2.1 and Lemma 2.6 (vii).

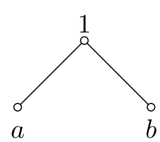
(iii) $\Rightarrow$ (i): We have $L(a, a * b) = L(a, b) \subseteq L(b)$ and hence $a * b \in M := \{x \in P : L(a, x) \subseteq L(b)\}$. If $c \in M$, then $c \leqslant a * b$, which yields that $a * b$ is the greatest element of M, i.e., it is the relative pseudocomplement of a with respect to b. $\square$

In some cases the existence of the supremum of any two elements can be expressed by means of relative pseudocomplementation. In the following we show that a relatively pseudocomplemented poset being already a join-semilattice follows from a simple identity in two variables.

Theorem 2.8. *If $(P, \leqslant, *) \in \mathcal{P}$ satisfies the identity $(x * y) * y \approx (y * x) * x$, then $(P, \leqslant)$ is a join-semilattice and $x \vee y = (x * y) * y$ for all $x, y \in P$.*

Proof. If $a, b, c \in P$ and $a, b \leqslant c$, then $a \leqslant (a * b) * b$ and $b \leqslant (b * a) * a = (a * b) * b$ according to (v) of Lemma 2.6 and $(a * b) * b \leqslant (c * b) * b = (b * c) * c = 1 * c = c$ according to (iii), (i) and (ii) of Lemma 2.6. $\square$

Example 2.9. Let $(P, \leqslant)$ denote the poset with the Hasse diagram



and $x * y$ the relative pseudocomplement of x with respect to y $(x, y \in P)$ and put $x \circ y := (x * y) * y$ for all $x, y \in P$. Then the operation tables for $*$ and $\circ$ look as follows:

$*$	a	b	1
a	1	b	1
b	a	1	1
1	a	b	1

$\circ$	a	b	1
a	a	1	1
b	1	b	1
1	1	1	1

Since $\circ$ is commutative, we have $\circ = \vee$.

The fact that the condition of Theorem 2.8 is not necessary for a pseudocomplemented poset to be a join-semilattice can be seen from the following example.

Example 2.10. Let $(C, \leqslant)$ be a nonempty chain without greatest element and a, b, 1 be elements not belonging to C. We extend the partial ordering on C to $P := C \cup \{a, b, 1\}$ as follows: For all $c \in C$ we have $c < a < 1$ and $c < b < 1$ and, moreover, a and b are incomparable. Let $d, e \in P$. If $d \leqslant e$ then $d * e = 1$ and if $d \nleqslant e$ then $d * e = e$. This shows $\mathbf{P} = (P, \leqslant) \in \mathcal{P}$. Moreover, $\mathbf{P}$ is a join-semilattice which is not a lattice since $a \wedge b$ does not exist. However, for $c \in C$ we have $(a * c) * c = c * c = 1 \neq a = 1 * a = (c * a) * a$. Hence the identity of Theorem 2.8 is violated.

3. Distributive posets

In this section we will show how the concept of distributivity in posets is connected with that of relative pseudocomplementation. For the convenience of the reader we recall the concept of a distributive poset which was defined in [11], see also [8].

Definition 3.1. A poset $(P, \leqslant)$ is called *distributive* if

$$U(L(x, y), L(x, z)) = U(L(x, U(y, z))) \tag{3.1}$$

for all $x, y, z \in P$.

Remark 3.2. For a lattice we obtain the usual notion of distributivity. This can be seen as follows: For a lattice $(P, \vee, \wedge)$ and $a, b \in P$ we have $L(a, b) = L(a \wedge b)$ and $U(a, b) = U(a \vee b)$. Hence we obtain:

$$U(L(x, y), L(x, z)) = U(L(x \wedge y), L(x \wedge z)) = U((x \wedge y) \vee (x \wedge z))$$

and

$$U(L(x, U(y, z))) = U(L(x, U(y \vee z))) = U(L(x \wedge (y \vee z))) = U(x \wedge (y \vee z)).$$

This shows that (3.1) is equivalent to the distributive identity

$$(x \wedge y) \vee (x \wedge z) \approx x \wedge (y \vee z).$$

Lemma 3.3. *A poset $(P, \leqslant)$ is distributive if and only if*

$$U(L(x,y), L(x,z)) \subseteq U(L(x, U(y,z))) \tag{3.2}$$

for all $x, y, z \in P$.

P r o o f. The converse inclusion holds in every poset $(P, \leqslant)$ since $a, b, c \in P$ and $d \in U(L(a, U(b,c)))$ together imply $L(a, U(b,c)) \subseteq L(d)$ and hence $L(a,b) \cup L(a,c) \subseteq L(d)$, i.e., $d \in U(L(a,b), L(a,c))$. □

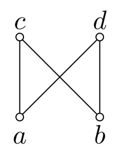
It is well-known that every relatively pseudocomplemented lattice is in fact distributive. Hence, the question arises if the same is true for relatively pseudocomplemented posets despite the fact that the distributive law (3.1) is formulated in a different way. The following theorem answers this question in the positive.

Theorem 3.4. *Every member of $\mathcal{P}$ is distributive.*

P r o o f. If $(P, \leqslant, *) \in \mathcal{P}$, $a, b, c \in P$ and $d \in U(L(a,b), L(a,c))$, then $L(a,b) \cup L(a,c) \subseteq L(d)$ and hence $a * d \in U(b,c)$, whence $d \in U(L(a, a*d)) \subseteq U(L(a, U(b,c)))$ proving (3.2). □

It is well-known that every finite distributive lattice is relatively pseudocomplemented. Unfortunately, this is not true for finite posets as can be seen from the following example:

E x a m p l e 3.5. Consider the poset **P** given by the following Hasse diagram:



It can be easily checked that **P** is distributive. However, it does not belong to $\mathcal{P}$ since it has no greatest element. For example, $a * c$ does not exist.

However, by adding one more natural condition to the finiteness of the distributive poset, we can show that it belongs to $\mathcal{P}$ as the following theorem shows. Moreover, this result also holds for infinite posets satisfying the ascending chain condition.

Theorem 3.6. *If $(P, \leqslant)$ is a distributive poset satisfying the ascending chain condition, then the following conditions are equivalent:*

(i) $(P, \leqslant) \in \mathcal{P}$.

(ii) *If $a, b \in P$ and c, d are maximal elements of $\{x \in P\colon L(a, x) \subseteq L(b)\}$, then $c \vee d$ exists.*

P r o o f. First assume (ii). Let $a, b \in P$ and put $M := \{x \in P\colon L(a, x) \subseteq L(b)\}$. Since $b \in M$ we have $M \neq \emptyset$. Because $(P, \leqslant)$ satisfies the ascending chain condition, every element of M lies under a maximal element of M. If M had at least two maximal elements, say c and d, then $c \vee d$ would exist and, using distributivity, we would conclude

$$\begin{aligned} L(a, c \vee d) &= L(U(L(a, c \vee d))) = L(U(L(a, U(c, d)))) = L(U(L(a, c), L(a, d))) \\ &\subseteq L(U(L(b))) = L(b), \end{aligned}$$

i.e., $c \vee d \in M$, contradicting the maximality of c. Hence M has only one maximal element which is then the greatest element and hence it is $a * b$. This shows (i). That (i) implies (ii) is obvious. $\square$

Let us finally note that the poset from Example 2.3 is distributive and satisfies condition (ii) of Theorem 3.6.

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