

GEOMETRIC PROPERTIES OF WRIGHT FUNCTION

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Abstract. In the present paper, we investigate certain geometric properties and inequalities for the Wright function and mention a few important consequences of our main results. A nonlinear differential equation involving the Wright function is also investigated.

Keywords: analytic function; univalent function; starlike function; strongly starlike function; convex function; close-to-convex function; Wright function; Bessel function; subordination of functions

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1. INTRODUCTION

The entire function (of z)

$$(1.1) \quad W_{\lambda, \mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \quad \lambda > -1, \mu \in \mathbb{C},$$

called the Wright function, has appeared for the first time in connection with the partitions of natural numbers, see [28]. Later on, it has been used in the asymptotic theory of partitions, Mikusinski operational calculus, integral transforms and in fractional differential equations (see [10], [13]). The Wright function can be represented in terms of familiar hypergeometric functions (see [10], page 389) and in terms of the Bessel functions J_ν (see [23], page 204).

Also, the Wright function generalizes various functions like array function, Whittaker function, (Wright-type) entire auxiliary functions, etc. The reader is referred to [10], [12] for details and many interesting results on the Wright function.

Let \mathcal{A} denote the class of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ having the form

$$(1.2) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

By \mathcal{S} , we denote the subclass of \mathcal{A} consisting of functions which are univalent in \mathbb{D} . For two analytic functions f and F in \mathbb{D} , we say that f is subordinated to F , and express this symbolically by $f(z) \prec F(z)$, if $f(z) = F(w(z))$ in \mathbb{D} , for some analytic function w in \mathbb{D} with $w(0) = 0$ and $|w(z)| < 1$. In particular, if $F \in \mathcal{S}$, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

A function $f \in \mathcal{A}$ is called starlike, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. The class of starlike functions in \mathcal{A} is denoted by \mathcal{S}^* . Analytically, a function $f \in \mathcal{A}$ is called starlike if and only if it satisfies $\Re\{zf'(z)/f(z)\} > 0$, $z \in \mathbb{D}$. A function $f \in \mathcal{A}$ which maps \mathbb{D} onto a convex domain is called a convex function and the class of such functions is denoted by \mathcal{K} . A function $f \in \mathcal{A}$ is called convex if and only if it satisfies $1 + \Re\{zf''(z)/f'(z)\} > 0$, $z \in \mathbb{D}$. Let $\tilde{\mathcal{S}}^*(\alpha)$, $0 < \alpha \leq 1$ be the class of strongly starlike functions of order α in \mathbb{D} , which is defined by

$$(1.3) \quad \tilde{\mathcal{S}}^*(\alpha) = \left\{ f \in \mathcal{A} : \left| \arg\left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D} \right\}.$$

Note that $\tilde{\mathcal{S}}^*(1) \equiv \mathcal{S}^*$. Further, a function $f \in \mathcal{A}$ is called close-to-convex in \mathbb{D} if the complement of $f(\mathbb{D})$ can be written as the union of non-intersecting half-lines. A function $f \in \mathcal{A}$ is close-to-convex with respect to a starlike function g , denoted by \mathcal{C}_g , if it satisfies $\Re\{zf'(z)/g(z)\} > 0$, $z \in \mathbb{D}$. For more details about these classes one can refer to [7], [9].

In this paper, we consider the following normalized form of the Wright function:

$$(1.4) \quad \mathbb{W}_{\lambda, \mu}(z) = z\Gamma(\mu)W_{\lambda, \mu}(z) := \sum_{n=0}^{\infty} \frac{\Gamma(\mu)z^{n+1}}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu > 0, \quad z \in \mathbb{D}.$$

The normalized Wright function $\mathbb{W}_{\lambda, \mu}$ was studied recently by the present author in [23] (see also [17]). Note that

$$(1.5) \quad \mathbb{W}_{1, \nu+1}(-z) = \mathbb{J}_{\nu}(z) = \Gamma(\nu + 1)z^{1-\nu/2}J_{\nu}(2\sqrt{z}).$$

Here, $\mathbb{J}_{\nu}(z)$ denotes the normalized Bessel function, investigated recently for the geometric properties in [2], [22], [25]. The function $J_{\nu}(z)$ is the well known Bessel function, defined by

$$(1.6) \quad J_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} W_{1, \nu+1}\left(\frac{-z^2}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+\nu}}{n!\Gamma(n + \nu + 1)}.$$

The special functions play an important role in function theory, especially the hypergeometric function, which appeared in De-Branges' solution of the famous Bieberbach conjecture (see [6]). Several researchers studied classes of analytic functions involving special functions $\mathcal{F} \subset \mathcal{A}$, to find different conditions such that the members of \mathcal{F} to have certain geometric properties such as univalence, starlikeness or convexity in \mathbb{D} . In this context many results are available in the literature regarding the hypergeometric functions (see [14], [24], [21], [20]), normalized Bessel functions (see [2], [4], [22], [25]), generalized Bessel functions (see [3], [16]), generalized Struve functions (see [30], [31]), Lommel functions (see [29]), Wright functions (see [23]) and Mittag-Leffler function (see [1]). In this paper, our main aim is to examine the geometric properties and inequalities of the Wright function $\mathbb{W}_{\lambda,\mu}$. We also investigate an initial value problem involving the Wright function.

2. CLOSE-TO-CONVEXITY AND STARLIKENESS OF $\mathbb{W}_{\lambda,\mu}$

In this section we obtain certain sufficient conditions for close-to-convexity and starlikeness of $\mathbb{W}_{\lambda,\mu}$ in \mathbb{D} . To prove our results, we shall need the following known results.

Lemma 2.1 (Fejér [8]). *Let $f \in \mathcal{A}$ be of the form (1.2) with $a_n \geq 0$. If the sequences $\{na_n\}$ and $\{na_n - (n+1)a_{n+1}\}$ are non-increasing, then f is starlike in \mathbb{D} .*

Lemma 2.2 (Ozaki [18]). *Let $f \in \mathcal{A}$ be of the form (1.2). If*

$$1 \geq 2a_2 \geq \dots \geq na_n \geq (n+1)a_{n+1} \dots \geq 0$$

or

$$1 \leq 2a_2 \leq \dots \leq na_n \leq (n+1)a_{n+1} \dots \leq 2,$$

then f is close-to-convex with respect to $g(z) = z/(1-z)$.

Lemma 2.3 (Halenbeck and Ruscheweyh [11]). *Let $G(z)$ be convex and univalent in \mathbb{D} and $F(z)$ be analytic in \mathbb{D} with $G(0) = F(0) = 1$. If $F(z) \prec G(z)$ in \mathbb{D} , then*

$$(n+1)z^{-n-1} \int_0^z t^n F(t) dt \prec (n+1)z^{-n-1} \int_0^z t^n G(t) dt, \quad n \in \mathbb{N} \cup \{0\}.$$

Our first result is given below by Theorem 2.1:

Theorem 2.1. *Let $\lambda \geq 1$ and $\mu \geq 1$. If $\Gamma(\lambda + \mu) \geq 2\Gamma(\mu)$, then $\mathbb{W}_{\lambda,\mu}$ is close-to-convex with respect to $g(z) = z/(1-z)$.*

Proof. By using Lemma 2.2, it is sufficient to show that

$$(2.1) \quad 1 \geq 2a_2 \geq \dots \geq na_n \geq (n+1)a_{n+1} \dots \geq 0.$$

From (1.4), we have

$$\begin{aligned} na_n - (n+1)a_{n+1} &= na_n - \frac{(n+1)\Gamma(\lambda(n-1) + \mu)}{n\Gamma(\lambda n + \mu)} a_n \\ &= \frac{a_n}{n\Gamma(\lambda n + \mu)} (n^2\Gamma(\lambda n + \mu) - (n+1)\Gamma(\lambda(n-1) + \mu)) \\ &= \frac{a_n}{n\Gamma(\lambda n + \mu)} X(n), \end{aligned}$$

where

$$X(n) = n^2\Gamma(\lambda n + \mu) - (n+1)\Gamma(\lambda(n-1) + \mu).$$

Under the hypothesis, it is clear that

$$\begin{aligned} n^2\Gamma(\lambda n + \mu) &= n^2\Gamma(\lambda(n-1) + \lambda + \mu) \geq n^2\Gamma(\lambda(n-1) + 1 + \mu) \\ &= n^2(\lambda(n-1) + \mu)\Gamma(\lambda(n-1) + \mu) \\ &\geq (n+1)\Gamma(\lambda(n-1) + \mu), \quad n \in \mathbb{N} \setminus \{1\}. \end{aligned}$$

Also, $X(1) \geq 0$ and $\Gamma(\lambda n + \mu) \geq \Gamma(\lambda(n-1) + \mu)$, $n \geq 2$. Hence $X(n) \geq 0$ for all $n \geq 1$. This shows that the inequality (2.1) holds. This completes the proof. \square

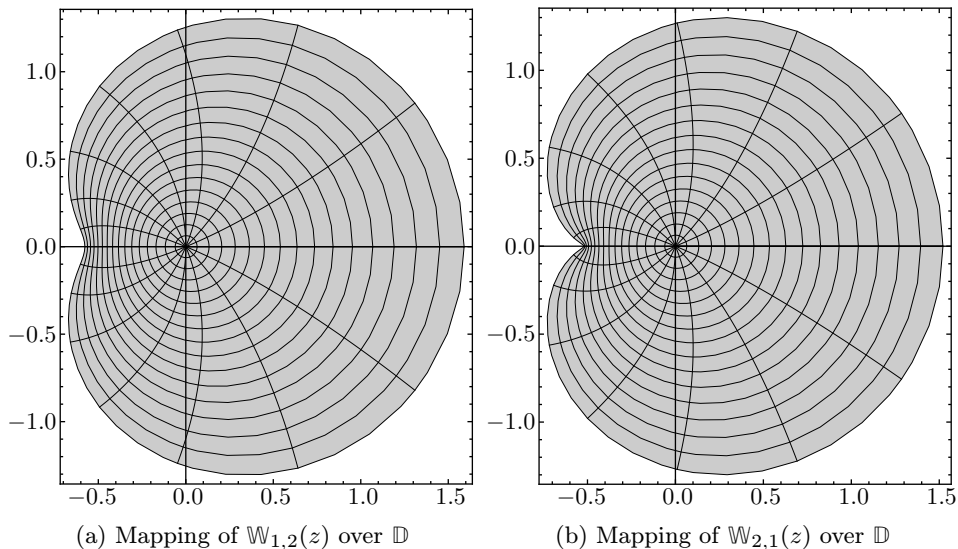
Taking $\lambda = 1$, $\mu = \nu + 1$ ($\nu > -1$) and replacing z by $-z$ in Theorem 2.1, we get the following result:

Corollary 2.1. *If $\nu \geq 1$, then \mathbb{J}_ν is close-to-convex in \mathbb{D} with respect to $g(z) = z/(1-z)$.*

Example 2.1. Taking $\lambda = 1$ in Theorem 2.1, we obtain that the function $\mathbb{W}_{1,\mu}$ is close-to-convex for $\mu \geq 2$. Also, we obtain that the function $\mathbb{W}_{2,\mu}$ is close-to-convex for $\mu \geq 1$. In particular, functions $\mathbb{W}_{1,2}$ and $\mathbb{W}_{2,1}$ are close to convex and their image domains under \mathbb{D} are given below in Figures (a) and (b).

Theorem 2.2. *Let $\lambda \geq 1$ and $\mu \geq 1$. If $\Gamma(\lambda + \mu) \geq 4\Gamma(\mu)$, then $\mathbb{W}_{\lambda,\mu}$ is starlike in \mathbb{D} .*

Proof. In view of Lemma 2.1, it is sufficient to prove that $\{na_n\}$ and $\{na_n -$



$(n+1)a_{n+1}$ are non-increasing sequences for all $n \geq 1$. Clearly, the sequence $\{na_n\}$ is non-increasing by Theorem 2.1. Therefore, it suffices to show that

$$(2.2) \quad na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} \geq 0 \quad \forall n \geq 1.$$

Under the hypothesis, we have

$$\begin{aligned} n^2\Gamma(\lambda n + \mu) &\geq n^2\Gamma(\lambda(n-1) + \mu + 1) = n^2(\lambda(n-1) + \mu)\Gamma(\lambda(n-1) + \mu) \\ &\geq 2(n+1)\Gamma(\lambda(n-1) + \mu), \quad n \in \mathbb{N}. \end{aligned}$$

Hence

$$\begin{aligned} &na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} \\ &= \frac{n\Gamma(\mu)}{\Gamma(\lambda(n-1) + \mu)(n-1)!} - \frac{2(n+1)\Gamma(\mu)}{\Gamma(\lambda n + \mu)n!} + \frac{(n+2)\Gamma(\mu)}{\Gamma(\lambda(n+1) + \mu)(n+1)!} \\ &= \frac{\Gamma(\mu)}{(n-1)!} \left(\frac{n}{\Gamma(\lambda(n-1) + \mu)} - \frac{2(n+1)}{\Gamma(\lambda n + \mu)n} + \frac{(n+2)}{\Gamma(\lambda(n+1) + \mu)(n+1)n} \right) \geq 0. \end{aligned}$$

This shows that the inequality (2.2) holds, hence $\mathbb{W}_{\lambda,\mu}(z)$ is starlike in \mathbb{D} . □

Taking $\lambda = 1$, $\mu = \nu + 1$, $\nu > -1$ and replacing z by $-z$ in Theorem 2.2, we get the following result:

Corollary 2.2. *If $\nu \geq 3$, then \mathbb{J}_ν is starlike in \mathbb{D} .*

Example 2.2. Taking $\lambda = 1$ in Theorem 2.2, we obtain that the function $\mathbb{W}_{1,\mu}$ is starlike for $\mu \geq 4$. Also, we obtain that the function $\mathbb{W}_{2,\mu}$ is starlike for $\mu \geq \frac{1}{2}(-1 + \sqrt{17})$. Further, we observe that, as λ increases, μ decreases to preserve the starlikeness of the function $\mathbb{W}_{\lambda,\mu}$.

Theorem 2.3. *If $\lambda \geq 1$ and $\mu \geq 1 + \sqrt{3}$, then $\mathbb{W}_{\lambda,\mu} \in \tilde{\mathcal{S}}^*(\alpha)$. Here α is given by*

$$(2.3) \quad \alpha = \frac{2}{\pi} \arcsin\left(\eta\sqrt{1 - \frac{1}{4}\eta^2} + \frac{1}{2}\eta\sqrt{1 - \eta^2}\right),$$

where $\eta = 2(\mu + 1)/\mu^2$.

Proof. Under the hypothesis, the inequality $\Gamma(\mu + n) \leq \Gamma(\lambda n + \mu)$, $n \in \mathbb{N}$ holds and is equivalent to

$$(2.4) \quad \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \leq \frac{\Gamma(\mu)}{\Gamma(n + \mu)} = \frac{1}{(\mu)_n}, \quad n \in \mathbb{N},$$

where $(x)_n$ is the well known Pochhammer symbol defined by

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)\dots(x+n-1), & n \in \mathbb{N}. \end{cases}$$

For $n \in \mathbb{N}$, we have

$$(2.5) \quad (x)_n = x(x+1)_{n-1}, \quad x^n \leq (x)_n.$$

Using (2.5) in (2.4), we have

$$(2.6) \quad \begin{aligned} |\mathbb{W}'_{\lambda,\mu}(z) - 1| &\leq \sum_{n=1}^{\infty} \frac{(n+1)\Gamma(\mu)}{n!\Gamma(\lambda n + \mu)} |z|^n < \frac{1}{\mu} \sum_{n=1}^{\infty} \frac{n+1}{n!} \frac{1}{(\mu)_{n-1}} \\ &= \frac{1}{\mu} \left(\sum_{n=0}^{\infty} \frac{n}{n!} \frac{1}{(\mu)_n} + \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(\mu)_n} \right) \leq \frac{1}{\mu} \left(\sum_{n=0}^{\infty} \frac{1}{(\mu)_n} + \sum_{n=0}^{\infty} \frac{1}{(\mu)_n} \right) \\ &< \frac{2}{\mu} \sum_{n=0}^{\infty} \frac{1}{(\mu+1)^n} = \frac{2(\mu+1)}{\mu^2} = \eta. \end{aligned}$$

Note that under the hypothesis $0 < \eta \leq 1$. From (2.6), we conclude that $\mathbb{W}'_{\lambda,\mu}(z) \prec 1 + \eta z$, $z \in \mathbb{D}$, which implies that

$$(2.7) \quad |\arg(\mathbb{W}'_{\lambda,\mu}(z))| < \arcsin \eta, \quad z \in \mathbb{D}.$$

Using Lemma 2.3, for $F(z) = \mathbb{W}'_{\lambda,\mu}(z)$, $G(z) = 1 + \eta z$ and $n = 0$, we obtain $\mathbb{W}_{\lambda,\mu}(z)/z \prec 1 + \frac{1}{2}\eta z$, $z \in \mathbb{D}$, and consequently

$$(2.8) \quad \left| \arg \left(\frac{\mathbb{W}_{\lambda,\mu}(z)}{z} \right) \right| < \arcsin \frac{\eta}{2}, \quad z \in \mathbb{D}.$$

Now from (2.7) and (2.8), we conclude that

$$\begin{aligned} \left| \arg \left(\frac{z\mathbb{W}'_{\lambda,\mu}(z)}{\mathbb{W}_{\lambda,\mu}(z)} \right) \right| &= \left| \arg \left(\frac{z}{\mathbb{W}_{\lambda,\mu}(z)} \right) + \arg(\mathbb{W}'_{\lambda,\mu}(z)) \right| \\ &\leq \left| \arg \left(\frac{z}{\mathbb{W}_{\lambda,\mu}(z)} \right) \right| + |\arg(\mathbb{W}'_{\lambda,\mu}(z))| \\ &< \arcsin \frac{\eta}{2} + \arcsin \eta \\ &= \arcsin \left(\eta \sqrt{1 - \frac{1}{4}\eta^2} + \frac{1}{2}\eta \sqrt{1 - \eta^2} \right), \end{aligned}$$

i.e., $\mathbb{W}_{\lambda,\mu}(z) \in \tilde{\mathcal{S}}^*(\alpha)$ for α given by (2.3). □

Corollary 2.3. *Let $\lambda \geq 1$ and $\mu \geq 1 + \sqrt{3}$. If $0 < \alpha \leq 1$ and*

$$(2.9) \quad \eta = \frac{2(\mu + 1)}{\mu^2} = 2\nu \sqrt{\frac{5 - 4\sqrt{1 - \nu^2}}{16\nu^2 + 9}},$$

where $\nu = \sin(\frac{1}{2}\alpha\pi)$, then $\mathbb{W}_{\lambda,\mu} \in \tilde{\mathcal{S}}^*(\alpha)$.

Proof. If we substitute the value of η from (2.9) to (2.3), we obtain the required α . Hence the result. □

Taking $\alpha = 1$ in Corollary 2.3, we get

$$\nu = 1 \Rightarrow \eta = \frac{2(\mu + 1)}{\mu^2} = \frac{2}{\sqrt{5}} \Rightarrow \frac{\mu + 1}{\mu^2} = \frac{1}{\sqrt{5}}.$$

Hence, we get the following result:

Corollary 2.4. *Let $\lambda \geq 1$ and $\mu = \mu^*$, where μ^* is the positive root of $\mu^2 - \sqrt{5}\mu - \sqrt{5} = 0$. Then $\mathbb{W}_{\lambda,\mu}$ is starlike in \mathbb{D} .*

3. A NONLINEAR DIFFERENTIAL EQUATION

In this section, we aim to study a nonlinear differential equation involving the Wright function. For this, we shall need the following lemmas:

Lemma 3.1 (Miller and Mocanu [15]). *Let $\Omega \subset \mathbb{C}$. Suppose that the function $\psi(z): \mathbb{C}^2 \times \mathbb{D} \rightarrow \mathbb{C}$ satisfies the condition $\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for all $K \geq M(M - |a|)/(M + |a|)$, $\theta \in \mathbb{R}$ and $z \in \mathbb{D}$. Let $p(z)$ be an analytic function of the form*

$$(3.1) \quad p(z) = a + a_1z + a_2z^2 + \dots, \quad z \in \mathbb{D},$$

such that $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{D}$. Then $|p(z)| < M$, where $0 \leq |a| < M$.

Lemma 3.2 (Tuneski [26]). *If $f \in \mathcal{A}$ and $|f''(z)| \leq 1$, $z \in \mathbb{D}$, then f is starlike in \mathbb{D} .*

Theorem 3.1. *For all $\lambda > -1$ and $\mu > 0$, let $\mathbb{W}_{\lambda, \mu}(z)$ satisfy the inequality*

$$(3.2) \quad |z\mathbb{W}_{\lambda, \mu}(z)| < \frac{M(M - |a|)}{(M + 1)(M + |a|)}, \quad 0 \leq |a| < M \leq 1; \quad z \in \mathbb{D}.$$

Let φ be the (unique) solution of the initial value problem

$$\begin{aligned} \varphi^{(n+1)}(z) + \mathbb{W}_{\lambda, \mu}(z)\varphi^{(n)}(z) &= \mathbb{W}_{\lambda, \mu}(z), \quad z \in \mathbb{D} \\ (n \in \mathbb{N} \cup \{0\}, \varphi(0) &= 0, \varphi'(0) = 1, \varphi^{(k)}(0) = 0, \quad k = 2, \dots, n-1, \varphi^{(n)}(0) = a), \end{aligned}$$

where $\varphi^{(n)}$ denotes the n th derivative with respect to z . Then the inequality $|\varphi^{(n)}(z)| < M$, $z \in \mathbb{D}$ holds.

Proof. Let the function $p(z)$ be defined by $p(z) = \varphi^{(n)}(z)$, $z \in \mathbb{D}$. Note that $p(z)$ has the form (3.1), and then it follows that

$$\frac{zp'(z)}{1 + p(z)} = \frac{z\varphi^{(n+1)}(z)}{1 + \varphi^{(n)}(z)} = z\mathbb{W}_{\lambda, \mu}(z), \quad z \in \mathbb{D}; \quad \varphi^{(n)}(z) \neq -1.$$

We denote $\psi(r, s; z)$ and Ω by

$$\psi(r, s; z) := \frac{s}{1 + r}, \quad r \neq -1$$

and

$$\Omega := \left\{ w \in \mathbb{C}: |w| < \frac{M(M - |a|)}{(M + 1)(M + |a|)}, \quad 0 \leq |a| < M \leq 1 \right\},$$

respectively. Then clearly

$$\psi(p(z), zp'(z); z) = \frac{zp'(z)}{1+p(z)} = \frac{z\varphi^{(n+1)}(z)}{1+\varphi^{(n)}(z)} \in \Omega, \quad z \in \mathbb{D}.$$

Further, for any $\theta \in \mathbb{R}$, $K \geq M(M - |a|)/(M + |a|)$ and $z \in \mathbb{D}$, we have

$$|\psi(Me^{i\theta}, Ke^{i\theta}; z)| = \left| \frac{Ke^{i\theta}}{1 + Me^{i\theta}} \right| \geq \frac{M(M - |a|)}{(M + 1)(M + |a|)},$$

which gives that

$$\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega.$$

Therefore, in view of Lemma 3.1 it follows that

$$|p(z)| = |\varphi^{(n)}(z)| < M, \quad z \in \mathbb{D}; \quad 0 \leq |a| < M \leq 1.$$

This completes the proof. □

By taking $n = 2$ in the above theorem we get the following result:

Corollary 3.1. *For all $\lambda > -1$ and $\mu > 0$, let $\mathbb{W}_{\lambda, \mu}(z)$ satisfy the inequality (3.2) in \mathbb{D} . Let φ be the (unique) solution of the initial value problem given by*

$$(3.3) \quad \begin{aligned} \varphi'''(z) + \mathbb{W}_{\lambda, \mu}(z)\varphi''(z) &= \mathbb{W}_{\lambda, \mu}(z), \quad z \in \mathbb{D} \\ (\varphi(0) = 0, \varphi'(0) = 1, \varphi''(0) = a). \end{aligned}$$

Then the inequality $|\varphi''(z)| < M$, $0 \leq |a| < M \leq 1$ holds.

Corollary 3.2. *If $\mathbb{W}_{\lambda, \mu}(z)$ satisfies the inequality*

$$|z\mathbb{W}_{\lambda, \mu}(z)| < \frac{1 - |a|}{2(1 + |a|)}, \quad 0 \leq |a| < 1$$

and the function $\varphi(z)$ is the (unique) solution of the initial value problem given by (3.3), then φ is starlike in \mathbb{D} .

Proof. The proof can be obtained easily by taking $M = 1$ in Corollary 3.1 and then using Lemma 3.2. □

4. INEQUALITIES

The following result by Fejér will be needed in this section.

Lemma 4.1 (Fejér [8]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_1 = 1$. If the sequence $\{a_n\}$ is convex decreasing, i.e., $0 \geq a_{n+2} - a_{n+1} \geq a_{n+1} - a_n$ for all $n \in \mathbb{N} \setminus \{1\}$, then*

$$\Re\left(\sum_{n=1}^{\infty} a_n z^{n-1}\right) > \frac{1}{2}, \quad z \in \mathbb{D}.$$

The convex hull of \mathcal{K} , denoted by $\overline{\text{co}}\mathcal{K}$, is the set of all convex combinations of functions belonging to \mathcal{K} . We recall from [5] that the closure of the set $\overline{\text{co}}\mathcal{K}$ is

$$(4.1) \quad \overline{\text{co}}\mathcal{K} = \left\{ f \in \mathcal{A}: \Re\left(\frac{f(z)}{z}\right) > \frac{1}{2}, \quad z \in \mathbb{D} \right\}.$$

It is well known (see [27]) that a sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence for the class $\mathcal{X} \subset \mathcal{A}$, whenever we have

$$(4.2) \quad \sum_{n=1}^{\infty} b_n a_n z^n \prec \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathbb{D}$$

for all $\sum_{n=1}^{\infty} a_n z^n \in \mathcal{X}$.

Lemma 4.2 (Piejko and Sokół [19]). *The function of the form (1.2) is in the set $\overline{\text{co}}\mathcal{K}$ if and only if a_2, a_3, \dots is a subordinating factor sequence for the class \mathcal{K} .*

Theorem 4.1. *For each $\lambda \geq 1$ and $\mu \geq 1$, we have*

$$|\mathbb{W}_{\lambda, \mu}(z)| \leq r {}_0F_1(-; \mu; r),$$

where ${}_0F_1(-; \mu; r)$ is the well known hypergeometric function and $|z| = r < 1$.

Proof. Using (2.4), we get

$$(4.3) \quad \begin{aligned} |\mathbb{W}_{\lambda, \mu}(z)| &\leq |z| + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)|z|^n}{\Gamma((n-1)\lambda + \mu)(n-1)!} \\ &\leq |z| + \sum_{n=1}^{\infty} \frac{|z|^{n+1}}{(\mu)_n n!} = r {}_0F_1(-; \mu; r). \end{aligned}$$

This proves the result. □

By using (1.4) and Theorem 4.1, we get the following result:

Corollary 4.1. *For each $\lambda \geq 1$ and $\mu \geq 1$, we have $|W_{\lambda,\mu}(z)| \leq {}_0F_1(-; \mu; r)/\Gamma(\mu)$, $|z| = r < 1$.*

Theorem 4.2. *Let $\lambda \geq 1$ and $\mu \geq 1$. If $\Gamma(\lambda + \mu) \geq 2\Gamma(\mu)$, then*

$$(4.4) \quad \Re\left(\frac{\mathbb{W}_{\lambda,\mu}(z)}{z}\right) > \frac{1}{2}, \quad z \in \mathbb{D}.$$

Proof. Under the hypothesis, the inequality

$$n!\Gamma(\lambda n + \mu) \geq (n-1)!\Gamma(\lambda(n-1) + \mu)$$

holds, which is equivalent to

$$(4.5) \quad \frac{1}{\Gamma(\lambda(n-1) + \mu)(n-1)!} \geq \frac{1}{\Gamma(\lambda n + \mu)n!}.$$

Now we need to show that

$$\{a_n\}_{n=1}^{\infty} = \left\{ \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1) + \mu)(n-1)!} \right\}_{n=1}^{\infty}$$

is a convex decreasing sequence. We observe that

$$\begin{aligned} a_{n+2} - 2a_{n+1} + a_n \\ = \frac{\Gamma(\mu)}{\Gamma(\lambda(n+1) + \mu)(n+1)!} - \frac{2\Gamma(\mu)}{\Gamma(\lambda n + \mu)n!} + \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1) + \mu)(n-1)!} \geq 0, \end{aligned}$$

which shows that $\{a_n\}_{n=1}^{\infty}$ is a convex decreasing sequence. Now applying Lemma 4.1, we get

$$\Re\left\{ \sum_{n=1}^{\infty} a_n z^{n-1} \right\} > \frac{1}{2}, \quad z \in \mathbb{D}$$

which is equivalent to (4.4). This proves the result. \square

Proceeding similarly as in Theorem 4.2, we get the following result:

Theorem 4.3. *Let $\lambda \geq 1$ and $\mu \geq 1$. If $\Gamma(\lambda + \mu) \geq 4\Gamma(\mu)$, then*

$$(4.6) \quad \Re\{\mathbb{W}'_{\lambda,\mu}(z)\} > \frac{1}{2}, \quad z \in \mathbb{D}.$$

Corollary 4.2. Let $\lambda \geq 1$ and $\mu \geq 1$. If $\Gamma(\lambda + \mu) \geq 2\Gamma(\mu)$, then the sequence

$$(4.7) \quad \left\{ \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)n!} \right\}_{n=1}^{\infty}$$

is a subordinating sequence for the class \mathcal{K} .

Proof. By using (4.1) and (4.4), we have $\mathbb{W}_{\lambda,\mu}(z) \in \overline{\text{co}}\mathcal{K}$. □

Now applying Lemma 4.2, we get the desired result.

Corollary 4.3. Let $\lambda \geq 1$ and $\mu \geq 1$. If $\Gamma(\lambda + \mu) \geq 4\Gamma(\mu)$, then the sequence

$$(4.8) \quad \left\{ \frac{(n+1)\Gamma(\mu)}{\Gamma(\lambda n + \mu)n!} \right\}_{n=1}^{\infty}$$

is a subordinating sequence for the class \mathcal{K} .

Proof. By using (4.1) and (4.6), we have $z\mathbb{W}'_{\lambda,\mu}(z) \in \overline{\text{co}}\mathcal{K}$. Now applying Lemma 4.2, we get the desired result. □

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