ON SOME LOCAL SPECTRAL THEORY AND BOUNDED LOCAL RESOLVENT OF OPERATOR MATRICES

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Abstract. We extend and generalize some results in local spectral theory for upper triangular operator matrices to upper triangular operator matrices with unbounded entries. Furthermore, we investigate the boundedness of the local resolvent function for operator matrices.

Keywords: local resolvent function; single-valued extension property; operator matrix

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1. Introduction

Many problems in mathematical physics are described by the system of partial or ordinary differential equations or linearizations thereof. In applications, the time evolution of a physical system is governed by block operator matrices. On the other hand, the spectral properties of the block operator matrices are of vital importance as they govern for instance the time evolution and hence the stability of the underlying physical systems. Especially, the study of upper triangular operator matrices and related topics is one of the hottest areas in operator theory. In the recent past, a number of mathematicians have studied 2×2 bounded upper triangular operator matrices, see [3], [4], [7], [11], [12]. In [8], [12], the authors studied some local spectral properties for a bounded upper triangular operator matrix acting on a Banach space. The aim of this paper is to extend these results to the unbounded case. On the other hand, we give sufficient conditions on its diagonal entries which ensure the boundedness of the local resolvent function of bounded upper triangular operator matrices.

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2. Preliminaries

Throughout, X denotes a complex Banach space. Let A be a closed linear operator on X with domain $D(A)$, we denote by A^*, $R(A)$, $N(A)$, $R^\infty(A) = \bigcap_{n\geqslant 0} R(A^n)$, $\sigma_{\mathrm{su}}(A)$, $\sigma(A)$, respectively, the adjoint, the range, the null space, the hyper-range, the surjectivity spectrum and the spectrum of A. Recall after [1]: for a closed linear operator A and $x \in X$ the local resolvent of A at x, $\varrho_A(x)$ defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f\colon U \to D(A)$ such that the equation $(A-\mu I)f(\mu) = x$ holds for all $\mu \in U$. The local spectrum $\sigma_A(x)$ of A at x is defined as $\sigma_A(x) = \mathbb{C} \setminus \varrho_A(x)$. Evidently $\sigma_A(x) \subseteq \sigma_{\mathrm{su}}(A) \subseteq \sigma(A)$, $\varrho_A(x)$ is open and $\sigma_A(x)$ is closed.

Next, let A be a closed linear operator, A is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open neighborhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function $f\colon U \to D(A)$ which satisfies the equation $(A - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. A is said to have the SVEP if A has the SVEP for every $\lambda \in \mathbb{C}$. Denote

$$S(A) = \{\lambda \in \mathbb{C}\colon\ A \text{ does not have the SVEP at } \lambda\}.$$

$S(A)$ is an open set of $\mathbb{C}$.

Now, we define the notion of localizable spectrum [9] for a closed operator.

Definition 2.1. Let $(A, D(A))$ be a closed operator with SVEP. The localizable spectrum $\sigma_{\mathrm{loc}}(A)$ of A is the set of all complex numbers λ_0 with the following property: for each open neighborhood U of λ_0 there exists a nonzero vector $x \in X$ such that $\sigma_A(x) \subseteq U$.

Clearly, $\sigma_{\mathrm{loc}}(A) \subseteq \sigma_{\mathrm{ap}}(A) \subseteq \sigma(A)$. For a large class of operators including normal operators, the localizable spectrum coincides with $\sigma(A)$, for more information see [9].

Let X and Y be Banach spaces and $\mathcal{B}(X,Y)$ denote the space of all bounded linear operators from X to Y, let $(A, D(A))$, $(B, D(B))$ and $(C, D(C))$ be closed linear operator, such that:

$$\begin{aligned} A&\colon\ D(A) \subseteq X \to X,\\ B&\colon\ D(B) \subseteq Y \to Y,\\ C&\colon\ D(C) \subseteq Y \to X. \end{aligned}$$

We denote by M_C the operator defined on $D(A) \oplus (D(C) \cap D(B))$, $M_C\colon\ D(A) \oplus (D(C) \cap D(B)) \to X \oplus Y$ by

$$\begin{pmatrix} A & C\\ 0 & B \end{pmatrix}.$$

Lemma 2.1. *Let $(A, D(A))$ and $(B, D(B))$ be closed linear operators. For every $C \in \mathcal{B}(Y, X)$, M_C is closed with domain $D(A) \oplus D(B)$.*

P r o o f. It is easy to see that $D(M_C) = D(A) \oplus D(B)$. Now, let $x_n \oplus y_n \in D(A) \oplus D(B)$ be such that $x_n \oplus y_n \to x \oplus y$ and $M_C(x_n \oplus y_n) \to z_1 \oplus z_2$. Thus, $Ax_n + Cy_n \to z_1$ and $By_n \to z_2$, since B is closed, $By = z_2$. Since $C \in \mathcal{B}(Y, X)$, $Cy_n \to Cy$, hence $Ax_n \to z_1 - Cy$. A is closed, thus $Ax = z_1 - Cy$. Therefore $M_C(x \oplus y) = z_1 \oplus z_2$, so M_C is closed with domain $D(A) \oplus D(B)$. □

3. SVEP and local spectrum for operator matrices

We start this section by the following proposition which extends the results of [12], Proposition 3.1.

Proposition 3.1. *Let $(A, D(A))$ and $(B, D(B))$ be closed linear operators. For every $C \in \mathcal{B}(Y, X)$ we have*

$$S(A) \subseteq S(M_C) \subseteq S(A) \cup S(B).$$

P r o o f. Assume that $\lambda \notin S(A) \cup S(B)$, let V_λ be a neighborhood of λ and take an analytic function $h\colon V_\lambda \to D(A) \oplus D(B)$ satisfying $(M_C - \mu)h(\mu) = 0$ for all $\mu \in V_\lambda$. Hence, $(A - \mu)h_1(\mu) + Ch_2(\mu) = 0$ and $(B - \mu)h_2(\mu) = 0$ for all $\mu \in V_\lambda$, where $h = h_1 \oplus h_2$. Since $\lambda \notin S(B)$, $h_2 \equiv 0$ on V_λ. Thus $(A - \mu)h_1(\mu) = 0$ on V_λ, since $\lambda \notin S(A)$, $h_1 \equiv 0$ on V_λ. Therefore $h \equiv 0$, so $\lambda \notin S(M_C)$. Now, suppose that $\lambda \notin S(M_C)$, let V_λ be a neighborhood of λ and $f\colon V_\lambda \to D(A)$ an analytic function satisfying $(A - \mu)f(\mu) = 0$ for all $\mu \in V_\lambda$. Let $h = f \oplus 0\colon V_\lambda \to D(A) \oplus D(B)$, h is an analytic function and we have $(M_C - \mu)h(\mu) = 0$, since $\lambda \notin S(M_C)$, $h \equiv 0$ on V_λ. Hence $f \equiv 0$, so $\lambda \notin S(A)$. □

R e m a r k 3.1. If $C = 0$, then $S(M_0) = S(A) \cup S(B)$.

Theorem 3.1. *Let $(A, D(A))$ and $(B, D(B))$ be closed linear operators. For every $C \in \mathcal{B}(Y, X)$ we have*

$$S(B) \cup \sigma_A(x) = S(B) \cup \sigma_{M_C}(x \oplus 0) \quad \forall\, x \in X.$$

P r o o f. Let $\lambda \notin S(B) \cup \sigma_{M_C}(x \oplus 0)$, thus there exists an open neighborhood V_λ of λ and an analytic function $h\colon V_\lambda \to D(A) \oplus D(B)$ such that $(M_C - \mu)h(\mu) = x \oplus 0$ for all $\mu \in V_\lambda$. Let $h_1\colon V_\lambda \to D(A)$ and $h_2\colon V_\lambda \to D(B)$ be analytic functions with

$h = h_1 \oplus h_2$, then $(A-\mu)h_1(\mu)+Ch_2(\mu) = x$ and $(B-\mu)h_2(\mu) = 0$. Since $\lambda \notin S(B)$, $h_2 \equiv 0$ on V_λ, thus $(A-\mu)h_1(\mu) = x$ for all $\mu \in V_\lambda$. So $\lambda \notin \sigma_A(x)$ for all $x \in X$. Conversely, if $\lambda \notin \sigma_A(x)$, then there exists an open neighborhood V_λ of λ and an analytic function $f\colon V_\lambda \to D(A)$ such that $(A-\mu)f(\mu) = x$ for all $\mu \in V_\lambda$. Let $h = f \oplus 0\colon V_\lambda \to D(A) \oplus D(B)$, h is an analytic function and $(M_C - \mu)h(\mu) = x \oplus 0$ for all $\mu \in V_\lambda$. So $\lambda \notin \sigma_{M_C}(x \oplus 0)$. $\square$

R e m a r k 3.2. Theorem 3.1 generalizes [8], Proposition 2.3. Note that

$$\sigma_{M_C}(x \oplus 0) \subseteq \sigma_A(x) \quad \text{and} \quad \sigma_B(y) \subseteq \sigma_{M_C}(x \oplus y), \quad x \in X,\ y \in Y$$

hold in the general case when A, B are closed and $C \in \mathcal{B}(Y, X)$.

As consequences of Theorem 3.1, we have the following corollaries.

Corollary 3.1. *Let $(A, D(A))$ and $(B, D(B))$ be closed linear operators, assume that B has the SVEP. Then for every $C \in \mathcal{B}(Y, X)$,*

$$\sigma_A(x) = \sigma_{M_C}(x \oplus 0) \quad \forall\, x \in X.$$

Corollary 3.2. *Let $(A, D(A))$ and $(B, D(B))$ be closed linear operators, assume that B has the SVEP. Then for every $C \in \mathcal{B}(Y, X)$,*

$$\sigma_{\rm su}(A) \cup \sigma_{\rm su}(B) \subseteq \sigma_{\rm su}(M_C).$$

P r o o f. We have $\sigma_{\rm su}(A) = \bigcup\limits_{x\in X} \sigma_A(x)$ and $\bigcup\limits_{x\in X} \sigma_{M_C}(x \oplus 0) \subseteq \sigma_{\rm su}(M_C)$, from Corollary 3.1, $\sigma_{\rm su}(A) \subseteq \sigma_{\rm su}(M_C)$. Since $\sigma_B(y) \subseteq \sigma_{M_C}(x \oplus y)$ for all $x \in X$ and $y \in Y$, $\sigma_{\rm su}(B) \subseteq \sigma_{\rm su}(M_C)$. Thus $\sigma_{\rm su}(A) \cup \sigma_{\rm su}(B) \subseteq \sigma_{\rm su}(M_C)$. $\square$

Lemma 3.1. *Let $(A, D(A))$ and $(B, D(B))$ be closed linear operators. For every $C \in \mathcal{B}(Y, X)$ we have*

$$\sigma_p(A) \subseteq \sigma_p(M_C) \subseteq \sigma_p(A) \cup \sigma_p(B).$$

P r o o f. Suppose that $\lambda \notin \sigma_p(M_C)$, if $(A-\lambda)x = 0$, then $(M_C-\lambda)(x\oplus 0) = 0\oplus 0$ implies that $x = 0$, hence $\lambda \notin \sigma_p(A)$. Now suppose that $\lambda \notin \sigma_p(A) \cup \sigma_p(B)$, if $(M_C - \lambda)(x \oplus y) = 0 \oplus 0$, then $(A-\lambda)x + Cy = 0$ and $(B-\lambda)y = 0$, since $\lambda \notin \sigma_p(B)$, $y = 0$, hence $(A-\lambda)x = 0$. Since $\lambda \notin \sigma_p(A)$, $x = 0$, so $\lambda \notin \sigma_p(M_C)$. $\square$

Theorem 3.2. *Let $(A, D(A))$ and $(B, D(B))$ be closed linear operators, assume that B has the SVEP. Then for every $C \in \mathcal{B}(Y, X)$ we have*

$$\sigma(A) \cup \sigma(B) = \sigma(M_C).$$

P r o o f. If A and B are invertible, then M_C is invertible with the inverse

$$\begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}.$$

Hence $\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B)$. Conversely, since B has the SVEP, $\sigma_{\mathrm{su}}(B) = \sigma(B)$, from Lemma 3.1 and Corollary 3.2 we have $\sigma_p(A) \cup \sigma_{\mathrm{su}}(A) \cup \sigma(B) \subseteq \sigma_p(M_C) \cup \sigma_{\mathrm{su}}(M_C) = \sigma(M_C)$, thus $\sigma(A) \cup \sigma(B) \subseteq \sigma(M_C)$. $\square$

4. Bounded local resolvent

It is well known that the resolvent mapping $R_T(z) = (T - z)^{-1}$, which is defined and analytic on the resolvent set $\varrho(T)$, is unbounded. On the other hand, as observed in [10], the behavior of local resolvent functions may be quite different. In [5], Bermudez and Gonzalez have shown that a normal operator N on a separable Hilbert space has a nontrivial bounded local resolvent function if and only if the interior of the spectrum of N is not empty. Neumman extended this result in [13] to non-separable Banach spaces. In [6], Bračič and Müller proved that for every operator T on a Banach space X, such that both its point spectrum and its localizable spectrum have nonempty interior, there is a vector x such that the local resolvent function at x, $R_T(\cdot, x)$ is bounded on $\varrho_T(x)$.

Let X, Y be Banach spaces and let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. We start this section by the following definitions.

Definition 4.1. An operator $T \in \mathcal{B}(X)$ is said to have the decomposition property δ if, given an arbitrary open cover $\{U_1, U_2\}$ of $\mathbb{C}$, every $x \in X$ admits a decomposition $x = x_1 + x_2$, where x_i, $i = 1, 2$ satisfies $x_i = (T - z)f_i(z)$ for all $z \in \mathbb{C} \setminus \overline{U_i}$ and some analytic function $f_i\colon \mathbb{C} \setminus \overline{U_i} \to X$.

Definition 4.2. An operator $T \in \mathcal{B}(X)$ is said to be decomposable if for any open covering U_1, U_2 of the complex plane $\mathbb{C}$ there are two closed T-invariant subspaces X_1 and X_2 of X such that $X_1 + X_2 = X$ and $\sigma(T|X_k) \subset U_k$, $k = 1, 2$.

Proposition 4.1. *If $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ are normal, then*

$$\operatorname{int}(\sigma(M_0)) \neq \emptyset \Leftrightarrow R_{M_0}(\cdot, z) \text{ is bounded for some } z \in X \oplus Y.$$

P r o o f. Of course, if A and B are normal, then M_0 is normal. Indeed:

$$M_0 M_0^* = AA^* \oplus BB^* = A^*A \oplus B^*B = M_0^* M_0.$$

According to [13], Proposition 3, we have the result. $\square$

In the following proposition, we give a sufficient conditions on A and B to prove that the local resolvent function of M_C is bounded.

Proposition 4.2. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ have the SVEP and the property δ. Suppose that $\operatorname{int}(\sigma(B)) \neq \emptyset$, then there exists $z \in X \oplus Y$ and $y \in Y$ such that $\sigma_{M_C}(z) = \sigma_B(y)$ and $R_{M_C}(\cdot, z), R_B(\cdot, y)$ are bounded on $\varrho_{M_C}(z)$ for every $C \in \mathcal{B}(Y, X)$.*

P r o o f. Let U be a nonempty open set such that $U \subseteq \sigma(B)$. Since A and B have the SVEP, from Proposition 3.1, M_C has the SVEP and $\sigma(B) \subseteq \sigma(M_C)$, then $\operatorname{int}(\sigma(M_C)) \neq \emptyset$. Since A and B have the property δ, according to [14], Proposition 3.4, M_C has the property δ. By [6], Theorem 1, there exist $z \in X \oplus Y$ and $y \in Y$ such that $\sigma_{M_C}(z) = \overline{U}$ and $\sigma_B(y) = \overline{U}$, hence $\sigma_{M_C}(z) = \sigma_B(y)$ and $R_{M_C}(\cdot, z), R_B(\cdot, y)$ are bounded on $\varrho_{M_C}(z)$. $\square$

Using the same arguments we can prove the following proposition.

Proposition 4.3. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ have the SVEP and the property δ. Suppose that $\operatorname{int}(\sigma(A)) \neq \emptyset$. Then there exist $z \in X \oplus Y$ and $x \in X$ such that $\sigma_{M_C}(z) = \sigma_A(x)$ and $R_{M_C}(\cdot, z), R_A(\cdot, x)$ are bounded on $\varrho_{M_C}(z)$.*

Proposition 4.4. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ be such that B has the property δ. Suppose that M_C has the SVEP and the property δ. Assume that $\operatorname{int}(\sigma(B)) \neq \emptyset$. Then there exist $z \in X \oplus Y$ and $y \in Y$ such that $\sigma_{M_C}(z) = \sigma_B(y)$ and $R_{M_C}(\cdot, z)$, $R_B(\cdot, y)$ are bounded on $\varrho_{M_C}(z)$.*

P r o o f. Since M_C has the property δ, by [14], Proposition 3.4, B has the property δ. Since M_C and B have the SVEP, $\sigma(M_C) = \sigma_s(M_C)$ and $\sigma(B) = \sigma_s(B)$, from the inclusion $\sigma_s(B) \subseteq \sigma_s(M_C)$ we have $\operatorname{int}(\sigma(M_C)) \neq \emptyset$. According to [6], Theorem 1, there exist $z \in X \oplus Y$ and $y \in Y$ such that $\sigma_{M_C}(z) = \overline{U}$ and $\sigma_B(y) = \overline{U}$, therefore $\sigma_{M_C}(z) = \sigma_B(y)$ and $R_{M_C}(\cdot, z)$, $R_B(\cdot, y)$ are bounded on $\varrho_{M_C}(z)$. $\square$

In particular, we obtain the following results.

Corollary 4.1. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ be decomposable. Assume that $\operatorname{int}(\sigma(A)) \neq \emptyset$. Then there exist $z \in X \oplus Y$ and $x \in X$ such that $\sigma_{M_C}(z) = \sigma_A(x)$ and $R_{M_C}(\cdot, z), R_A(\cdot, x)$ are bounded on $\varrho_{M_C}(z)$.*

Corollary 4.2. *Let $(A,B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ be decomposable. Assume that* $\operatorname{int}(\sigma(B)) \neq \emptyset$. *Then there exist $z \in X \oplus Y$ and $y \in Y$ such that $\sigma_{M_C}(z) = \sigma_B(y)$ and $R_{M_C}(\cdot, z), R_B(\cdot, y)$ are bounded on $\varrho_{M_C}(z)$ for every $C \in \mathcal{B}(Y,X)$.*

Now we give some information about the localizable spectrum for operator matrices.

Theorem 4.1. *Let $(A,B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$. Then*

$$\sigma_{\mathrm{loc}}(M_0) = \sigma_{\mathrm{loc}}(A) \cap \sigma_{\mathrm{loc}}(B).$$

Proof. Let $\lambda \in \sigma_{\mathrm{loc}}(M_0)$. Then for each open neighborhood V of λ there exists $z \in X \oplus Y$ such that $\sigma_{M_0}(z) \subseteq V$. With $z = x \oplus y$, where $x \in X$ and $y \in Y$, since $\sigma_{M_0}(x \oplus y) = \sigma_A(x) \cup \sigma_B(y)$, $\sigma_A(x) \subseteq V$ and $\sigma_B(y) \subseteq V$, hence $\lambda \in \sigma_{\mathrm{loc}}(A) \cap \sigma_{\mathrm{loc}}(B)$. Conversely, assume that $\lambda \in \sigma_{\mathrm{loc}}(A) \cap \sigma_{\mathrm{loc}}(B)$. Then for each open neighborhood V of λ there exist $x \in X$ and $y \in Y$ such that $\sigma_A(x) \cup \sigma_B(y) \subseteq V$, thus $\sigma_{M_0}(x \oplus y) \subseteq V$, and therefore $\lambda \in \sigma_{\mathrm{loc}}(M_0)$. It follows that $\sigma_{\mathrm{loc}}(M_0) = \sigma_{\mathrm{loc}}(A) \cap \sigma_{\mathrm{loc}}(B)$. □

In the general case when $C \neq 0$ we have the following theorem.

For $A \in \mathcal{B}(X)$ we denote by R_A the right multiplication operator given by $R_A(X) = XA$.

Theorem 4.2. *Let $(A,B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$. If $C \in N(R_B^n)$ for some $n \geqslant 1$, then*

$$\sigma_{\mathrm{loc}}(A) \cap \sigma_{\mathrm{loc}}(B) \subseteq \sigma_{\mathrm{loc}}(M_C).$$

Proof. Let $\lambda \in \sigma_{\mathrm{loc}}(A) \cap \sigma_{\mathrm{loc}}(B)$. Then for each open neighborhood V of λ there exist $x \in X$ and $y \in Y$ such that $\sigma_A(x) \cup \sigma_B(y) \subseteq V$. According to [4], Theorem 2.3, we have $\sigma_{M_C}(x \oplus B^n y) \subseteq \sigma_A(x) \cup \sigma_B(y)$ and then $\sigma_{M_C}(x \oplus B^n y) \subseteq V$, consequently $\lambda \in \sigma_{\mathrm{loc}}(M_C)$. Therefore $\sigma_{\mathrm{loc}}(A) \cap \sigma_{\mathrm{loc}}(B) \subseteq \sigma_{\mathrm{loc}}(M_C)$. □

Theorem 4.3. *Let $(A,B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$, assume that* $\operatorname{int}(\sigma_p(A) \cup \sigma_p(B)) = \emptyset$ *and $\sigma_{\mathrm{loc}}(A)$ has nonempty interior. Let U be a nonempty open subset of $\sigma_{\mathrm{loc}}(A)$ and let $u \in X$ be a vector with $\sigma_A(u) \subseteq U$. Then for every ε there exists some $z \in X \oplus Y$ such that $\|u \oplus 0 - z\| \leqslant \varepsilon$, $\sigma_{M_C}(z) = \overline{U}$ and $R_{M_C}(\cdot, z)$ is bounded.*

Proof. Since $\operatorname{int}(\sigma_p(A) \cup \sigma_p(B)) = \emptyset$, $\operatorname{int}(\sigma_p(M_C)) = \emptyset$. From Theorem 4.2, $\sigma_{\mathrm{loc}}(A) \subseteq \sigma_{\mathrm{loc}}(M_C)$, then $\sigma_{\mathrm{loc}}(M_C)$ has nonempty interior. Because B has the SVEP, by [12], Proposition 2.2, we have $\sigma_A(u) = \sigma_{M_C}(u \oplus 0) \subseteq U$. By [6], Theorem 4, we conclude that for every ε there exists some $z \in X \oplus Y$ such that $\|u \oplus 0 - z\| \leqslant \varepsilon$, $\sigma_{M_C}(z) = \overline{U}$ and $R_{M_C}(\cdot, z)$ is bounded. □

5. Application to Hamiltonian operator

A Hamiltonian operator matrix H is a block operator matrix

$$\begin{pmatrix} A & C \\ B & -A^* \end{pmatrix}$$

acting on the product space $X \times X$ of some complex Hilbert space X with closed densely defined operators A, B, C such that B, C are self-adjoint and H is densely defined. For an upper triangular Hamiltonian operator defined on a Hilbert space X

$$\begin{pmatrix} A & C \\ 0 & -A^* \end{pmatrix},$$

where A is closed and C is bounded on X, we have the following proposition.

Proposition 5.1. *Let $(A, D(A))$ be a closed operator. If A^* has the SVEP, then*

$$\sigma(H) = -\sigma(A^*) \cup \sigma(A).$$

Proof. The proof follows from Theorem 3.2. □

If A is bounded, from Corollary 4.1 the following holds.

Proposition 5.2. *Let $A \in \mathcal{B}(X)$. If A is decomposable and $\operatorname{int}(\sigma(A)) \neq \emptyset$, then*

(1) *H has a bounded local resolvent function;*
(2) *there exist $z \in X \oplus X$ and $x \in X$ such that $\sigma_H(z) = \sigma_A(x)$.*

Example 5.1. Consider the plate bending equation

$$D\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)^2 \omega = 0 \quad \text{for } 0 < x < 1 \text{ and } 0 < y < 1,$$

where $D > 0$ is a constant, with boundary conditions

$$\omega(x,0) = \omega(x,1) = 0, \quad \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = 0, \quad y = 0, 1.$$

We introduce the rotation θ, the Lagrange parametric function q, and the moment m as

$$\theta = \frac{\partial \omega}{\partial x}, \quad q = D\Big(\frac{\partial^3 \omega}{\partial x^3} + \frac{\partial^3 \omega}{\partial x \partial y^2}\Big), \quad m = -D\Big(\frac{\partial \theta}{\partial x} + \frac{\partial^2 \omega}{\partial y^2}\Big).$$

The equation becomes (see [15])

$$\frac{\partial}{\partial x}\begin{pmatrix}\omega\\ \theta\\ q\\ m\end{pmatrix}=\begin{pmatrix}0 & 1 & 0 & 0\\ -\partial^2/\partial y^2 & 0 & 0 & -1/D\\ 0 & 0 & 0 & \partial^2/\partial y^2\\ 0 & 0 & -1 & 0\end{pmatrix}\begin{pmatrix}\omega\\ \theta\\ q\\ m\end{pmatrix}.$$

The corresponding Hamiltonian operator matrix is given by

$$H=\begin{pmatrix}0 & 1 & 0 & 0\\ -\partial^2/\partial y^2 & 0 & 0 & -1/D\\ 0 & 0 & 0 & \partial^2/\partial y^2\\ 0 & 0 & -1 & 0\end{pmatrix}=\begin{pmatrix}A & C\\ 0 & -A^*\end{pmatrix}$$

with domain $D(A)\oplus D(A^*)\subseteq X\oplus X$, $X=L^2(0,1)\oplus L^2(0,1)$, $\mathcal{A}=AC[0,1]$, and

$$A=\begin{pmatrix}0 & 1\\ -\partial^2/\partial y^2 & 0\end{pmatrix},\quad B=\begin{pmatrix}0 & 0\\ 0 & -1/D\end{pmatrix},$$

$$D(A)=\{(\omega,\theta)\in X,\ \omega(0)=\omega(1)=0,\ \omega'\in\mathcal{A},\ \omega''\in X\}.$$

From [2], Example 4.1, $\sigma_{\rm asc}(A^*)=\emptyset$, where $\sigma_{\rm asc}(\cdot)$ is the ascent spectrum, and by [1], Corollary 4.9, we have $S(A^*)=\emptyset$. According to Proposition 5.1,

$$\sigma(H)=-\sigma(A^*)\cup\sigma(A).$$

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