EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A FRACTIONAL p-LAPLACIAN PROBLEM OF KIRCHHOFF TYPE VIA KRASNOSELSKII'S GENUS

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Abstract. We use the genus theory to prove the existence and multiplicity of solutions for the fractional p-Kirchhoff problem

$$\begin{cases} -\Big[M\Big(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\Big)\Big]^{p-1}(-\Delta)_p^s u = \lambda h(x,u) & \text{in } \Omega,\\ u=0 \quad \text{on } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where Ω is an open bounded smooth domain of $\mathbb{R}^N$, $p>1$, $N>ps$ with $s\in(0,1)$ fixed, $Q=\mathbb{R}^{2N}\setminus(C\Omega\times C\Omega)$, $\lambda>0$ is a numerical parameter, M and h are continuous functions.

Keywords: existence results; genus theory; fractional p-Kirchhoff problem

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1. Introduction

The purpose of this work is to investigate the existence and multiplicity of solutions of the fractional p-Kirchhoff problem

$$\text{(1)}\qquad \begin{cases} -\Big[M\Big(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\Big)\Big]^{p-1}(-\Delta)_p^s u = \lambda h(x,u) & \text{in } \Omega,\\ u=0 \quad \text{on } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where $p>1$, $N>ps$ with $s\in(0,1)$ fixed, $\Omega\subset\mathbb{R}^N$ is an open bounded smooth domain, $Q=\mathbb{R}^{2N}\setminus(C\Omega\times C\Omega)$ with $C\Omega=\mathbb{R}^N\setminus\Omega$, $\lambda>0$ is a numerical parameter and $M\colon\mathbb{R}^+\to\mathbb{R}^+$, $h\colon\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ are continuous functions that satisfy some

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suitable conditions which will be given later. Here $(-\Delta)_p^s$ is the fractional p-Laplacian operator which (up to normalization factors) can be defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,\mathrm{d}y \tag{2}$$

for $x \in \mathbb{R}^N$, where $B(x,\varepsilon)$ is the ball centered at $x \in \mathbb{R}^N$ with radius ε.

Fractional Laplacians have attracted much interest since they are connected with different applications and sometimes from the mathematical point of view the nonlocal character introduces difficulties that need some new approaches, see for instance [11], [22] and the references therein. Also, these operators arise in many different contexts, such as optimization, finance, flame propagation, minimal surfaces and water waves. For more details see [2].

The fractional Laplacian is nonlocal, that means that it does not act by pointwise differentiation but by a global integral with respect to a singular kernel, which causes the main difficulty when we want to study problems involving it. The fractional p-Laplacian operator becomes the p-Laplacian when $s = 1$ and in this case, problem (1) reduces to a p-Kirchhoff type problem, where different methods were proposed to study the existence of solutions (see [1], [8], [12], [17] and references therein). A natural question is whether or not the existence results obtained in this classical context can be extended to the nonlocal framework of the fractional Laplacian type operators. In this spirit, we study the existence of weak solutions for problem (1). For more details about problems related to it see for example [4], [6], [23], [24]. To the best of our knowledge, the literature for fractional Laplacian equations is still expanding and rather young.

The natural space to look for solutions is the fractional Sobolev space $W_0^{s,p}(\Omega)$ (see [11], [16]). To study problem (1), it is important to encode the boundary condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ in the weak formulation. Inspired by [5], [13], [21], we define the function space

$$X = \Big\{u\colon \mathbb{R}^N \to \mathbb{R},\ u \text{ is measurable},\ u|_\Omega \in L^p(\Omega),\ \frac{u(x)-u(y)}{|x-y|^{N/p+s}} \in L^p(Q)\Big\}.$$

The space X is endowed with a norm defined as

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \Big(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p} \tag{3}$$

and also

$$X_0 = \{u \in X\colon\ u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

with the norm

$$\|u\| = \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}, \tag{4}$$

which is equivalent to the norm in X (see [13], [16], [20]). We note that

$$C_c^2(\Omega) \subseteq X_0$$

(see [16], Lemma 1.20). Thus, the spaces X and X_0 are nonempty.

Denote by $L^\nu(\Omega)$ for $\nu \in [1, p_s^\star]$ the Lebesgue space of measurable functions on Ω, endowed with the norm $|u|_\nu = \left(\int_\Omega |u(x)|^\nu\,\mathrm{d}x\right)^{1/\nu}$, which is denoted by $|\cdot|_\nu$.

By [13], Lemma 2.4, the space $(X_0, \|\cdot\|)$ is a reflexive Banach space. These spaces for the case $p = 2$ are studied in [20]. Note that in (3) and (4), the integrals can be extended to all $\mathbb{R}^{2N}$ since $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.

The aim of this work is to study the existence and multiplicity of solutions of the p-Kirchhoff type problem in the fractional case, using the genus theory introduced by Krasnoselskii (see [3], [15]). Inspired by the ideas given in [9], [10], where in [10] the authors proved the existence and multiplicity of solutions to the following problem

$$\begin{cases} -\left[M\left(\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)\right]^{p-1} \Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded smooth domain of $\mathbb{R}^N$, $N > p > 1$, M and f are continuous functions, we will use the same type of assumptions to prove the existence and multiplicity of solutions to the p-Kirchhoff problem in the fractional case.

In view of our problem (1), we assume that:

$$|h(x,t)| \leqslant \gamma(x) + \delta(x)|t|^{q-1} \quad \text{for any } (x,t) \in \overline{\Omega} \times \mathbb{R}, \tag{h$_1$}$$

where $\gamma \in L^{\nu_1}(\Omega)$, $\delta \in L^{\nu_2}(\Omega)$ for some positive numbers ν_1, ν_2, q with

$$\nu_1 \geqslant \frac{p_s^\star}{p_s^\star - 1}, \quad \nu_2 \geqslant \frac{p_s^\star}{p_s^\star - q}, \quad q \in [1, p_s^\star), \tag{h$_2$}$$
$$h(x,t) = -h(x,-t) \quad \forall\, t \in \mathbb{R},\ \forall\, x \in \overline{\Omega},$$

and

$$H(x,u) = \int_0^u h(x,t)\,\mathrm{d}t > 0 \quad \text{for every } (x,u) \in \overline{\Omega} \times \mathbb{R} \setminus \{0\}. \tag{h$_3$}$$

Also, we assume that M satisfies the following condition:

There are positive constants A_0, A_1, B_0, B_1 and α, with $\alpha > q/p$ such that:

$$A_0 + A_1 t^\alpha \leqslant [M(t)]^{p-1} \leqslant B_0 + B_1 t^\alpha \quad \forall\, t \geqslant 0. \tag{M}$$

The contents of the paper are: In Section 2, we present preliminaries with the main tools on fractional Sobolev spaces and genus theory. In Section 3, we introduce a variational setting of the problem and we prove Theorem 1.1.

Our main result can be stated as follows.

Theorem 1.1. *Assume that* (h_1), (h_2), (h_3) *and* (M) *are satisfied. Then for any* $k \in \mathbb{N}$ *there exists* λ_k *such that when* $\lambda > \lambda_k$, *problem* (1) *has at least* k *distinct pairs of nontrivial solutions.*

2. Preliminary results

In this section, we collect some information to be used in the paper. Suppose that Ω is an open domain of $\mathbb{R}^N$, $s \in (0,1)$, $p \in [1,\infty)$. Define the fractional Sobolev space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) = \Big\{u \in L^p(\Omega)\colon \ \frac{|u(x)-u(y)|}{|x-y|^{N/p+s}} \in L^p(\Omega\times\Omega)\Big\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \Big(\int_\Omega |u|^p\,\mathrm{d}x + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p},$$

where the term

$$[u] = \Big(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}$$

is the so-called Gagliardo (semi) norm of u. $W_0^{s,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,p}(\Omega)}$. To study fractional Sobolev spaces in detail, we refer to [16], [11]. We remark that the norm in (3) and the Gagliardo norm are not the same because $\Omega\times\Omega$ is strictly contained in Q (this makes the classical fractional Sobolev space approach not sufficient for studying the problem).

In order to prove our main result, we need the following propositions:

Proposition 2.1 ([16]). *If* Ω *has a continuous boundary, then the embedding* $X_0 \hookrightarrow L^\nu(\Omega)$ *is compact for any* $\nu \in [1, p_s^\star)$ *and continuous for* $\nu = p_s^\star$.

Proposition 2.2 ([16])**.** *Let $p \in [1, \infty)$, $s \in (0, 1)$ and let Ω be a smooth open set in $\mathbb{R}^N$. Then the embedding*

$$W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$$

is continuous. Hence, there exists a constant $c(N, s, p) \geqslant 1$ such that

$$\|u\|_{W^{s,p}(\Omega)} \leqslant c(N, s, p)\|u\|_{W^{1,p}(\Omega)}$$

for any $u \in W^{1,p}(\Omega)$.

We also recall some preliminaries on genus theory (see for instance [14], [19]).

Definition 2.1. Let E be a real Banach space. Define the following set

$$\Lambda(E) = \{A \subset E\colon\ A \text{ is closed, } A \neq \emptyset,\ 0 \notin A \ \text{ and } \ -A = A\}.$$

If $A \in \Lambda(E)$, we call the genus of A the number $\gamma(A)$ defined as

$$\gamma(A) = \inf\{n \geqslant 1\colon\ \exists\, \varphi\colon\ A \to \mathbb{R}^n \setminus \{0\} \text{ continuous and odd}\}.$$

For the sake of convenience, $\gamma(\emptyset) = 0$. As always, if such an integer n does not exist, we have $\gamma(A) = \infty$.

Theorem 2.1 ([14])**.** *Let $E = \mathbb{R}^N$ and Ω be an open, bounded and symmetric subset of E with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Corollary 2.1. *Let S be the unit sphere in E. Then*
(1) *$\gamma(S^{N-1}) = N$,*
(2) *if E is of infinite dimension and separable, we have $\gamma(S) = \infty$.*

Now, we list the essential properties of the genus that we will be using in the proof of Theorem 1.1.

Proposition 2.3 ([14])**.** *Let A, A_1, $A_2 \in \Lambda$. Then:*
- ▷ *$\gamma(A) \geqslant 0$, $\gamma(A) = 0 \Leftrightarrow A = \emptyset$,*
- ▷ *if $A_1 \subset A_2$, then $\gamma(A_1) \leqslant \gamma(A_2)$,*
- ▷ *$\gamma(A_1 \cup A_2) \leqslant \gamma(A_1) + \gamma(A_2)$,*
- ▷ *if there exists an odd map $g \in C(A_1, A_2)$, then $\gamma(A_1) \leqslant \gamma(A_2)$,*
- ▷ *if there exists an odd homeomorphism $g\colon A_1 \to A_2$, then $\gamma(A_1) = \gamma(A_2)$.*

In order to prove the Palais-Smale compactness condition, we recall the following definition:

Definition 2.2. Let E be a Banach space and let $J \in C^1(E, \mathbb{R})$. If a sequence $(u_n) \subset E$ for which $(J(u_n))$ is bounded and $J'(u_n) \to 0$ when $n \to \infty$ in E', possesses a convergent subsequence, then we say that J satisfies the Palais-Smale condition (denoted as (PS)).

We now state a theorem due to Clarke.

Theorem 2.2 ([7], [19])**.** *Let $J \in C^1(E, \mathbb{R})$ be a functional satisfying the Palais-Smale condition. Also suppose that:*

▷ *J is bounded from below and even,*

▷ *there is a compact set $K \in \Lambda$ such that $\gamma(K) = k$ and $\sup\limits_{u \in K} J(u) < J(0)$.*

Then J possesses at least k pairs of distinct critical points and their corresponding critical values are less than $J(0)$.

3. Main result

In this section we will discuss the existence of weak solutions for problem (1).

Definition 3.1. We say that u is a weak solution of problem (1) if u satisfies

$$[M(\|u\|^p)]^{p-1} \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} (v(x)-v(y))\,\mathrm{d}x\,\mathrm{d}y = \lambda \int_\Omega h(x,u) v\,\mathrm{d}x$$

for all $v \in X_0$.

Looking for a solution of problem (1) is equivalent to finding a critical point of the associated Euler-Lagrange functional $J\colon X_0 \to \mathbb{R}$ defined as

$$J(u) = \frac{1}{p} M^\star(\|u\|^p) - \lambda \int_\Omega H(x, u(x))\,\mathrm{d}x,$$

where $M^\star(t) = \int_0^t [M(\tau)]^{p-1}\,\mathrm{d}\tau$.

Note that J is a $C^1(X_0, \mathbb{R})$ functional and

$$J'(u)v = [M(\|u\|^p)]^{p-1} \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} (v(x)-v(y))\,\mathrm{d}x\,\mathrm{d}y - \lambda \int_\Omega h(x,u) v\,\mathrm{d}x$$

for any $v \in X_0$. Thus, critical points of J are weak solutions of (1).

To prove Theorem 1.1, we will need the following two lemmas.

Lemma 3.1. *J is bounded from below.*

P r o o f. Using conditions (M), (h_1), Proposition 2.1 and Hölder's inequality, we have

$$\begin{aligned} J(u) &\geqslant \frac{1}{p}\int_0^{\|u\|^p}(A_0+A_1\tau^\alpha)\,\mathrm{d}\tau-\lambda\int_\Omega\gamma(x)|u(x)|\,\mathrm{d}x-\lambda\int_\Omega\delta(x)\frac{|u(x)|^q}{q}\,\mathrm{d}x\\ &\geqslant \frac{A_0}{p}\|u\|^p+\frac{A_1}{p(\alpha+1)}\|u\|^{p(\alpha+1)}-\lambda|\gamma|_{\nu_1}|u|_{\nu_1'}-\frac{\lambda}{q}|\delta|_{\nu_2}|u|^q_{q\nu_2'}\\ &\geqslant \frac{A_0}{p}\|u\|^p+\frac{A_1}{p(\alpha+1)}\|u\|^{p(\alpha+1)}-\lambda c_1\|u\|-\lambda c_2\|u\|^q, \end{aligned}$$

where $c_1=c_3|\gamma|_{\nu_1}$, $c_2=c_4^q|\delta|_{\nu_2}q^{-1}$ and c_3, c_4 are constants of continuous embeddings of X_0 in $L^\nu(\Omega)$ for all $\nu\in[1,p_s^\star]$. Since $\alpha>qp^{-1}$, we get $p(\alpha+1)>q$, so J is bounded from below. □

Lemma 3.2. *J satisfies the* (PS) *condition.*

P r o o f. Let (u_n) be a Palais-Smale sequence for J, that is $(J(u_n))$ is bounded and $J'(u_n)\to 0$ when $n\to\infty$ in X_0'. Thus, there exists a positive constant c such that

$$|J(u_n)|\leqslant c\quad \forall\, n\in\mathbb{N}.$$

Using the above lemma, we obtain

$$c\geqslant J(u_n)\geqslant\frac{A_0}{p}\|u_n\|^p+\frac{A_1}{p(\alpha+1)}\|u_n\|^{p(\alpha+1)}-\lambda c_1\|u_n\|-\lambda c_2\|u_n\|^q.$$

Since $\alpha>qp^{-1}$, we have $p(\alpha+1)>q$ and then we conclude that the sequence (u_n) is bounded in X_0. Thus, passing to a subsequence if necessary, still denoted by (u_n), we have

$$\|u_n\|^p\to t_0 \tag{5}$$

and there exists $u\in X_0$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } X_0,\\ u_n &\to u \quad \text{in } L^\nu(\Omega),\ \forall\,\nu\in[1,p_s^\star),\\ u_n(x) &\to u(x) \quad \text{a.e. } x\in\Omega. \end{aligned} \tag{6}$$

If $t_0 = 0$, then the proof is finished. If $t_0 > 0$, then from (5) and since M is a continuous function we get

$$M(\|u_n\|^p) \to M(t_0) \tag{7}$$

as $n \to \infty$. Thus, for n sufficiently large, $M(\|u_n\|^p) \geqslant \overline{c} > 0$ for a constant $\overline{c}$.

Let us now consider the sequence

$$P_n = J'(u_n)u_n + \lambda \int_\Omega h(x, u_n)u_n \,\mathrm{d}x - J'(u_n)u - \lambda \int_\Omega h(x, u_n)u \,\mathrm{d}x.$$

From (h_1), the Lebesgue dominated convergence theorem and Proposition 2.1 we get

$$\int_\Omega h(x, u_n)u_n \,\mathrm{d}x \to \int_\Omega h(x, u)u \,\mathrm{d}x,$$
$$\int_\Omega h(x, u_n)u \,\mathrm{d}x \to \int_\Omega h(x, u)u \,\mathrm{d}x,$$

so we have that $P_n \to 0$ as $n \to \infty$ and it is easy to see that

$$P_n = [M(\|u_n\|^p)]^{p-1}\|u_n\|^p - [M(\|u_n\|^p)]^{p-1} \int_Q \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{N+ps}}(u(x) - u(y)) \,\mathrm{d}x \,\mathrm{d}y.$$

Also, we set

$$S_n = -[M(\|u_n\|^p)]^{p-1}\Big[\int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}(u_n(x) - u_n(y)) \,\mathrm{d}x \,\mathrm{d}y - \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \,\mathrm{d}x \,\mathrm{d}y\Big].$$

By the weak convergence of the sequence (u_n), we obtain that $S_n \to 0$. Hence

$$P_n + S_n = [M(\|u_n\|^p)]^{p-1} \int_Q \Big[\frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{N+ps}} - \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}\Big]((u_n(x) - u_n(y)) - (u(x) - u(y))) \,\mathrm{d}x \,\mathrm{d}y.$$

Using the elementary inequalities from [18], we have for all $x, y \in \mathbb{R}$,

$$(|x|^{p-2}x - |y|^{p-2}y)(x - y) \geqslant c_p|x - y|^p \quad \text{if } p \geqslant 2, \tag{8}$$

or

$$(|x|^{p-2}x-|y|^{p-2}y)(x-y)\geqslant c_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}}\quad\text{if } 2>p>1. \tag{9}$$

We distinguish two cases:

Case 1: If $p\geqslant 2$, then using (8) we obtain

$$P_n+S_n\geqslant(\overline{c})^{p-1}c_p\int_Q\frac{|(u_n(x)-u(x))-(u_n(y)-u(y))|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y,$$

and then

$$P_n+S_n\geqslant(\overline{c})^{p-1}c_p\|u_n-u\|^p.$$

Thus, we conclude that $\|u_n-u\|\to 0$ as $n\to\infty$ in X_0.

Case 2: If $1<p<2$, then by (9) we get

$$[(|x|^{p-2}x-|y|^{p-2}y)(x-y)]^{p/2}\geqslant c_p^{\star}\frac{|x-y|^p}{(|x|^p+|y|^p)^{(2-p)/2}}, \tag{10}$$

where $c_p^{\star}=c_p^{p/2}/\sqrt{2}$. Then using (10), we have

$$\begin{aligned}(11)\quad & c_p^{\star}|(u_n(x)-u_n(y))-(u(x)-u(y))|^p\\ &\leqslant(|u_n(x)-u_n(y)|^p+|u(x)-u(y)|^p)^{1-p/2}\\ &\quad\times[(|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u(x)-u(y)|^{p-2}(u(x)-u(y)))\\ &\quad\times((u_n(x)-u_n(y))-(u(x)-u(y)))]^{p/2}.\end{aligned}$$

Dividing the two sides of inequality (11) by $|x-y|^{N+ps}$ and integrating on Q, we obtain from Hölder's inequality

$$\begin{aligned}&c_p^{\star}\int_Q\frac{|(u_n(x)-u_n(y))-(u(x)-u(y))|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant\Big[\int_Q\Big(\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}+\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\Big)\,\mathrm{d}x\,\mathrm{d}y\Big]^{(2-p)/2}\\ &\quad\times\Big[\int_Q\Big(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\Big)\\ &\quad\times\big((u_n(x)-u_n(y))-(u(x)-u(y))\big)\,\mathrm{d}x\,\mathrm{d}y\Big]^{p/2}.\end{aligned}$$

Then

$$c_p^{\star}\varrho_n^{p/2}\|u_n-u\|^p\leqslant[\|u_n\|^p+\|u\|^p]^{(2-p)/2}(P_n+S_n)^{p/2},$$

where $\varrho_n = [M(\|u_n\|^p)]^{p-1}$. Since $P_n + S_n \to 0$ as $n \to \infty$ and using (7) and the fact that $[\|u_n\|^p + \|u\|^p]$ is bounded, we obtain

$$\|u_n - u\| \to 0 \quad \text{as} \quad n \to \infty.$$

In both cases we deduce that (u_n) satisfies the (PS) condition. $\square$

P r o o f of Theorem 1.1. Using Lemma 1.24 in [16], we have

$$X_0 = \{u \in W^{s,p}(\mathbb{R}^N)\colon\ u = 0 \ \text{ a.e. in } C\Omega\},$$

and set

$$S = \{u \in W^{1,p}(\mathbb{R}^N)\colon\ u = 0 \ \text{ a.e. in } C\Omega\}.$$

Clearly, by Proposition 2.2, we have $S \subset X_0$. Let $\{e_1, e_2, \ldots\}$ be a Schauder basis of the space S and for each $k \in \mathbb{N}$ consider X_k the subspace of S generated by the k vectors $\{e_1, e_2, \ldots, e_k\}$. For $\varrho > 0$ consider

$$K_k(\varrho) = \Big\{u \in X_k\colon\ \|u\|^2 = \sum_{i=1}^{k} \xi_i^2 = \varrho^2\Big\}.$$

It follows from hypothesis (h_3) that $\int_\Omega H(x, u(x))\,\mathrm{d}x > 0$ for any $u \in K_k(\varrho)$. Then

$$\mu_k = \inf_{u \in K_k(\varrho)} \int_\Omega H(x, u(x))\,\mathrm{d}x$$

is strictly positive, because $K_k(\varrho)$ is compact. Let

$$\lambda_k = \frac{1}{\mu_k}\Big(\frac{B_0}{p}\varrho^p + \frac{B_1 \varrho^{p(\alpha+1)}}{p(\alpha+1)}\Big),$$

and note $\lambda_k > 0$. Then using (M), for $\lambda > \lambda_k$ and for any $u \in K_k(\varrho)$ we have

$$J(u) \leqslant \frac{B_0}{p}\varrho^p + \frac{B_1 \varrho^{p(\alpha+1)}}{p(\alpha+1)} - \lambda\mu_k < \frac{B_0}{p}\varrho^p + \frac{B_1 \varrho^{p(\alpha+1)}}{p(\alpha+1)} - \lambda_k\mu_k = 0,$$

which implies that

$$\sup_{K_k(\varrho)} J(u) < 0 = J(0).$$

On the other hand, we consider the following odd homeomorphism $\chi\colon\ K_k(\varrho) \to S^{k-1}$ defined as $\chi(u) = (\theta_1, \theta_2, \ldots, \theta_k)$, where S^{k-1} is the sphere in $\mathbb{R}^k$. From Theorem 2.1 and Proposition 2.3, we conclude that $\gamma(K_k(\varrho)) = k$. Moreover, from (h_2), J is even. Thanks to Theorem 2.2, J has at least k ditinct pairs of nontrivial solutions. $\square$

E x a m p l e 3.1. We consider $\Omega \subset \mathbb{R}^N$ an open bounded smooth domain, $s = \frac{1}{2}$, $N > 1$, $q \in [1, 2^\star_{1/2})$ and $\alpha > \frac{1}{2}q$. Set $h(x,t) = \delta(x)t|t|^{q-2}$ and $M(t) = 1 + t^\alpha$. Then for any $k \in \mathbb{N}$ there exists λ_k such that when $\lambda > \lambda_k$, the problem

$$\begin{cases} -(1 + \|u\|^{2\alpha})(-\Delta)^{1/2}u = \lambda\delta(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N/\Omega, \end{cases}$$

has at least k distinct pairs of nontrivial solutions.

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