ITÔ-HENSTOCK INTEGRAL AND ITÔ'S FORMULA FOR THE OPERATOR-VALUED STOCHASTIC PROCESS

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Abstract. In this paper, we introduce the Itô-Henstock integral of an operator-valued stochastic process and formulate a version of Itô's formula.

Keywords: Itô-Henstock integrable function; Itô's formula; Q-Wiener process

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1. Introduction

The Riemann integral is perhaps the most widely known integral. It is the first integral learned in elementary calculus. However, this integral has several limitations and the class of Riemann-integrable functions is quite limited. One of the attempts to solve some of the limitations of the Riemann integral was done by Henri Lebesgue (1875–1941). He formulated the Lebesgue integral which turns out to be the correct one for almost all uses and is the one used almost exclusively by professional mathematicians, see [13]. However, the Lebesgue integral is technically involved especially for non-mathematicians and requires an extensive study of measure theory. The Henstock integral, which was studied independently by Henstock and Kurzweil in the 1950s and later known as the Henstock-Kurzweil integral, is one of the notable integrals that were introduced which in some sense is more general than the Lebesgue integral. Since then, Henstock-Kurzweil integration has been deeply studied and investigated by numerous authors (see [8], [10], [11], [12], [13], [14]). The

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Henstock-Kurzweil integral is a Riemann-type definition of integral which is more explicit and minimizes the technicalities in the classical approach of the Lebesgue integral. This approach to integration is known as the generalized Riemann approach or Henstock approach.

In stochastic calculus, it is impossible to define stochastic integrals using the Riemann approach since the integrators have paths of unbounded variation and integrands are highly oscillatory, see [22]. The most common integrator is the Brownian motion. For the same reason, it is not even possible to define the stochastic integral as a Riemann-Stieltjes integral (see [16]). In the classical approach to stochastic integration, the stochastic integral of an adapted stochastic process is obtained from the mean square limit of stochastic integrals for simple processes. This approach is similar to defining the Lebesgue integral of a measurable function. Hence, to give a more explicit definition and reduce the technicalities in the classical way of defining the stochastic integral, Henstock approach to stochastic integration had already been studied in several papers (see [2], [15], [18], [22], [23]).

The concept of stochastic integral has been extended to infinite-dimensional spaces, mainly Hilbert and Banach spaces (see [4], [6], [19]). The stochastic integral in Hilbert spaces is presented in a manner similar to the real-valued case. The integrand is an operator-valued stochastic process and the integrator is a Q-Wiener process, a Hilbert space-valued Wiener process which is dependent on a symmetric nonnegative definite trace class operator Q. However, there seems to be no unifying treatment of stochastic integration in a general Banach space. In [21], Riedle introduced a stochastic integral with respect to cylindrical Wiener processes and asserts that this definition is a straightforward extension of the real-valued situation which results in simple conditions on the integrand. The strength in defining this stochastic integral is that there is no need to put any geometric constraints on the Banach space being considered.

One of the highlights in stochastic calculus after defining the stochastic integral is the formulation of Itô's formula. This formula is named after the Japanese mathematician Kiyosi Itô and is considered as the stochastic counterpart of the classical chain rule of differentiation. In 1973, Black and Scholes used Itô formula to find the price of an option. Different versions and extensions of Itô's formula in Hilbert spaces can be found in literature (see [3], [4], [6], [19], [24]).

In this paper, we define the Itô-Henstock integral of an operator-valued stochastic process with respect to a Hilbert space-valued Q-Wiener process. We also formulate a version of Itô's formula. We do not claim here that the Itô-Henstock integral is equivalent to the classical stochastic integral in Hilbert spaces (see [4]). However, this will serve as a starting point in giving an alternative definition of stochastic integrals in infinite-dimensional spaces using Henstock approach.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a sequence $\{\mathcal{F}_t\colon\ 0 \leqslant t \leqslant T\}$ of σ-subfields of $\mathcal{F}$ such that $\mathcal{F}_t \subseteq \mathcal{F}$ and $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ for $t_1 < t_2$, called a *filtration*. A probability space together with $\{\mathcal{F}_t\colon\ 0 \leqslant t \leqslant T\}$, or simply $\{\mathcal{F}_t\}$, is called a *filtered probability space* and is denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

Let H be a separable Banach space and $\mathcal{B}(H)$ be the Borel σ-field of H, i.e., the smallest σ-field containing all closed (or open) subsets of H. A mapping $h\colon\ \Omega \to H$ such that $\{\omega \in \Omega\colon\ h(\omega) \in A\} \in \mathcal{F}$ for every $A \in \mathcal{B}(H)$ is called a *measurable mapping* or a *random variable*. In this case, h is said to be $\mathcal{F}$-measurable. Given a random variable $h\colon\ \Omega \to H$, consider $\mathcal{L}(h)$ defined by $\mathcal{L}(h)(A) = \mathbb{P}(\{\omega \in \Omega\colon\ h(\omega) \in A\})$ for all $A \in \mathcal{B}(H)$. The probability measure $\mathcal{L}(h)$ on $(H, \mathcal{B}(H))$ is called the *probability distribution* or the *law* of h (see [4], page 18).

A *stochastic process*, or simply a *process*, on $[0, T]$ is a mapping $f\colon\ [0, T] \times \Omega \to H$ such that $f(t, \cdot)\colon\ \Omega \to H$ is a random variable for every $t \in [0, T]$. More often, we ignore the dependence of f on ω and write f_t for $f(t, \omega)$. The process f is usually denoted by $\{f_t\}_{0 \leqslant t \leqslant T}$. A process $f\colon\ [0, T] \times \Omega \to H$ is said to be *adapted* to $\{\mathcal{F}_t\}$ if f_t is $\mathcal{F}_t$-measurable for all $t \in [0, T]$. When no confusion arises, we may refer to a process adapted to $\{\mathcal{F}_t\}$ as simply an adapted process.

Let $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_U)$ be separable Hilbert spaces with respective inner products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$. Denote by $L(U, V)$ the space of all bounded linear operators $Q\colon\ U \to V$ equipped with the norm $\|Q\|_{L(U,V)} = \sup\limits_{u \in U, \|u\|_U = 1} \|Q(u)\|_V$. For convenience, we shall write Qu for $Q(u)$ and $L(U)$ for $L(U, U)$. Denote by $L^2(\Omega, V)$ the space of all square-integrable random variables $g\colon\ \Omega \to V$ equipped with the norm

$$\|g\|_{L^2(\Omega,V)} = \sqrt{\int_\Omega \|g(\omega)\|_V^2 \, \mathrm{d}\mathbb{P}(\omega)}.$$

The *adjoint* or *dual* Q^* of an operator $Q \in L(U)$ is the unique map $Q^* \in L(U)$ such that $\langle Q^* u, u' \rangle_U = \langle u, Qu' \rangle_U$ for all $u, u' \in U$. An operator $Q \in L(U)$ is said to be *self-adjoint* or *symmetric* if for all $u, u' \in U$, $\langle Qu, u' \rangle_U = \langle u, Qu' \rangle_U$ and is said to be *nonnegative definite* if for every $u \in U$, $\langle Qu, u \rangle_U \geqslant 0$. If $Q \in L(U)$ is nonnegative definite, by the square-root lemma (see [20], page 196) there exists a unique operator $S \in L(U)$ such that S is nonnegative definite and $S^2 = Q$. The operator S is denoted by $Q^{1/2}$.

Let $\{e_j\}_{j=1}^\infty$, or simply $\{e_j\}$, be an orthonormal basis in U. If $Q \in L(U)$ is nonnegative definite, then we define the trace of Q by

$$\operatorname{tr} Q = \sum_{j=1}^\infty \langle Qe_j, e_j \rangle_U.$$

It is shown in [20], page 206, that $\operatorname{tr} Q$ is well-defined and independent of the choice of orthonormal basis. An operator $Q\colon U \to U$ is said to be *trace-class* if $\operatorname{tr}[Q] := \operatorname{tr}(QQ^*)^{1/2} < \infty$. Denote by $L_1(U)$ the class of all trace-class operators on U. It is known from [20], page 209, that $L_1(U)$ equipped with the norm $\|Q\|_1 = \operatorname{tr}[Q]$ is a Banach space. If $Q \in L(U)$ is a symmetric nonnegative definite trace-class operator, then there exists an orthonormal basis $\{e_j\} \subset U$ and a sequence of positive real numbers $\{\lambda_j\}$ such that

$$Qe_j = \lambda_j e_j \quad \forall\, j \in \mathbb{N}$$

and $\lambda_j \to 0$ as $j \to \infty$, see [20], page 203. We shall call the sequence of pairs $\{\lambda_j, e_j\}$ an *eigensequence defined by* Q.

Let $Q\colon U \to U$ be either a symmetric nonnegative definite trace-class operator or $Q = 1_U$, where 1_U is the identity operator on U. If Q is a trace-class operator, let $\{\lambda_j, e_j\}$ be an eigensequence defined by Q. Then the subspace $U_Q := Q^{1/2}U$ of U equipped with the inner product

$$\langle u, v\rangle_{U_Q} = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \langle u, e_j\rangle_U \langle v, e_j\rangle_U$$

is a separable Hilbert space with $\{\sqrt{\lambda_j} e_j\}$ as its orthonormal basis (see [4], page 90, and [6], page 23).

Let $\{f_j\}$ be an orthonormal basis in U_Q. An operator $S \in L(U_Q, V)$ is said to be *Hilbert-Schmidt* if

$$\sum_{j=1}^{\infty} \|Sf_j\|_V^2 = \sum_{j=1}^{\infty} \langle Sf_j, Sf_j\rangle_V < \infty.$$

Denote by $L_2(U_Q, V)$ the class of all Hilbert-Schmidt operators from U_Q to V. It is known from [19], page 112, that $L_2(U_Q, V)$ is a separable Hilbert space equipped with the norm

$$\|S\|_{L_2(U_Q,V)} = \sqrt{\sum_{j=1}^{\infty} \|Sf_j\|_V^2}.$$

The Hilbert-Schmidt operator $S \in L_2(U_Q, V)$ and the norm $\|S\|_{L_2(U_Q,V)}$ are independent of the choice of the orthonormal basis (see [4], page 418 and [19], page 111). It is shown in [6], page 25, that $L(U, V)$ is properly contained in $L_2(U_Q, V)$.

A real-valued random variable $X\colon \Omega \to \mathbb{R}$ is called *normal* or *Gaussian*, with *mean* m and *variance* σ^2, if its density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{(x-m)^2}{2\sigma^2}\Big), \quad x \in \mathbb{R}.$$

In this case, the probability distribution of X is called a *Gaussian distribution* and we write $\mathcal{L}(X) \sim \mathcal{N}(m, \sigma^2)$.

Let X be a real-valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The expected value of X, denoted by $\mathbb{E}[X]$, is defined as the Lebesgue integral

$$\mathbb{E}[X] := \int_{\Omega} X(\omega)\, \mathrm{d}\mathbb{P}.$$

Let U be a separable Hilbert space and $Q\colon U \to U$ be a symmetric nonnegative definite trace-class operator. Assume that $\widetilde{X}\colon U \to L^2(\Omega, \mathbb{R})$ satisfies the following conditions:

(i) $\widetilde{X}$ is linear on U;
(ii) for every $u \in U$, $\widetilde{X}(u)$ is a real-valued Gaussian random variable with mean zero;
(iii) for all $u, u' \in U$, $\mathbb{E}[\widetilde{X}(u)\widetilde{X}(u')] = \langle Qu, u'\rangle_U$.

By Kolmogorov's two series theorem, see [25], page 46, there exists a random variable $X\colon \Omega \to U$ such that

$$\widetilde{X}(u)(\omega) = \langle X(\omega), u\rangle_U \quad \mathbb{P}\text{-almost surely (abbrev. as } \mathbb{P}\text{-a.s.).} \tag{2.1}$$

We call X a *U-valued Gaussian random variable with covariance Q*. We remark that if Q is not assumed to be a symmetric nonnegative definite trace-class operator, then by the Strong law of large numbers, see [7], page 489, it is not always true that there exists a random variable $X\colon \Omega \to U$ satisfying (2.1). A U-valued process $\{Y_t\}_{0 \leqslant t \leqslant T}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *Gaussian* if for any $n \in \mathbb{N}$ and $0 \leqslant t_1, t_2, \ldots, t_n \leqslant T$, $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})$ is a U^n-valued Gaussian random variable.

Let $B = \{B_t\}_{0 \leqslant t \leqslant T}$ be an adapted real-valued process. Then B is called a *Brownian motion* or *real-valued Wiener process* if the following properties are satisfied:

(i) $B(0, \omega) = 0$ for all $\omega \in \Omega$;
(ii) for $0 \leqslant s < t \leqslant T$, the increment $B_t - B_s$ is Gaussian with $\mathcal{L}(B_t - B_s) \sim \mathcal{N}(0, t - s)$;
(iii) for $0 \leqslant s < t \leqslant T$, $B_t - B_s$ is independent of $\mathcal{F}_s$;
(iv) $B(\cdot, \omega)\colon [0, T] \to \mathbb{R}$ is continuous for almost all $\omega \in \Omega$.

The next defintion extends the concept of a Brownian motion to a Hilbert space-valued Wiener process.

Definition 2.1 ([6]). Let U be a separable Hilbert space, $Q\colon U \to U$ be a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ be an eigensequence defined by Q, and $\{B_j\}$ be a sequence of independent Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. The process

$$\widetilde{W}_t := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) e_j \tag{2.2}$$

is called a *Q-Wiener process in U*.

We note that the series (2.2) converges in $L^2(\Omega, U)$. For each $u \in U$, denote

$$\widetilde{W}_t(u) := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) \langle e_j, u \rangle_U,$$

with the series converging in $L^2(\Omega, \mathbb{R})$. Similarly, it is not always true that there exists a U-valued process W such that

$$\widetilde{W}_t(u)(\omega) = \langle W_t(\omega), u \rangle_U \quad \mathbb{P}\text{-a.s.} \tag{2.3}$$

However, given a symmetric nonnegative definite trace-class operator Q, a U-valued process satisfying (2.3) can be defined. We call the process W a *U-valued Q-Wiener process*. When no confusion arises, we may write $W_t(u)(\omega)$ instead of $\langle W(t, \omega), u \rangle_U$. It should be noted that $W_t(e_j)/\sqrt{\lambda_j}$, $j = 1, 2, \ldots$, is a sequence of real-valued Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ (see [4], page 87).

The next theorem enumerates some of the properties of a U-valued Q-Wiener process.

Theorem 2.2 ([6]). *A U-valued Q-Wiener process $W = \{W_t\}_{0 \leqslant t \leqslant T}$ has the following properties:*

(i) *For each $\omega \in \Omega$, $W(0, \omega) = 0_U$, where 0_U is the additive identity in U.*

(ii) *W has $\mathbb{P}$-a.s. continuous trajectories, i.e.,*

$$W(\cdot, \omega)\colon\ [0, T] \to U \text{ is continuous for almost all } \omega \in \Omega.$$

(iii) *W is a Gaussian process with covariance operator Q, i.e., for any $u, u' \in U$ and $0 \leqslant s, t \leqslant T$,*

$$\mathbb{E}[W_t(u) W_s(u')] = (t \wedge s) \langle Qu, u' \rangle_U.$$

(iv) *W has independent increments, i.e., for $0 \leqslant t_1 < t_2 < \ldots < t_n \leqslant T$, $n \in \mathbb{N}$, U-valued random variables*

$$W_{t_1},\ W_{t_2} - W_{t_1},\ W_{t_3} - W_{t_2},\ \ldots,\ W_{t_n} - W_{t_{n-1}}$$

are independent.

(v) *For an arbitrary* $u \in U$, *the law*

$$\mathcal{L}((W_t - W_s)(u)) \sim \mathcal{N}(0, (t-s)\langle Qu, u\rangle_U)$$

holds.

Next, we define a Q-Wiener process with respect to a filtration.

A filtration $\{\mathcal{F}_t\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *normal* if

(i) $\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$, and

(ii) $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0, T]$.

A Q-Wiener process W_t, $t \in [0, T]$ is called a *Q-Wiener process with respect to a filtration* if

(i) W_t is adapted to $\{\mathcal{F}_t\}$, $t \in [0, T]$ and

(ii) $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $0 \leqslant s \leqslant t \leqslant T$.

It is shown in [19], page 16, that any U-valued Q-Wiener process $W(t)$, $t \in [0, T]$, is a Q-Wiener process with respect to a normal filtration. From now onwards, a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ shall mean a probability space equipped with a normal filtration.

3. Itô-Henstock integral

In [2], Chew et al. introduced the Itô-Henstock integral of a real-valued process with respect to a Brownian motion. We shall use the same definition of belated partial division employed by the authors in [2] to define the Itô-Henstock integral of an $L(U, V)$-valued stochastic process with respect to a U-valued Q-Wiener process.

Definition 3.1 ([2]). Let δ be a positive function on $[0, T]$. A finite collection D of interval-point pairs $\{((\xi_i, v_i], \xi_i)\}_{i=1}^{n}$ is a *δ-fine belated partial division* of $[0, T]$ if

(i) $(\xi_i, v_i]$, $i = 1, 2, \ldots, n$, are disjoint left-open subintervals of $[0, T]$; and

(ii) each $(\xi_i, v_i]$ is δ-fine belated, that is, $(\xi_i, v_i] \subset [\xi_i, \xi_i + \delta(\xi_i))$.

The term *partial* is used in Definition 3.1 since the finite collection of disjoint left-open subintervals of $[0, T]$ may not cover the entire interval $[0, T]$. Using the Vitali covering lemma, the following concept can be defined.

Definition 3.2 ([2]). Given $\eta > 0$, a given δ-fine belated partial division $D = \{((\xi, v], \xi)\}$ is said to be a *(δ, η)-fine belated partial division* of $[0, T]$ if it fails to cover $[0, T]$ by at most length η, that is,

$$\Bigl| T - (D) \sum (v - \xi) \Bigr| \leqslant \eta.$$

This type of partial division is the basis to which we define the Itô-Henstock integral.

Throughout the following discussions, assume that U and V are separable Hilberts spaces, $Q\colon U \to U$ is a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ is an eigensequence defined by Q, and W is a U-valued Q-Wiener process.

Definition 3.3. Let $f\colon [0,T]\times\Omega \to L(U,V)$ be an adapted process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Then f is said to be *Itô-Henstock integrable*, or $\mathcal{I}H$-integrable, on $[0,T]$ with respect to W if there exists $A \in L^2(\Omega, V)$ such that for every $\varepsilon > 0$, there is a positive function δ on $[0,T]$ and a positive number $\eta > 0$ such that for any (δ,η)-fine belated partial division $D = \{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ of $[0,T]$, we have

$$\mathbb{E}[\|S(f,D,\delta,\eta) - A\|_V^2] < \varepsilon,$$

where

$$S(f,D,\delta,\eta) := (D)\sum f_\xi(W_v - W_\xi) := \sum_{i=1}^n f_{\xi_i}(W_{v_i} - W_{\xi_i}).$$

In this case, f is $\mathcal{I}H$-integrable to A on $[0,T]$ and A is called the $\mathcal{I}H$-integral of f which will be denoted by $(\mathcal{I}H)\int_0^T f_t \,\mathrm{d}W_t$ or $(\mathcal{I}H)\int_0^T f \,\mathrm{d}W$.

Before giving an example of an $\mathcal{I}H$-integrable process, we need to present Lemma 3.5 and Lemma 3.6 first. To prove these two lemmas, we shall use the following proposition.

Proposition 3.4 ([19])**.** *Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ be two measurable spaces and $\Psi\colon E_1 \times E_2 \to \mathbb{R}$ be a bounded measurable function. Let X_1 and X_2 be two random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$, respectively, and let $\mathcal{G} \subset \mathcal{F}$ be a fixed σ-field. Assume that X_1 is $\mathcal{G}$-measurable and X_2 is independent of $\mathcal{G}$. Then*

$$\mathbb{E}[\Psi(X_1, X_2)|\mathcal{G}] = \widehat{\Psi}(X_1),$$

where $\widehat{\Psi}(x_1) = \mathbb{E}[\Psi(x_1, X_2)]$, $x_1 \in E_1$.

When we speak of a subinterval of $[0,T]$, we shall mean that the subinterval is either a closed interval $[\xi, v]$ or half-open interval $(\xi, v]$.

Lemma 3.5. *Let $f\colon [0,T] \times \Omega \to L(U,V)$ be an adapted process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and $\{[\xi_i, v_i]\}_{i=1}^n$ be a finite collection of disjoint subintervals of $[0,T]$. Then*

$$\mathbb{E}\Bigg[\sum_{i<j}\langle f_{\xi_i}(W_{v_i} - W_{\xi_i}), f_{\xi_j}(W_{v_j} - W_{\xi_j})\rangle_V\Bigg] = 0.$$

P r o o f. It is enough to show that

$$\mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i}), f_{\xi_j}(W_{v_j} - W_{\xi_j})\rangle_V] = 0 \quad \text{for } i < j.$$

By Proposition 3.4,

$$\begin{aligned}\mathbb{E}[\langle f_{\xi_j}^* f_{\xi_i}(W_{v_i} - W_{\xi_i}), W_{v_j} - W_{\xi_j}\rangle_U|\mathcal{F}_{\xi_j}](\omega) \\ = \mathbb{E}[\langle f_{\xi_j}^*(\omega) f_{\xi_i}(\omega)(W_{v_i}(\omega) - W_{\xi_i}(\omega)), W_{v_j} - W_{\xi_j}\rangle_U]\end{aligned}$$

hence

$$\mathbb{E}[\langle f_{\xi_j}^*(\omega) f_{\xi_i}(\omega)(W_{v_i}(\omega) - W_{\xi_i}(\omega)), W_{v_j} - W_{\xi_j}\rangle_U] = 0,$$

since $\mathbb{E}[\langle W_t - W_s, u\rangle_U] = 0$ for all $u \in U$. Thus,

$$\begin{aligned}\mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i}), f_{\xi_j}(W_{v_j} - W_{\xi_j})\rangle_V] \\ = \mathbb{E}[\mathbb{E}[\langle f_{\xi_j}^* f_{\xi_i}(W_{v_i} - W_{\xi_i}), W_{v_j} - W_{\xi_j}\rangle_U|\mathcal{F}_{\xi_j}]] = 0.\end{aligned}$$

This completes the proof. □

Lemma 3.6. *Let $f\colon [0,T] \times \Omega \to L(U,V)$ be an adapted process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and $\{[\xi_i, v_i]\}_{i=1}^n$ be a finite collection of disjoint subintervals of $[0,T]$. Then*

$$\begin{aligned}\mathbb{E}\left[\left\|\sum_{i=1}^n f_{\xi_i}(W_{v_i} - W_{\xi_i})\right\|_V^2\right] &= \sum_{i=1}^n \mathbb{E}[\|f_{\xi_i}(W_{v_i} - W_{\xi_i})\|_V^2] \\ &= \sum_{i=1}^n (v_i - \xi_i)\mathbb{E}[\|f_{\xi_i}\|_{L_2(U_Q,V)}^2].\end{aligned}$$

P r o o f. By Lemma 3.5,

$$\begin{aligned}\mathbb{E}\left[\left\|\sum_{i=1}^n f_{\xi_i}(W_{v_i} - W_{\xi_i})\right\|_V^2\right] &= \mathbb{E}\Bigg[\sum_{i=1}^n \langle f_{\xi_i}(W_{v_i} - W_{\xi_i}), f_{\xi_i}(W_{v_i} - W_{\xi_i})\rangle_V \\ &\quad + 2\sum_{i<j}\langle f_{\xi_i}(W_{v_i} - W_{\xi_i}), f_{\xi_j}(W_{v_j} - W_{\xi_j})\rangle_V\Bigg] \\ &= \sum_{i=1}^n \mathbb{E}[\|f_{\xi_i}(W_{v_i} - W_{\xi_i})\|_V^2].\end{aligned}$$

Let $S_m = \sum_{l=1}^m \langle f_{\xi_i}(W_{v_i} - W_{\xi_i}), b_l\rangle_V^2$, where $\{b_l\}$ is an orthonormal basis in V. Note that

$$S_m \to \sum_{l=1}^\infty \langle f_{\xi_i}(W_{v_i} - W_{\xi_i}), b_l\rangle_V^2 := S \quad \text{as } m \to \infty$$

and $S_m(\omega) \leqslant S_{m+1}(\omega)$ for all $m \in \mathbb{N}$. By the monotone convergence theorem, we have

$$\int_\Omega S(\omega)\,\mathrm{d}\mathbb{P} = \lim_{m\to\infty}\int_\Omega S_m(\omega)\,\mathrm{d}\mathbb{P},$$

so that

$$\begin{aligned}\mathbb{E}\Bigg[\sum_{l=1}^{\infty}\langle f_{\xi_i}(W_{v_i}-W_{\xi_i}),b_l\rangle_V^2\Bigg] &= \lim_{m\to\infty}\mathbb{E}\Bigg[\sum_{l=1}^{m}\langle f_{\xi_i}(W_{v_i}-W_{\xi_i}),b_l\rangle_V^2\Bigg]\\ &= \sum_{l=1}^{\infty}\mathbb{E}[\langle f_{\xi_i}(W_{v_i}-W_{\xi_i}),b_l\rangle_V^2]\\ &= \sum_{l=1}^{\infty}\mathbb{E}[\mathbb{E}[\langle f_{\xi_i}(W_{v_i}-W_{\xi_i}),b_l\rangle_V^2|\mathcal{F}_{\xi_i}]]\\ &= \sum_{l=1}^{\infty}\mathbb{E}[\mathbb{E}[\langle W_{v_i}-W_{\xi_i},f_{\xi_i}^*b_l\rangle_U^2|\mathcal{F}_{\xi_i}]].\end{aligned}$$

Using Proposition 3.4,

$$\mathbb{E}[\langle W_{v_i}-W_{\xi_i},f_{\xi_i}^*b_l\rangle_U^2|\mathcal{F}_{\xi_i}](\omega) = \mathbb{E}[\langle W_{v_i}-W_{\xi_i},f_{\xi_i}(\omega)^*b_l\rangle_U^2].$$

Since $\mathbb{E}[\langle W_t-W_s,u\rangle_U^2] = (t-s)\langle Qu,u\rangle$ for all $u\in U$,

$$\mathbb{E}[\langle W_{v_i}-W_{\xi_i},f_{\xi_i}(\omega)^*b_l\rangle_U^2] = (v_i-\xi_i)\langle Qf_{\xi_i}(\omega)^*b_l,f_{\xi_i}(\omega)^*b_l\rangle_U.$$

It follows that

$$\mathbb{E}\Bigg[\sum_{l=1}^{\infty}\langle f_{\xi_i}(W_{v_i}-W_{\xi_i}),b_l\rangle_U^2\Bigg] = \sum_{l=1}^{\infty}(v_i-\xi_i)\mathbb{E}[\langle Qf_{\xi_i}^*b_l,f_{\xi_i}^*b_l\rangle_U]. \tag{3.1}$$

Let $\{\lambda_j,e_j\}$ be an eigensequence defined by Q. Then

$$\mathbb{E}[\langle Qf_{\xi_i}^*b_l,f_{\xi_i}^*b_l\rangle_U] = \mathbb{E}\Bigg[\sum_{j=1}^{\infty}\lambda_j\langle f_{\xi_i}^*b_l,e_j\rangle_U^2\Bigg] = \mathbb{E}\Bigg[\sum_{j=1}^{\infty}\langle f_{\xi_i}(\sqrt{\lambda_j}e_j),b_l\rangle_U^2\Bigg]. \tag{3.2}$$

Thus, using (3.1) and (3.2), we have

$$\begin{aligned}\mathbb{E}\Bigg[\sum_{i=1}^{n}\langle f_{\xi_i}(W_{v_i}-W_{\xi_i}),f_{\xi_i}(W_{v_i}-W_{\xi_i})\rangle_V\Bigg] &= \sum_{i=1}^{n}\sum_{l=1}^{\infty}(v_i-\xi_i)\mathbb{E}\Bigg[\sum_{j=1}^{\infty}\langle f_{\xi_i}(\sqrt{\lambda_j}e_j),b_l\rangle_V^2\Bigg]\\ &= \sum_{i=1}^{n}(v_i-\xi_i)\mathbb{E}\Bigg[\sum_{j=1}^{\infty}\|f_{\xi_i}(\sqrt{\lambda_j}e_j)\|_V^2\Bigg]\\ &= \sum_{i=1}^{n}(v_i-\xi_i)\mathbb{E}[\|f_{\xi_i}\|_{L_2(U_Q,V)}^2],\end{aligned}$$

which completes the proof. □

E x a m p l e 3.7. Let $g\colon \Omega \to L(U,V)$ be a random variable bounded in $L_2(U_Q,V)$, that is, there exists $M>0$ such that $\|g(\omega)\|_{L_2(U_Q,V)} \leqslant M$ for all $\omega\in\Omega$, and let $\hat{\theta}\colon \Omega \to L(U,V)$ be a random variable such that for all $\omega\in\Omega$, $\hat{\theta}(\omega)$ is the zero operator in $L(U,V)$, that is, $\hat{\theta}(\omega)\colon U\to V$ defined by $\hat{\theta}(\omega)(k)=0_V$ for all $k\in U$, where 0_V is the additive identity of V. Let $s\in[0,T]$ be fixed. Let $f\colon [0,T]\times\Omega\to L(U,V)$ be an adapted process on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ such that for $t\in[0,T]$,

$$f_t = \begin{cases} g & \text{if } t=s,\\ \hat{\theta} & \text{if } t\neq s. \end{cases}$$

Then f is $\mathcal{I}H$-integrable to the zero random variable $\theta\in L^2(\Omega,V)$ on $[0,T]$, i.e.,

$$(\mathcal{I}H)\int_0^T f_t\,\mathrm{d}W_t = \theta,$$

where $\theta\colon \Omega\to V$ is defined by $\theta(\omega)=0_V$ for all $\omega\in\Omega$.

P r o o f. Given $\varepsilon>0$, choose $\delta(\xi)=\frac{1}{2}\varepsilon M^{-2}>0$ for all $\xi\in[0,T]$. Let $D=\{((\xi_i,v_i],\xi_i)\}_{i=1}^n$ be any δ-fine belated partial division of $[0,T]$. If $s\neq\xi_i$ for all i, then we are done. Suppose that $s=\xi_i$ for some i. Then by Lemma 3.6,

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg\|\sum_{i=1}^n f_{\xi_i}(W_{v_i}-W_{\xi_i})-\theta\Bigg\|_V^2\Bigg] &= \sum_{i=1}^n (v_i-\xi_i)\mathbb{E}[\|f_{\xi_i}\|^2_{L_2(U_Q,V)}]\\ &= (v_i-s)\mathbb{E}[\|g\|^2_{L_2(U_Q,V)}] < \frac{\varepsilon}{2M^2}M^2<\varepsilon.\end{aligned}$$

The above inequality holds for any δ-fine belated partial division of $[0,T]$. Hence, it also holds for any (δ,η)-fine belated partial division of $[0,T]$. Thus, f is $\mathcal{I}H$-integrable to θ on $[0,T]$. □

The following statements show that the Itô-Henstock integral posseses the standard properties of an integral. The proofs are analogous to the proofs in [12].

(1) The Itô-Henstock integral is uniquely determined, in the sense that if A_1 and A_2 are two Itô-Henstock integrals of f in Definition 3.3, then $\|A_1-A_2\|_{L^2(\Omega,V)}=0$.

(2) Let $\alpha\in\mathbb{R}$. If f and g are $\mathcal{I}H$-integrable on $[0,T]$, then

(i) $f+g$ is $\mathcal{I}H$-integrable on $[0,T]$, and

$$(\mathcal{I}H)\int_0^T (f+g)\,\mathrm{d}W = (\mathcal{I}H)\int_0^T f\,\mathrm{d}W + (\mathcal{I}H)\int_0^T g\,\mathrm{d}W;$$

(ii) αf is $\mathcal{I}H$-integrable on $[0,T]$, and

$$(\mathcal{I}H)\int_0^T (\alpha f)\,\mathrm{d}W = \alpha(\mathcal{I}H)\int_0^T f\,\mathrm{d}W.$$

(3) If $f\colon\ [0,T]\times\Omega\to L(U,V)$ is $\mathcal{I}H$-integrable on $[0,c]$ and $[c,T]$ where $c\in(0,T)$, then f is $\mathcal{I}H$-integrable on $[0,T]$ and

$$(\mathcal{I}H)\int_0^T f\,\mathrm{d}W=(\mathcal{I}H)\int_0^c f\,\mathrm{d}W+(\mathcal{I}H)\int_c^T f\,\mathrm{d}W.$$

(4) If $f\colon\ [0,T]\times\Omega\to L(U,V)$ is $\mathcal{I}H$-integrable on $[0,T]$, then f is also $\mathcal{I}H$-integrable on every subinterval $[c,d]$ of $[0,T]$.

(5) A process $f\colon\ [0,T]\times\Omega\to L(U,V)$ is $\mathcal{I}H$-integrable on $[0,T]$ if and only if there exist $A\in L^2(\Omega,V)$, a decreasing sequence $\{\delta_n\}$ of positive functions defined on $[0,T]$, and a decreasing sequence of positive numbers $\{\eta_n\}$ such that for any (δ_n,η_n)-fine belated partial division D_n of $[0,T]$, we have

$$\lim_{n\to\infty}\mathbb{E}[\|S(f,D_n,\delta_n,\eta_n)-A\|_V^2]=0.$$

In this case,

$$A=(\mathcal{I}H)\int_0^T f_t\,\mathrm{d}W_t.$$

(6) *Cauchy criterion.* A process $f\colon\ [0,T]\times\Omega\to L(U,V)$ is $\mathcal{I}H$-integrable on $[0,T]$ if and only if for every $\varepsilon>0$, there exist a positive function δ on $[0,T]$ and a positive number η such that for any two (δ,η)-fine belated partial divisions D and D' of $[0,T]$, we have

$$\mathbb{E}[\|S(f,D,\delta,\eta)-S(f,D',\delta,\eta)\|_V^2]<\varepsilon.$$

(7) *Henstock lemma.* Let f be $\mathcal{I}H$-integrable on $[0,T]$ and $F(u,v):=(\mathcal{I}H)\int_u^v f_t\,\mathrm{d}W_t$ for any $(u,v]\subseteq[0,T]$. Then for every $\varepsilon>0$, there exist a positive function δ on $[0,T]$ and a positive number η such that for any (δ,η)-fine belated partial division D of $[0,T]$, we have

$$\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_v-W_\xi)-F(\xi,v)\}\right\|_V^2\right]<\varepsilon.$$

In this section, we present a version of Itô's formula (for Q-Wiener process) which involves the Itô-Henstock integral. We begin defining the first and second Fréchet derivatives, regulated mapping, primitive, and bilinear mapping. Refer to [1] and [5] for more details.

Definition 4.1. Let K and H be Banach spaces, O be an open subset of K, and x_0 be a particular point in K. A continuous linear operator $g\colon K \to H$ is called the *Fréchet derivative* of a mapping $f\colon O \to H$ at x_0 if

$$\lim_{x \to 0_K} \frac{\|f(x_0 + x) - f(x_0) - g(x)\|_H}{\|x\|_K} = 0. \tag{4.1}$$

The Fréchet derivative of f at x_0 is usually denoted by $f'(x_0)$ or $Df(x_0)$. A mapping f is called *Fréchet differentiable* on O if $f'(x_0)$ exists at every $x_0 \in O$.

The following definitions describe a regulated mapping on a closed interval and its primitive.

Definition 4.2. Let $[a, b]$ be a closed interval of $\mathbb{R}$ and let H be a Banach space. We say that a mapping $f\colon [a, b] \to H$ is a *step function* if there exists an increasing finite sequence $\{x_i\}_{i=0}^n$ of points in $[a, b]$, where $x_0 = a$, $x_n = b$, and f is constant in each of the open intervals (x_i, x_{i+1}), $0 \leqslant i \leqslant n-1$.

Definition 4.3. Let $f\colon [a, b] \to H$ be a mapping and $x \in [a, b)$. We say that f has a *limit on the right* or *right limit* if

$$f(x+) := \lim_{\substack{y \in [a,b],\, y > x \\ y \to x}} f(y)$$

exists. Similarly, given $x \in (a, b]$, we say that f has a *limit on the left* or *left limit* if

$$f(x-) := \lim_{\substack{y \in [a,b],\, y < x \\ y \to x}} f(y)$$

exists.

Definition 4.4. A mapping $f\colon [a, b] \to H$ is said to be *regulated* if for all $t \in [a, b]$, both the right and left limits $f(t+)$ and $f(t-)$ exist.

It was established in [5] that a mapping $f\colon [a, b] \to H$ is regulated if and only if f is a limit of a uniformly convergent sequence of step functions. Moreover, a continuous mapping is also regulated.

Definition 4.5. Let H be a Banach space and let $f\colon [a,b] \to H$ be a mapping. We say that a continuous mapping $g\colon [a,b] \to H$ is a *primitive* of f in $[a,b]$ if there exists a denumerable set $D \subset [a,b]$ such that for all $t \in [a,b] \setminus D$, g is Fréchet differentiable at t and $g'(t) = f(t)$.

We remark that a primitive is not unique and if g_1 and g_2 are two primitives of f in $[a,b]$, then $g_1 - g_2$ is constant in $[a,b]$ (see [5], page 165).

Theorem 4.6 ([5])**.** *Any regulated function $f\colon [a,b] \to H$ has a primitive in $[a,b]$.*

Definition 4.7. Let $f\colon [a,b] \to H$ be a regulated function and let g be a primitive of f. The *integral of f between α and β* is defined by

$$\int_\alpha^\beta f(t)\,\mathrm{d}t := g(\beta) - g(\alpha).$$

The definition of $\int_\alpha^\beta f(t)\,\mathrm{d}t$ is independent of the choice of the primitive (see [5], page 166).

Definition 4.8. If $f\colon K \to H$ is Fréchet differentiable on an open set $O \subseteq K$ and the first Fréchet derivative f' at $x_0 \in O$ is Fréchet differentiable at x_0, then the Fréchet derivative of f' at x_0 is called the *second Fréchet derivative* of f at x_0 and is denoted by $f''(x_0)$ or $D^2 f(x_0)$. A mapping f is said to be *twice Fréchet differentiable* on O if $f''(x_0)$ exists at every $x_0 \in O$.

Observe that if the second Fréchet derivative $f''(x_0)$ exists, then $f''(x_0) \in L(K, L(K,H))$. If f'' exists at every point in O, then

$$f''\colon O \subseteq K \to L(K, L(K,H)).$$

Definition 4.9. A mapping $f\colon K \times K \to H$ is said to be a *bilinear mapping* if it is linear in each of the two variables, that is, for any $\alpha, \beta \in \mathbb{R}$,

$$f(\alpha x_1 + \beta x_1', x_2) = \alpha f(x_1, x_2) + \beta f(x_1', x_2)$$

and

$$f(x_1, \alpha x_2 + \beta x_2') = \alpha f(x_1, x_2) + \beta f(x_1, x_2').$$

A bilinear mapping $f\colon K \times K \to H$ is said to be bounded or continuous if there exists $M > 0$ such that

$$\|f(x_1, x_2)\|_H \leqslant M \|x_1\|_K \|x_2\|_K$$

for all $(x_1, x_2) \in K \times K$. The vector space of all continuous bilinear mappings $f\colon K \times K \to H$ is denoted by $L(K \times K, H)$.

It is known (see [17], page 64) that $L(K \times K, H)$ is a normed space with

$$\|f\|_{L(K\times K,H)} = \inf\{M\colon\ \|f(x_1,x_2)\|_H \leqslant M\|x_1\|_K\|x_2\|_K,\ (x_1,\ x_2) \in K \times K\}.$$

Moreover, $L(K, L(K, H)) = L(K \times K, H)$ (see [17], page 65). Hence the second Fréchet derivative $f''(x_0)$, if it exists, is an element of $L(K \times K, H)$.

Theorem 4.10 ([5]). *If f is twice Fréchet differentiable at x_0, then the bilinear mapping $f''(x_0)$ is symmetric, that is,*

$$f''(x_0)(x_1, x_2) = f''(x_0)(x_2, x_1)$$

for all $x_1, x_2 \in K$.

Next, we define a Hilbert-Schmidt bilinear mapping.

Lemma 4.11 ([1]). *Let K and H be separable Hilbert spaces and $f\colon\ K\times K \to H$ be a bilinear mapping. If the series*

$$\sum_{j,l=1}^{\infty} \|f(e_j, e_l)\|_H^2 = \sum_{j,l=1}^{\infty} \langle f(e_j, e_l), f(e_j, e_l)\rangle_H < \infty$$

for the orthonormal basis $\{e_j\}$, then

$$\sum_{j,l=1}^{\infty} \|f(e_j, e_l)\|_H^2 = \sum_{j,l,m=1}^{\infty} \langle f(e_j, e_l), b_m\rangle_H^2 = \sum_{j=1}^{\infty} \|g(e_j)\|_{L_2(K,H)}^2,$$

no matter what orthonormal bases $\{e_j\} \subseteq K$ and $\{b_m\} \subseteq H$ are chosen, where $g \in L_2(K, L_2(K, H))$ is defined by

$$(g(x_1))(x_2) = f(x_1, x_2) \quad \forall\, x_1, x_2 \in K.$$

Definition 4.12. A bilinear mapping $f\colon\ K \times K \to H$ is said to be Hilbert-Schmidt if

$$\sqrt{\sum_{j,l,m=1}^{\infty} \langle f(e_j, e_l), b_m\rangle_H^2} < \infty$$

for arbitrary orthonormal bases $\{e_j\}$ and $\{b_m\}$ of K and H, respectively. The space of all Hilbert-Schmidt bilinear mappings is denoted by $L_2(K \times K, H)$, which is a normed space (see [1], page 294) with norm

$$\|f\|_{L_2(K\times K,H)} = \sqrt{\sum_{j,l,m=1}^{\infty} \langle f(e_j, e_l), b_m\rangle_H^2}.$$

It is also known (see [1], page 294) that $L_2(K, L_2(K,H)) = L_2(K \times K, H)$.

Theorem 4.13 (*Taylor's formula* [5])**.** *Let K and H be Banach spaces, O be an open subset of K, and $f\colon O \to H$ be twice continuously Fréchet differentiable. If the segment joining x and $x+h$ is in O, then we have*

$$f(x+h) = f(x) + f'(x)(h) + \int_0^1 (1-t) f''(x+th)(h,h)\,\mathrm{d}t.$$

We note that in a normed space E, the *segment* joining two points a and b is defined as the set of points $a + t(b-a)$ with $0 \leqslant t \leqslant 1$.

We next define the Riemann integral of a bounded Banach-valued function on $[0,T]$. Note that the given closed interval $[0,T]$ in the following definition can be replaced with any closed interval $[a,b]$.

Definition 4.14. Consider a bounded function $f\colon [0,T] \to H$, where H is a Banach space. Let $P\colon\ 0 = a_1 < b_1 = a_2 < b_2 = \ldots < b_n = T$ be a partition of $[0,T]$. Take a set of points $\mathcal{T} = \{t_i\}_{i=1}^n$, $t_i \in [a_i, b_i)$. The *Riemann sum* of the function f is defined by

$$S(f, P, \mathcal{T}) = \sum_{i=1}^n f(t_i)(b_i - a_i).$$

Call $d(P) = \max\limits_{1 \leqslant i \leqslant n} (b_i - a_i)$ the diameter of the partition P. As in the usual real-valued case, we will say that f is *Riemann integrable* if the sums $S(f,P,\mathcal{T})$ tend to some limit when $d(P) \to 0$. The limit is called the *Riemann integral* of the function f and is denoted by $(\mathcal{R}) \int_0^T f(t)\,\mathrm{d}t$ or $(\mathcal{R}) \int_0^T f\,\mathrm{d}t$.

In Definition 4.14, the limit is taken in the sense that, for every $\varepsilon > 0$, there exists a positive number δ such that for any partition P with $d(P) < \delta$,

$$\left\| S(f,P,\mathcal{T}) - (\mathcal{R}) \int_0^T f(t)\,\mathrm{d}t \right\|_H < \varepsilon.$$

The Riemann integral is equivalent to the Henstock-Kurzweil integral provided that the positive function δ on $[0,T]$ is a constant function. It also adopts the standard properties of an integral in the real-valued case. The reader may refer to [9] for the proof and discussions of the other properties.

We shall now derive a version of Itô's formula for the Itô-Henstock integral. Throughout the discussion, assume that U and V are separable Hilbert spaces and $\{\lambda_j, e_j\}$ is an eigensequence defined by a symmetric nonnegative definite trace class operator Q in Definition 2.1. We also introduce the following concept.

Definition 4.15. A function $f\colon [0,T]\to\mathbb{R}$ is said to be $AC^*[0,T]$ if for every $\varepsilon>0$, there exists a $\delta>0$ such that for any finite collection $D=\{[\xi,v]\}$ of disjoint subintervals of $[0,T]$ with

$$(D)\sum(v-\xi)<\delta,$$

we have

$$(D)\sum|f(\xi)|(v-\xi)<\varepsilon.$$

Next, we consider several lemmas involving limits of sums over full or partial divisions.

Lemma 4.16. *Let $f\colon [0,T]\times\Omega\to L(U,V)$ be an $\mathcal{I}H$-integrable process and suppose that $\mathbb{E}[\|f_t\|^2_{L_2(U_Q,V)}]$ is $AC^*[0,T]$. Given $\varepsilon>0$, there exist a positive function δ on $[0,T]$ and a positive number η such that if $D=\{((\xi,v],\xi)\}$ is a (δ,η)-fine belated partial division of $[0,T]$, then*

$$\mathbb{E}\Bigl[\Bigl\|S(f,D\cup D^{\rm c},\delta,\eta)-(\mathcal{I}H)\int_0^T f_t\,{\rm d}W_t\Bigr\|^2_V\Bigr]<\varepsilon,$$

where $\{(\xi,v]\colon ((\xi,v],\xi)\in D^{\rm c}\}$ is the collection of all subintervals of $[0,T]$ which are not included in the set $\{(\xi,v]\colon ((\xi,v],\xi)\in D\}$, and

$$S(f,D\cup D^{\rm c},\delta,\eta):=(D\cup D^{\rm c})\sum f_\xi(W_v-W_\xi).$$

P r o o f. Let $\varepsilon>0$ be given. Then there exist a positive function δ on $[0,T]$ and a positive number η_1 such that for any (δ,η_1)-fine belated partial division $D_1=\{((\xi,v],\xi)\}$ of $[0,T]$, we have

$$\mathbb{E}\Bigl[\Bigl\|S(f,D_1,\delta,\eta_1)-(\mathcal{I}H)\int_0^T f_t\,{\rm d}W_t\Bigr\|^2_V\Bigr]<\frac{\varepsilon}{4}.$$

Since $\mathbb{E}[\|f_t\|^2_{L_2(U_Q,V)}]$ is $AC^*[0,T]$, then for the given ε, there exists $\eta_2>0$ such that for any collection of disjoint subintervals $D_2=\{((\xi,v],\xi)\}$ of $[0,T]$ with $(D_2)^{\rm c}\sum(v-\xi)<\eta_2$, we have

$$(D_2)^{\rm c}\sum\mathbb{E}[\|f_\xi\|^2_{L_2(U_Q,V)}](v-\xi)<\frac{\varepsilon}{4}.$$

Let $\eta=\min\{\eta_1,\eta_2\}$. Then for any (δ,η)-fine belated partial division $D=\{((\xi,v],\xi)\}$ of $[0,T]$, we have

$$\mathbb{E}\Bigl[\Bigl\|S(f,D,\delta,\eta)-(\mathcal{I}H)\int_0^T f_t\,{\rm d}W_t\Bigr\|^2_V\Bigr]<\frac{\varepsilon}{4}.$$

Moreover, it follows that

$$D^{\mathrm{c}} \sum (v-\xi) \leqslant \eta \leqslant \eta_2,$$

hence by Lemma 3.6,

$$\mathbb{E}\left[\left\|(D^{\mathrm{c}}) \sum f_\xi (W_v - W_\xi)\right\|_V^2\right] = (D^{\mathrm{c}}) \sum \mathbb{E}[\|f_\xi\|^2_{L_2(U_Q,V)}](v-\xi) < \frac{\varepsilon}{4}.$$

Thus, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|S(f, D\cup D^{\mathrm{c}}, \delta, \eta) - (\mathcal{I}H)\int_0^T f_t \,\mathrm{d}W_t\right\|_V^2\right] \\
&\leqslant 2\mathbb{E}\left[\left\|S(f, D, \delta, \eta) - (\mathcal{I}H)\int_0^T f_t \,\mathrm{d}W_t\right\|_V^2\right] + 2\mathbb{E}\left[\left\|(D^{\mathrm{c}}) \sum f_\xi (W_v - W_\xi)\right\|_V^2\right] < \varepsilon.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.17. *Let $f\colon [0,T]\times\Omega \to L(U\times U, V)$ be an adapted process such that $f_t(\omega)$ is symmetric, $\mathbb{E}[\|f_t\|^2_{L_2(U_Q\times U_Q,V)}]$ is bounded on $[0,T]$ and $M_t := \sum\limits_{j=1}^{\infty} \lambda_j f_t(e_j,e_j)$ is Riemann integrable on $[0,T]$. Then for any partition $P=\{[\xi_i,v_i]\}_{i=1}^n$ of $[0,T]$*

$$\mathbb{E}\left[\left\|(P)\sum f_\xi(W_v-W_\xi, W_v-W_\xi) - (\mathcal{R})\int_0^T M_t \,\mathrm{d}t\right\|_V^2\right] \to 0$$

as $d(P) := \max\limits_{1\leqslant i\leqslant n}(v_i-\xi_i) \to 0$.

P r o o f. Let $(W_v-W_\xi)^{(2)} := (W_v-W_\xi, W_v-W_\xi)$. Since M_t is Riemann integrable on $[0,T]$,

$$\mathbb{E}\left[\left\|(P)\sum M_\xi(v-\xi) - (\mathcal{R})\int_0^T M_t \,\mathrm{d}t\right\|_V^2\right] \to 0 \quad \text{as } d(P)\to 0.$$

Hence, it is enough to show that

$$\mathbb{E}\left[\left\|(P)\sum \{f_\xi(W_v-W_\xi)^{(2)} - M_\xi(v-\xi)\}\right\|_V^2\right] \to 0 \quad \text{as } d(P)\to 0.$$

Observe that

$$\mathbb{E}\left[\left\|(P)\sum\{f_\xi(W_v - W_\xi)^{(2)} - M_\xi(v-\xi)\}\right\|_V^2\right]$$
$$\begin{aligned}
&= \sum_{i=1}^{n}(\mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}, f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}\rangle_V] \\
&\quad - 2\mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}, M_{\xi_i}(v_i - \xi_i)\rangle_V] + \mathbb{E}[\langle M_{\xi_i}(v_i-\xi_i), M_{\xi_i}(v_i-\xi_i)\rangle_V]) \\
&\quad + 2\sum_{i<p}(\mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}, f_{\xi_p}(W_{v_p} - W_{\xi_p})^{(2)}\rangle_V] \\
&\quad - \mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}, M_{\xi_p}(v_p - \xi_p)\rangle_V] - \mathbb{E}[\langle M_{\xi_i}(v_i-\xi_i), f_{\xi_p}(W_{v_p} - W_{\xi_p})^{(2)}\rangle_V] \\
&\quad + \mathbb{E}[\langle M_{\xi_i}(v_i-\xi_i), M_{\xi_p}(v_p-\xi_p)\rangle_V]).
\end{aligned}$$

Then consider the following claims.

Claim 1:

$$\begin{aligned}
&\mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}, f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}\rangle_V] \\
&\quad = 2(v_i-\xi_i)^2\mathbb{E}[\|f_{\xi_i}\|^2_{L_2(U_Q\times U_Q,V)}] + (v_i-\xi_i)^2\mathbb{E}[\langle M_{\xi_i}, M_{\xi_i}\rangle_V].
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}, f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}\rangle_V] \\
&\quad = \sum_{j=1}^{\infty}\mathbb{E}[\langle f_{\xi_i}(e_j,e_j), f_{\xi_i}(e_q,e_q)\rangle_V\mathbb{E}[\langle W_{v_i} - W_{\xi_i}, e_j\rangle_U^4|\mathcal{F}_{\xi_i}]] \\
&\qquad + \sum_{\substack{j=j',l=l'\\ j\neq l}}\mathbb{E}[\langle f_{\xi_i}(e_j,e_l), f_{\xi_i}(e_j,e_l)\rangle_V\mathbb{E}[\langle W_{v_i} - W_{\xi_i}, e_j\rangle_U^2\langle W_{v_i} - W_{\xi_i}, e_l\rangle_U^2|\mathcal{F}_{\xi_i}]] \\
&\qquad + \sum_{\substack{j=l',l=j'\\ j\neq l}}\mathbb{E}[\langle f_{\xi_i}(e_j,e_l), f_{\xi_i}(e_l,e_j)\rangle_V\mathbb{E}[\langle W_{v_i} - W_{\xi_i}, e_j\rangle_U^2\langle W_{v_i} - W_{\xi_i}, e_l\rangle_U^2|\mathcal{F}_{\xi_i}]] \\
&\qquad + \sum_{\substack{j=l,j'=l'\\ j\neq j'}}\mathbb{E}[\langle f_{\xi_i}(e_j,e_j), f_{\xi_i}(e_{j'},e_{j'})\rangle_V\mathbb{E}[\langle W_{v_i} - W_{\xi_i}, e_j\rangle_U^2\langle W_{v_i} - W_{\xi_i}, e_{j'}\rangle_U^2|\mathcal{F}_{\xi_i}]].
\end{aligned}$$

Since $\langle W_t, e_j\rangle_U/\sqrt{\lambda_j}$ is a Brownian motion,

$$\mathbb{E}\left[\left(\frac{\langle W_v - W_\xi, e_j\rangle_U}{\sqrt{\lambda_j}}\right)^4\right] = 3(v-\xi)^2 \quad\text{and}\quad \mathbb{E}\left[\left(\frac{\langle W_v - W_\xi, e_j\rangle_U}{\sqrt{\lambda_j}}\right)^2\right] = v-\xi.$$

Hence,

$$\begin{aligned}
&\mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}, f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}\rangle_V] \\
&\qquad\qquad = 2(v_i-\xi_i)^2\mathbb{E}[\|f_{\xi_i}\|^2_{L_2(U_Q\times U_Q,V)}] + (v_i-\xi_i)^2\mathbb{E}[\langle M_{\xi_i}, M_{\xi_i}\rangle_V].
\end{aligned}$$

This proves the first claim.

Similarly, the following claims hold:

Claim 2: For $i < p$,

$$\mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}, f_{\xi_p}(W_{v_p} - W_{\xi_p})^{(2)}\rangle_V] = \mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}, M_{\xi_p}(v_p - \xi_p)\rangle_V].$$

Claim 3: $\mathbb{E}[\langle f_{\xi_i}(W_{v_i} - W_{\xi_i})^{(2)}, M_{\xi_i}(v_i - \xi_i)\rangle_V] = (v_i - \xi_i)^2 \mathbb{E}[\langle M_{\xi_i}, M_{\xi_i}\rangle_V]$.

Claim 4: For $i < p$,

$$\mathbb{E}[\langle M_{\xi_i}(v_i - \xi_i), f_{\xi_p}(W_{v_p} - W_{\xi_p})^{(2)}\rangle_V] = \mathbb{E}[\langle M_{\xi_i}(v_i - \xi_i), M_{\xi_p}(v_p - \xi_p)\rangle_V].$$

Using the four claims, we have

$$\mathbb{E}\biggl[\biggl\|(P)\sum\{f_\xi(W_v - W_\xi)^{(2)} - M_\xi(v - \xi)\}\biggr\|_V^2\biggr] \leqslant 2d(P)T\mathbb{E}[\|f_{\xi_i}\|^2_{L_2(U_Q\times U_Q,V)}].$$

Since $\mathbb{E}[\|f_{\xi_i}\|^2_{L_2(U_Q\times U_Q,V)}]$ is bounded,

$$\mathbb{E}\biggl[\biggl\|(P)\sum\{f_\xi(W_v - W_\xi)^{(2)} - M_\xi(v - \xi)\}\biggr\|_V^2\biggr] \to 0$$

as $d(P) \to 0$. □

Let E_1 and E_2 be normed spaces. A function $f\colon E_1 \to E_2$ is said to be uniformly continuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E_1$ with $\|x - y\|_{E_1} < \delta$, we have $\|f(x) - f(y)\|_{E_2} < \varepsilon$.

Lemma 4.18. *Let $f\colon U \to L(U \times U, V)$ be a uniformly continuous function. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition $P = \{[\xi, v]\}$ of $[0, T]$ with $d(P) < \delta$,*

$$\|\overline{S}(f, P, \delta)\|_{L^2(\Omega,V)} < \varepsilon,$$

where

$$\overline{S}(f, P, \delta) := (P)\sum\biggl\{\int_0^1 (1 - t)(f(\widehat{W}) - f(W_\xi))(W_v - W_\xi)^{(2)}\,\mathrm{d}t\biggr\},$$

$\widehat{W} := W_\xi + t(W_v - W_\xi)$ and $t \in [0, 1]$.

P r o o f. Let $\varepsilon > 0$ be given. For any $[\xi, v] \in [0, T]$,

$$\begin{aligned}\mathbb{E}[\|W_v - W_\xi\|_U^4] &= \sum_{j=1}^\infty \mathbb{E}[\langle W_v - W_\xi, e_j\rangle_U^4] + \sum_{j\neq l}\mathbb{E}[\langle W_v - W_\xi, e_j\rangle_U^2\langle W_v - W_\xi, e_l\rangle_U^2]\\ &= \sum_{j=1}^\infty 3\lambda_j^2(v - \xi)^2 + \sum_{j\neq l}\lambda_j\lambda_l(v - \xi)^2 = (v - \xi)^2\biggl(2\sum_{j=1}^\infty \lambda_j^2 + p^2\operatorname{tr} Q\biggr).\end{aligned}$$

Since $\operatorname{tr} Q < \infty$, let M be a positive real number such that $2\sum_{j=1}^{\infty} \lambda_j^2 + (\operatorname{tr} Q)^2 \leqslant M$. Since f is uniformly continuous on U, there exists $\eta > 0$ such that if $\|u - u'\|_U < \eta$, we have

$$\|f(u) - f(u')\|_{L(U\times U,V)} < \frac{2\varepsilon}{T\sqrt{M}}.$$

For this η, there exists $\delta > 0$ such that whenever $|v - \xi| < \delta$, $\xi, v \in [0,T]$, we have $\|W_v - W_\xi\|_U < \eta$. But this implies that $\|\widehat{W} - W_\xi\|_U = t\|W_v - W_\xi\|_U < \eta$ for $t \in [0,1]$. It follows that

$$\|f(\widehat{W}) - f(W_\xi)\|_{L(U\times U,V)} < \frac{2\varepsilon}{T\sqrt{M}}.$$

Let $P = \{[\xi, v]\}$ of $[0,T]$ be a partition of $[0,T]$ with $d(P) < \delta$. Then

$$\begin{aligned}
&\|\overline{S}(f,P,\delta)\|_{L^2(\Omega,V)} \\
&\quad\leqslant (P)\sum \int_0^1 (1-t)\|(f(\widehat{W}) - f(W_\xi))(W_v - W_\xi)^{(2)}\|_{L^2(\Omega,V)}\,\mathrm{d}t \\
&\quad\leqslant (P)\sum \int_0^1 (1-t)\sqrt{\mathbb{E}[\|f(\widehat{W}) - f(W_\xi)\|^2_{L(U\times U,V)}\|(W_v - W_\xi)\|_U^4]}\,\mathrm{d}t \\
&\quad\leqslant \frac{1}{2}(P)\sum \sqrt{\Big(\frac{2\varepsilon}{T\sqrt{M}}\Big)^2 (v-\xi)^2 M} \leqslant \frac{1}{2}\frac{2\varepsilon}{T}(P)\sum (v-\xi) \leqslant \frac{1}{2}\frac{2\varepsilon}{T} T = \varepsilon.
\end{aligned}$$

This completes the proof. □

The following lemmas are respective consequences of the previous lemmas.

Lemma 4.19. *Let $f\colon [0,T]\times\Omega \to L(U,V)$ be an $\mathcal{IH}$-integrable process and suppose that $\mathbb{E}[\|f_t\|^2_{L_2(U_Q,V)}]$ is $AC^*[0,T]$. Then there exist a sequence of positive functions $\{\delta_n\}$ on $[0,T]$ and a sequence of positive numbers $\{\eta_n\}$ such that*

$$\lim_{n\to\infty} S(f, D_n \cup D_n^{\mathrm{c}}, \delta_n, \eta_n) = (\mathcal{IH})\int_0^T f_t\,\mathrm{d}W_t \quad \textit{in probability},$$

where D_n is any (δ_n,η_n)-fine belated partial division of $[0,T]$.

Lemma 4.20. *Let $f\colon [0,T]\times\Omega \to L(U\times U,V)$ be an adapted process such that such that $f_t(\omega)$ is symmetric, $\mathbb{E}[\|f_t\|^2_{L_2(U_Q\times U_Q,V)}]$ is bounded on $[0,T]$ and $M_t := \sum_{j=1}^{\infty} \lambda_j f_t(e_j,e_j)$ is Riemann integrable on $[0,T]$. Then there exists a sequence* *of positive numbers* $\{\delta_n\}$ *such that for any partition* $P_n = \{[\xi, v]\}$ *of* $[0, T]$ *with* $d(P_n) < \delta_n$*,*

$$\lim_{n\to\infty} \widehat{S}(f, P_n, \delta_n) = (\mathcal{R}) \int_0^T M_t \,\mathrm{d}t \quad \textit{in probability,}$$

where

$$\widehat{S}(f, P_n, \delta_n) := (P_n) \sum f_\xi(W_v - W_\xi, W_v - W_\xi).$$

Lemma 4.21. *Let* $f\colon U \to L(U \times U, V)$ *be a uniformly continuous function. Then there exists a sequence of positive numbers* $\{\delta_n\}$ *such that for any partition* $P_n = \{[\xi, v]\}$ *of* $[0, T]$ *with* $d(P_n) < \delta_n$*,*

$$\lim_{n\to\infty} \overline{S}(f, P_n, \delta_n) = 0 \quad \textit{in probability,}$$

where

$$\overline{S}(f, P, \delta) := (P) \sum \left\{ \int_0^1 (1 - t)(f(\widehat{W}) - f(W_\xi))(W_v - W_\xi)^{(2)} \,\mathrm{d}t \right\},$$

$\widehat{W} := W_\xi + t(W_v - W_\xi)$ *and* $t \in [0, 1]$.

We now state Itô's formula and give its proof using the above-mentioned lemmas.

Theorem 4.22 (*Itô's formula*)**.** *Let* $f\colon U \to V$ *be a function such that the first Fréchet derivative of* f *is continuous on* U *and the second Fréchet derivative of* f *is uniformly continuous on* U*. Suppose that*

(i) $f'(W_t)$ *is* $\mathcal{IH}$*-integrable on* $[0, T]$*;*
(ii) $\mathbb{E}[\|f'(W_t)\|^2_{L_2(U_Q,V)}]$ *is* $AC^*[0, T]$*;*
(iii) $\mathbb{E}[\|f''(W_t)\|^2_{L_2(U_Q\times U_Q,V)}]$ *is bounded on* $[0, T]$*;*
(iv) $M_t = \sum_{j=1}^{\infty} \lambda_j f''(W_t)(e_j, e_j)$ *is Riemann integrable on* $[0, T]$*.*

Then

$$f(W_T) - f(W_0) = (\mathcal{IH}) \int_0^T f'(W_t) \,\mathrm{d}W_t + \frac{1}{2}(\mathcal{R}) \int_0^T M_t \,\mathrm{d}t$$

for almost all $\omega \in \Omega$.

P r o o f. By Taylor's formula (Theorem 4.13), for any $v > \xi$,

$$\begin{aligned} f(W_v) - f(W_\xi) &= f'(W_\xi)(W_v - W_\xi) + \int_0^1 (1-t) f''(\widehat{W})(W_v - W_\xi)^{(2)}\,\mathrm{d}t \\ &= f'(W_\xi)(W_v - W_\xi) + \int_0^1 (1-t) f''(\widehat{W})(W_v - W_\xi)^{(2)}\,\mathrm{d}t \\ &\quad + \frac{1}{2} f''(W_\xi)(W_v - W_\xi)^{(2)} - \frac{1}{2} f''(W_\xi)(W_v - W_\xi)^{(2)} \\ &= f'(W_\xi)(W_v - W_\xi) + \frac{1}{2} f''(W_\xi)(W_v - W_\xi)^{(2)} \\ &\quad + \int_0^1 (1-t)(f''(\widehat{W}) - f''(W_\xi))(W_v - W_\xi)^{(2)}\,\mathrm{d}t, \end{aligned}$$

where $\widehat{W} := W_\xi + t(W_v - W_\xi)$ and $t \in [0,1]$. By Lemma 4.19, there exist a sequence of positive functions $\{\delta'_n\}$ on $[0,T]$ and a sequence of positive numbers $\{\eta'_n\}$ such that for any (δ'_n, η'_n)-fine belated partial division D'_n of $[0,T]$

$$(D'_n \cup {D'_n}^{\mathrm{c}}) \sum f'(W_\xi)(W_v - W_\xi) \to (\mathcal{IH}) \int_0^T f'(W_t)\,\mathrm{d}W_t$$

in probability. By Lemma 4.20, there exists a sequence of positive numbers $\{\delta^*_n\}$ such that for any partition $P_n = \{[\xi, v]\}$ of $[0,T]$ with $d(P_n) < \delta^*_n$,

$$(P_n) \sum f''(W_\xi)(W_v - W_\xi, W_v - W_\xi) \to (\mathcal{R}) \int_0^T M_t\,\mathrm{d}t$$

in probability. Moreover, by Lemma 4.21, there exists a sequence of positive numbers $\{\hat{\delta}_n\}$ such that for any partition $\widehat{P}_n = \{[\xi, v]\}$ of $[0,T]$ with $d(\widehat{P}_n) < \hat{\delta}_n$,

$$(\widehat{P}_n) \sum \left\{ \int_0^1 (1-t)(f''(\widehat{W}) - f''(W_\xi))(W_v - W_\xi)^{(2)}\,\mathrm{d}t \right\} \to 0$$

in probability. Choose $\delta_n(\xi) = \min\{\delta'_n(\xi), \delta^*_n, \hat{\delta}_n\}$ for all $\xi \in [0,T]$ and $\eta_n = \min\{\eta'_n, \delta^*_n, \hat{\delta}_n\}$. Then any (δ_n, η_n)-fine belated partial division of $[0,T]$ is also a (δ'_n, η'_n)-fine belated partial division of $[0,T]$. Let $D_n = \{(\xi, (\xi, v])\}$ be a (δ_n, η_n)-fine belated partial division of $[0,T]$. Then $(D_n \cup D_n^{\mathrm{c}})$ is a partition of $[0,T]$ with

$d(D_n \cup D_n^{\mathrm{c}}) < \delta_n^*$, $d(D_n \cup D_n^{\mathrm{c}}) < \hat{\delta}_n$. Hence,

$$(D_n \cup D_n^{\mathrm{c}}) \sum f'(W_\xi)(W_v - W_\xi) \to (\mathcal{IH}) \int_0^T f'(W_t)\,\mathrm{d}W_t,$$
$$(D_n \cup D_n^{\mathrm{c}}) \sum f''(W_\xi)(W_v - W_\xi, W_v - W_\xi) \to (\mathcal{R}) \int_0^T M_t\,\mathrm{d}t,$$

and

$$(D_n \cup D_n^{\mathrm{c}}) \sum \biggl\{ \int_0^1 (1-t)(f''(\widehat{W}) - f''(W_\xi))(W_v - W_\xi)^{(2)}\,\mathrm{d}t \biggr\} \to 0$$

in probability. Moreover,

$$(D_n \cup D_n^{\mathrm{c}}) \sum \{f(W_v) - f(W_\xi)\} \to f(W_T) - f(W_0).$$

Thus, the assertion holds for some subsequence. □

5. Conclusion and recommendations

In this paper, we use the generalized Riemann approach to define the integral of an operator-valued stochastic process with respect to a Hilbert space-valued Q-Wiener process. This newly defined integral, called Itô-Henstock integral, is more direct and minimizes the technicalities in the classical approach to stochastic integration in Hilbert spaces. We also establish the standard properties of the Itô-Henstock integral and formulate a version of Itô's formula. A worthwhile direction for further investigation is to verify the equivalence of the Itô-Henstock integral and the classical stochastic integral in Hilbert spaces defined in [4]. Another way of extending Itô's formula established in this paper is by considering functions of the form $f(t, X_t)$, where X_t is an Itô process.

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