SOME EQUIVALENT METRICS FOR BOUNDED NORMAL OPERATORS

Mohammad Reza Jabbarzadeh, Rana Hajipouri, Tabriz

Received December 5, 2016. First published October 16, 2017.
Communicated by Vladimír Müller

Abstract. Some stronger and equivalent metrics are defined on $\mathcal{M}$, the set of all bounded normal operators on a Hilbert space H and then some topological properties of $\mathcal{M}$ are investigated.

Keywords: Hilbert space; normal operator; equivalent metrics; composition operator

MSC 2010: 47A05

1. Introduction and preliminaries

Let H be a separable, infinite dimensional, complex Hilbert space with inner product $\langle\,,\rangle$ and let $\mathcal{B}(H)$ denote the algebra of all bounded linear operators on H. The problem of the topological structure of $\mathcal{C}(H)$, the subsets of closed and densely defined linear operators on H has been considered starting with the paper by Cordes and Labrousse [2]; see also [7]. They prove that the metric distance between two densely defined unbounded operators A and B may be taken as $\|(I+AA^*)^{-1}-(I+BB^*)^{-1}\|$. As the authors show, this metric defines the same topology for bounded operators as the ordinary metric $\|A-B\|$. For $A \in \mathcal{C}(H)$, let α denote the contraction defined as $\alpha(T) = A(1+A^*A)^{-1/2}$. Kaufman [5] studies a metric δ on $\mathcal{C}(H)$ defined as $\delta(A,B) = \|\alpha(A)-\alpha(B)\|$ and then the author discusses connections between δ-convergence and strong-operator-topology convergence. Also, he shows that this metric is stronger than the gap metric d (see [4], page 197) and not equivalent to it. In [6], Kittaneh presents quantitative improvements of the result of Kaufman [5] concerning equivalence of three metrics on the space of bounded linear operators on a Hilbert space. In [1], Benharrat and Messirdi defined some new strictly stronger metrics than the gap metric d and characterized the closure with respect to these metrics of the subset $\mathcal{B}(H)$ of bounded elements of $\mathcal{C}(H)$.

DOI: 10.21136/MB.2017.0101-16

Let $\mathcal{M}$ be the subset of bounded normal operators in $\mathcal{B}(H)$, $A \in \mathcal{M}$ and let $0 < a < \|A\|^{-1}$. In this paper, by motivation of the above mentioned results, we shall replace $1 + A^*A$ with $I + a^2A^*A + a^4(A^*)^2A^2 + \ldots$, and then we obtain some analogous results on topological properties of $\mathcal{M}$.

In Section 2, we show that $K_a(A) := \sum\limits_{n=0}^{\infty} a^{2n}A^{*n}A^n$ is positive, invertible and then we obtain the relation between the operators $K_a(A)$, $K_a^{-1}(A)$ and $(K_a(A))^{-1/2}$ in the case when A is normal. Moreover, we introduce some special types of metrics on normal operators in $\mathcal{B}(H)$ and then we compare the topologies induced by these metrics.

In Section 3, inspired by definition of bisecting for $A \in \mathcal{C}(H)$ in [8], we define $\widetilde{A}_a$ for $A \in \mathcal{M}$. Then using $\widetilde{A}_a$ and the metrics defined in Section 2, we introduce the $F_1, \ldots, F_4$ maps on $\mathcal{M}$ with different metrics into $\mathcal{M}$ with the aid of usual operator norm. Then we will proceed on investigating the continuity of these maps. At the end, as an example we determine $K_a(C_\varphi)$, $R_a(C_\varphi)$, $S_a(C_\varphi)$, $(\widetilde{C}_\varphi)_a$ for $C_\varphi \in \mathcal{M}$, where $C_\varphi(f) = f \circ \varphi$ is the composition operator on $L^2(\Sigma)$.

2. Stronger and equivalent metrics on $\mathcal{M}$

For $A \in \mathcal{B}(H)$, let A^*, $\mathcal{N}(A)$, $\mathcal{R}(A)$, $r(A)$ and $\|A\|$ denote the adjoint, the null space, the range, the spectral radius and the usual operator norm of A, respectively. Note that $r(A) = \lim\limits_{n\to\infty} \|A^n\|^{1/n} \leqslant \|A\|$ and that the equality holds if A is normal. A is called positive if $\langle Ax, x\rangle \geqslant 0$ holds for every $x \in H$ in which case we write $A \geqslant 0$. For an operator $A \in \mathcal{B}(H)$ let $0 < a < (r(A))^{-1}$ be an arbitrary but fixed number. Define $K_a(A) = \sum\limits_{n=0}^{\infty} a^{2n}A^{*n}A^n$. The definition of $K_a(A)$ is due to Gilfeather [3], Lambert and Petrovic [9].

Lemma 2.1. *Let $A \in \mathcal{B}(H)$. Then $0 \leqslant K_a(A) \in \mathcal{B}(H)$ and $K_a(A)$ is invertible with $\|K_a^{-1}(A)\| \leqslant 1$.*

Proof. Since $\lim\limits_{n\to\infty} \|a^{2n}A^{*n}A^n\|^{1/n} < (r(A))^{-2} \lim\limits_{n\to\infty} \|A^n\|^{2/n} = 1$, so the infinite series $K_a(A)$ converges absolutely. Also, for all $x \in H$ we have

$$\langle K_a(A)(x), x\rangle = \sum_{n=0}^{\infty} a^{2n}\|A^n(x)\|^2 \geqslant 0.$$

Thus,

$$\big\|\sqrt{K_a(A)(x)}\big\|^2 = \langle K_a(A)(x), x\rangle = \|x\|^2 + \sum_{n=1}^{\infty} a^{2n}\|A^n(x)\|^2 \geqslant \|x\|^2,$$

and so

$$R(\sqrt{K_a(A)}) = \overline{R(\sqrt{K_a(A)})} = N(\sqrt{K_a(A)})^{\perp} = H.$$

It follows that $\sqrt{K_a(A)}$ and hence $K_a(A)$ is invertible. Now, replacing x by $(K_a(A))^{-1/2}(x)$ we obtain $\|(K_a(A))^{-1/2}(x)\| \leqslant \|x\|$. This implies that

$$\frac{1}{\|K_a(A)\|} \leqslant \frac{1}{\|\sqrt{K_a(A)}\|^2} \leqslant 1.$$

□

For $A \in \mathcal{B}(H)$ set $R_a(A) = (K_a(A))^{-1}$ and $S_a(A) = \sqrt{R_a(A)}$. Then by Lemma 2.1, $R_a(A)$ and $S_a(A) = (K_a(A))^{-1/2}$ are positive and $S_a(A)$ is a contraction.

Moreover, when A is a normal operator, i.e. $AA^* = A^*A$, then $R_a(A) = R_a(A^*)$, $AR_a(A) = R_a(A)A$ and $A^*R_a(A) = R_a(A)A^*$.

Recall that for $A \in \mathcal{C}(H)$, the fundamental properties of $R_A = (I + A^*A)^{-1}$ and $S_A = (I + A^*A)^{-1/2}$ have been investigated by many authors, e.g. [2], [1]. In the following lemma we obtain a relationship between the concepts of $R_a(A)$ and $S_a(A)$ when $A \in \mathcal{B}(H)$ is a normal operator.

Lemma 2.2. *Let $A \in \mathcal{B}(H)$ be a normal operator and let $n \in \mathbb{N} \cup \{0\}$. Then the following assertions hold.*

(a) $A^n R_a(A) = R_a(A)A^n$;
(b) $A^n S_a(A) = S_a(A)A^n$;
(c) $S_a(A)(K_a(A) - I)S_a(A) = I - R_a(A)$;
(d) $\sqrt{K_a(A) - I} = a|A|(S_a(A))^{-1}$;
(e) $R_a(A) = I - a^2|A|^2$;
(f) $\mathcal{N}(S_a(A)) \cap \mathcal{N}(A) = \{0\}$.

Proof. (a) Since A is normal, from direct computations we obtain that

$$\begin{aligned} A^n K_a(A) &= A^n(I + a^2A^*A + a^4(A^*)^2A^2 + \ldots) \\ &= A^n + a^2A^nA^*A + a^4A^n(A^*)^2A^2 + \ldots \\ &= (I + a^2AA^* + a^4A^2(A^*)^2 + \ldots)A^n = K_a(A^*)A^n = K_a(A)A^n. \end{aligned}$$

Therefore, the inverse of $K_a(A)$ is also commute with all A^n.

(b) Since $A^nR_a(A) = R_a(A)A^n$, it follows that $A^nP(R_a(A)) = P(R_a(A))A^n$, where P is any polynomial. Now let $\{P_m\}$ be a sequence of polynomials converging uniformly to a continuous function g. Then for each $x, y \in H$ we have

$$\begin{aligned} \langle A^n g(R_a(A))(x), y\rangle &= \lim_{m\to\infty} \langle P_m(R_a(A))(x), (A^n)^*y\rangle \\ &= \lim_{m\to\infty} \langle P_m(R_a(A))A^n(x), y\rangle \quad \text{(by part (a))} \\ &= \langle g(R_a(A))A^n(x), y\rangle. \end{aligned}$$

Thus, $A^n g(R_a(A)) = g(R_a(A))A^n$. Let g be a square root function. Consequently, $A^n\sqrt{R_a(A)} = \sqrt{R_a(A)}A^n$, and so $A^n S_a(A) = S_a(A)A^n$.

(c) Since $R_a(A) = S_a^2(A)$, then

$$\begin{aligned}I - R_a(A) &= (R_a^{-1}(A) - I)R_a(A) = a^2A^*AR_a(A) + a^4(A^*)^2A^2R_a(A) + \dots\\ &= a^2A^*S_a(A)S_a(A)A + a^4(A^*)^2S_a(A)S_a(A)A^2 + \dots\\ &= a^2A^*S_a(A)AS_a(A) + a^4(A^*)^2S_a(A)A^2S_a(A) + \dots\\ &= \sum_{n=1}^{\infty} a^{2n}(A^*)^nS_a(A)A^nS_a(A) = S_a(A)(K_a(A) - I)S_a(A).\end{aligned}$$

(d) Normality of A implies that

$$K_a(A) - I = a^2A^*A(I + a^2A^*A + a^4(A^*)^2A^2 + \dots) = a^2|A|^2K_a(A).$$

Thus, $\sqrt{K_a(A) - I} = a|A|\sqrt{K_a(A)} = a|A|(S_a(A))^{-1}$.

(e) It follows from (c) and (d).

(f) It suffices to show that $\|S_a(A)u\|^2 + \|a|A|u\|^2 = \|u\|^2$ for all $u \in H$. For this, let $u \in H$. Then by (e) we have

$$\begin{aligned}\|S_a(A)u\|^2 + \|a|A|u\|^2 &= \langle S_a(A)u, S_a(A)u\rangle + \langle a|A|u, a|A|u\rangle\\ &= \langle u, R_a(A)u\rangle + \langle u, a^2|A|^2u\rangle\\ &= \langle u, R_a(A)u\rangle + \langle u, (I - R_a(A))u\rangle = \langle u, u\rangle = \|u\|^2.\end{aligned}$$

□

Lemma 2.3 ([2]). *Let A be closed. Then*

$$\Pi_{\mathcal{G}(A)} = \begin{bmatrix} R_A & A^*R_A^* \\ AR_A & I - R_A^* \end{bmatrix},$$

where $\Pi_{\mathcal{G}(A)}$ denotes the orthogonal projection onto $\mathcal{G}(A) = \{(x, Ax)\colon\ x \in \mathcal{D}(A)\}$.

Now inspired by matrix $\Pi_{\mathcal{G}(A)}$, we define $\Pi_a(A) \in \mathcal{B}(H \otimes H)$ for $A \in \mathcal{M}$:

$$\Pi_a(A) = \begin{bmatrix} R_a(A) & a|A|S_a(A) \\ a|A|S_a(A) & I - R_a(A) \end{bmatrix}.$$

In [1], Benharrat and Messirdi introduced metrics $g_G(T, S)$, $p_G(T, S)$, $q_G(T, S)$ and $\Sigma_G(T, S)$ for $S, T \in \mathcal{C}(H_1, H_2)$ and a positive bijection $G \in \mathcal{L}^+(H_1)$.

Now, inspired by these metrics we define special types of metrics on $\mathcal{M}$:

$$\begin{aligned}
d^{[1]}_{(a,b)}(A,B) &= \|\Pi_a(A) - \Pi_b(B)\|;\\
d^{[2]}_{(a,b)}(A,B) &= \sqrt{\|R_a(A) - R_b(B)\|^2 + \|a|A|S_a(A) - b|B|S_b(B)\|^2};\\
d^{[3]}_{(a,b)}(A,B) &= \|a|A| - b|B|\|;\\
d^{[4]}_{(a,b)}(A,B) &= \sqrt{2\|a|A| - b|B|\|^2 + 2\|S_a(A) - S_b(B)\|^2},
\end{aligned}$$

where $0 < a < \|A\|^{-1}$ and $0 < b < \|B\|^{-1}$ are arbitrary but fixed numbers, whenever A and B are nonzero elements of $\mathcal{M}$. Note that $d^{[3]} \leqslant d^{[4]}$. Hence, the topology induced from the metric $d^{[4]}$ on $\mathcal{M}$ is stronger than that induced from $d^{[3]}$.

Lemma 2.4 ([6]).

(a) *If $A, B \in \mathcal{B}(H)$ are positive, then*

$$\|A - B\| \leqslant \sqrt{\|A^2 - B^2\|}.$$

(b) *If $T \in \mathcal{B}(H \oplus H)$ and*

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

then $\|T\|^2 \leqslant \|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2$.

It was proved that in [1] the topology induced from the metric $g_G(T,S)$ on $\mathcal{C}(H_1, H_2)$ is strictly stronger than that induced from $p_G(T,S)$. But the following proposition proves that the metrics $d^{[1]}$ and $d^{[2]}$ on $\mathcal{M}$ generate the same topology.

Proposition 2.5. *The topology induced from the metric $d^{[1]}$ on $\mathcal{M}$ is equivalent to the topology induced from $d^{[2]}$ on $\mathcal{M}$.*

Proof. Let $A, B \in \mathcal{M}$. Evidently, $d^{[2]}_{(a,b)}(A,B) \leqslant d^{[1]}_{(a,b)}(A,B)$. On the other hand, by Lemma 2.4 (b) we have

$$\|\Pi_a(A) - \Pi_b(B)\|^2 \leqslant 2\|R_a(A) - R_b(B)\|^2 + 2\|a|A|S_a(A) - b|B|S_b(B)\|^2.$$

Thus, $d^{[1]}_{(a,b)}(A,B) \leqslant \sqrt{2} d^{[2]}_{(a,b)}(A,B)$. □

Lemma 2.6. *Let A and B be two nonzero elements of $\mathcal{B}(H)$. Then*

$$\left\| \frac{A}{\|A\|} - \frac{B}{\|B\|} \right\| \leqslant \frac{2\|A - B\|}{\|A\|}.$$

P r o o f. Since $\|B\| - \|A\|$ is not greater than $\|A-B\|$, so

$$\|B\|\|A\|\Big\|\frac{A}{\|A\|} - \frac{B}{\|B\|}\Big\| \leqslant \|B\|\|A-B\| + \|B\|(\|B\| - \|A\|) \leqslant 2\|B\|\|A-B\|.$$

The result follows. □

Now, let A and B be two nonzero normal elements of $\mathcal{B}(H)$. Then $r(A) = \|A\|$ and $r(B) = \|B\|$. For $0 < \alpha < 1$ put $a_\alpha = \alpha\|A\|^{-1}$ and $b_\alpha = \alpha\|B\|^{-1}$. By Lemma 2.6 we obtain

$$\|a_\alpha A - b_\alpha B\| = \Big\|\frac{\alpha A}{\|A\|} - \frac{\alpha B}{\|B\|}\Big\| \leqslant \frac{2\alpha\|A-B\|}{\|A\|}.$$

In the following theorem, we show that $d^{[i]}_{(a_\alpha,b_\alpha)} < \|\cdot\|$ for $i = 3,4$ on $\mathcal{M}$. This is why, in the study carried out by Benharrat and Messirdi, it was found that the restriction of the metric $q_G(T,S)$ to $\mathcal{L}(H_1,H_2)$ is equivalent to the operator norm.

Theorem 2.7. *The topology induced from the operator norm on $\mathcal{M}$ is strictly stronger than that induced from $d^{[i]}_{(a_\alpha,b_\alpha)}$ for $i = 3,4$ on $\mathcal{M}$.*

P r o o f. Let $A, B \in \mathcal{M}$. Let $A \neq 0$ and $B = 0$. Then by Lemma 2.4 (a) we have

$$\|S_{a_\alpha}(A) - I\| = \|\sqrt{I - a_\alpha^2|A|^2} - I\| \leqslant \sqrt{\|a_\alpha^2|A|^2\|} \leqslant a_\alpha\|A\|$$

and $\|a_\alpha|A|\| = a_\alpha\|A\|$. It follows that $d^{[3]}_{(a_\alpha,b_\alpha)}(A,0) = a_\alpha\|A\|$ and

$$d^{[4]}_{(a_\alpha,b_\alpha)}(A,0) = \sqrt{2(\|a_\alpha|A|\|)^2 + 2\|S_{a_\alpha}(A) - I\|^2} \leqslant 2a_\alpha\|A\|.$$

Now, let A and B be two nonzero elements of $\mathcal{M}$. Then by Lemma 2.4 (a) and Lemma 2.6 we have

$$\begin{aligned} d^{[3]}_{(a_\alpha,b_\alpha)}(A,B) &= \|a_\alpha|A| - b_\alpha|B|\| \leqslant \sqrt{\|a_\alpha^2 A^*A - b_\alpha^2 B^*B\|} \\ &\leqslant \sqrt{\|a_\alpha A^* - b_\alpha B^*\|\|a_\alpha A\| + \|b_\alpha B^*\|\|a_\alpha A - b_\alpha B\|} \\ &= \sqrt{(\|a_\alpha A\| + \|b_\alpha B\|)\|a_\alpha A - b_\alpha B\|} \\ &\leqslant \sqrt{\|a_\alpha A\| + \|b_\alpha B\|}\sqrt{\frac{2\alpha\|A-B\|}{\|A\|}}. \end{aligned}$$

Also, since

$$\begin{aligned} \|S_{a_\alpha}(A) - S_{b_\alpha}(B)\| &= \|\sqrt{I - a_\alpha^2|A|^2} - \sqrt{I - b_\alpha^2|B|^2}\| \\ &\leqslant \sqrt{\|(I - a_\alpha^2|A|^2) - (I - b_\alpha^2|B|^2)\|} \\ &= \sqrt{\|a_\alpha^2 A^*A - b_\alpha^2 B^*B\|} \leqslant \sqrt{\|a_\alpha A\| + \|b_\alpha B\|}\sqrt{\frac{2\alpha\|A-B\|}{\|A\|}}, \end{aligned}$$

we get that

$$d^{[4]}_{(a_\alpha,b_\alpha)}(A,B) \leqslant \sqrt{4(\|a_\alpha A\| + \|b_\alpha B\|)\frac{2\alpha\|A-B\|}{\|A\|}}.$$

This completes the proof. □

Recall that in the study carried out by Benharrat and Messirdi in [1], it was proved that the topology induced from the metric $q_G(T,S)$ on $\mathcal{C}(H_1,H_2)$ is strictly stronger than that induced from $g_G(T,S)$. However, in the following theorem we show that $d^{[1]} \cong d^{[3]}$.

Theorem 2.8. *The topology induced from the metric $d^{[1]}$ on $\mathcal{M}$ is equivalent to the topology induced from to the metric $d^{[3]}$ on $\mathcal{M}$.*

Proof. Let $A,B \in \mathcal{M}$. Then by Lemma 2.4 (a) and the definition of $d^{[i]}$ for $i=1,3$ we have

$$\begin{aligned}
d^{[3]}_{(a,b)}(A,B) &= \|a|A| - b|B|\| = \|a|A|S_a(A)S_a^{-1}(A) - b|B|S_b(B)S_b^{-1}(B)\| \\
&\leqslant \|a|A|S_a(A) - b|B|S_b(B)\|\|S_a^{-1}(A)\| \\
&\quad + \|b|B|S_b(B)\|\|S_a^{-1}(A) - S_b^{-1}(B)\| \\
&\leqslant d^{[1]}_{(a,b)}(A,B)\|S_a^{-1}(A)\| + \|b|B|S_b(B)\|\sqrt{\|S_a^{-2}(A) - S_b^{-2}(B)\|} \\
&= d^{[1]}_{(a,b)}(A,B)\|S_a^{-1}(A)\| + \|b|B|S_b(B)\|\sqrt{\|R_a^{-1}(A) - R_b^{-1}(B)\|} \\
&= d^{[1]}_{(a,b)}(A,B)\|S_a^{-1}(A)\| \\
&\quad + \|b|B|S_b(B)\|\sqrt{\|R_a^{-1}(A)(R_a(A) - R_b(B))R_b^{-1}(B)\|} \\
&\leqslant d^{[1]}_{(a,b)}(A,B)\|S_a^{-1}(A)\| \\
&\quad + \|b|B|S_b(B)\|\sqrt{\|R_a^{-1}(A)\|}\sqrt{\|R_b^{-1}(B)\|d^{[1]}_{(a,b)}(A,B)}.
\end{aligned}$$

Conversely, by Lemma 2.2 (e) and Lemma 2.4 (a) we obtain

$$\begin{aligned}
\|R_a(A) - R_b(B)\| &= \|(I - R_a(A)) - (I - R_b(B))\| = \|a^2|A|^2 - b^2|B|^2\| \\
&\leqslant \|a|A| - b|B|\|(\|a|A|\| + \|b|B|\|) = d^{[3]}_{(a,b)}(A,B)(\|a|A|\| + \|b|B|\|)
\end{aligned}$$

and

$$\begin{aligned}
\|a|A|S_a(A) - b|B|S_b(B)\| &\leqslant \|a|A| - b|B|\|\|S_a(A)\| + \|b|B|\|\|S_a(A) - S_b(B)\| \\
&\leqslant \|a|A| - b|B|\| + \|b|B|\|\|\sqrt{R_a(A)} - \sqrt{R_b(B)}\| \\
&\leqslant d^{[3]}_{(a,b)}(A,B) + \|b|B|\|\sqrt{\|R_a(A) - R_b(B)\|} \\
&\leqslant d^{[3]}_{(a,b)}(A,B) + \|b|B|\|\sqrt{d^{[3]}_{(a,b)}(A,B)(\|a|A|\| + \|b|B|\|)}.
\end{aligned}$$

But

$$(d^{[1]}_{(a,b)}(A,B))^2 \leqslant 2\|R_a(A) - R_b(B)\|^2 + 2\|a|A|S_a(A) - b|B|S_b(B)\|^2.$$

This completes the proof. □

3. Some operator transformations

The following lemma will be used in this section to obtain a new operator transform.

Lemma 3.1. *Let $A \in \mathcal{B}(H)$ be a normal operator. Then*

$$\|(I + S_a(A))^{-1}\| \leqslant 1.$$

Proof. For all $x \in H$ we have

$$\begin{aligned} \big\|\sqrt{(I + S_a(A))(x)}\big\|^2 &= \big\langle \sqrt{I + S_a(A)}(x), \sqrt{I + S_a(A)}x \big\rangle \\ &= \langle (I + S_a(A))(x), x\rangle = \langle x, x\rangle + \langle (S_a(A))x, x\rangle \geqslant \|x\|^2, \end{aligned}$$

and $R\big(\sqrt{I + S_a(A)}\big) = N\big(\sqrt{I + S_a(A)}\big)^{\perp} = H$. Thus, $\sqrt{I + S_a(A)}$ and hence $I + S_a(A)$ is invertible. Now, replacing x by $\sqrt{I + S_a(A)}(x)$ we obtain

$$\|\sqrt{I + S_a(A)}(x)\| \leqslant \|x\|.$$

It follows that

$$\big\|(I + S_a(A))^{-1}\big\| \leqslant \big\|\sqrt{I + S_a(A)}\big\|^2 \leqslant 1.$$

□

Definition 3.2. For $A \in \mathcal{M}$ and $0 < a < \|A\|^{-1}$ the bisecting of A, in the sense of Lambert and Petrovic, is the operator $\widetilde{A}_a$ defined as

$$\widetilde{A}_a = a|A|(I + S_a(A))^{-1}.$$

The bisecting of A was originally introduced in [8] by Labrousse in order to study closed operators. By Lemma 3.1, $I + S_a(A)$ is invertible and so $\widetilde{A}_a$ as a positive operator is well defined. Moreover, $\|\widetilde{A}_a\| \leqslant \|a|A|\|\|(I + S_a(A))^{-1}\| \leqslant 1.$

Now we consider the maps

$$\begin{aligned}
&F_1\colon\ (\mathcal{M},\|\cdot\|)\to(\mathcal{M},\|\cdot\|),\quad A\to(I+S_a(A))^{-1};\\
&F_2\colon\ (\mathcal{M},\|\cdot\|)\to(\mathcal{M},\|\cdot\|),\quad A\to\widetilde{A}_a;\\
&F_3\colon\ (\mathcal{M},d^{[3]})\to(\mathcal{M},\|\cdot\|),\quad A\to\widetilde{A}_a;\\
&F_4\colon\ (\mathcal{M},d^{[4]})\to(\mathcal{M},\|\cdot\|),\quad A\to\widetilde{A}_a.
\end{aligned}$$

We note that in $(\mathcal{M},\|\cdot\|)$, $\|\cdot\|$ is the norm of H. We pose the following question:

For which operators $A\in\mathcal{M}$ is the map F_i continuous?

Theorem 3.3. *The maps F_1, F_2, F_3 and F_4 are continuous.*

Proof. Let $A\in\mathcal{M}$ and $\|A\|\to 0$. By Theorem 2.7 and Lemma 3.1 we obtain

$$\begin{aligned}
\|F_1(A)-F_1(0)\| &= \|(I+S_{a_\alpha}(A))^{-1}-(I+I)^{-1}\|\\
&\leqslant \|(I+S_{a_\alpha}(A))^{-1}\|\|I+S_{a_\alpha}(A)-2I\|\|(2I)^{-1}\|\\
&\leqslant \|S_{a_\alpha}(A)-I\|\leqslant a_\alpha\|A\|\to 0.
\end{aligned}$$

Now, let A and B be two nonzero elements of $\mathcal{M}$ and $\|A-B\|\to 0$. We show that $\|F_1(A)-F_1(B)\|\to 0$. Again by Theorem 2.7 and Lemma 3.1, if $\|A-B\|\to 0$, we have

$$\begin{aligned}
\|F_1(A)-F_1(B)\| &= \|(I+S_{a_\alpha}(A))^{-1}-(I+S_{b_\alpha}(B))^{-1}\|\\
&\leqslant \|(I+S_{a_\alpha}(A))^{-1}\|\|S_{a_\alpha}(A)-S_{b_\alpha}(B)\|\|(I+S_{b_\alpha}(B))^{-1}\|\\
&\leqslant \sqrt{\|a_\alpha A\|+\|b_\alpha B\|}\sqrt{\frac{2\alpha\|A-B\|}{\|A\|}}\to 0.
\end{aligned}$$

Thus, F_1 is continuous.

Let $A\in\mathcal{M}$ and $\|A\|\to 0$. By Lemma 3.1 we have

$$\|F_2(A)-F_2(0)\|=\|\widetilde{A}_a-\widetilde{0}\|=\|a_\alpha|A|(I+S_{a_\alpha}(A))^{-1}\|\leqslant\|a_\alpha|A|\|=a_\alpha\|A\|\to 0.$$

Now, let A and B be two nonzero elements of $\mathcal{M}$ and $\|A-B\|\to 0$. Then from Theorem 2.7 we obtain

$$\begin{aligned}
\|F_2(A)-F_2(B)\| &= \|\widetilde{A}_a-\widetilde{B}_b\|=\|a_\alpha|A|(I+S_{a_\alpha}(A))^{-1}-b_\alpha|B|(I+S_{b_\alpha}(B))^{-1}\|\\
&\leqslant \sqrt{\|a_\alpha^2A^*A-b_\alpha^2B^*B\|}\,\|(I+S_{a_\alpha}(A))^{-1}\|\\
&\quad+\sqrt{\|b_\alpha B^*B\|}\,\|(I+S_{a_\alpha}(A))^{-1}-(I+S_{b_\alpha}(B))^{-1}\|\\
&\leqslant \sqrt{\|a_\alpha A\|+\|b_\alpha B\|}\sqrt{\frac{2\alpha\|A-B\|}{\|A\|}}\bigl(1+\sqrt{\|b_\alpha B^*B\|}\bigr)\to 0.
\end{aligned}$$

This implies that F_2 is continuous.

Let $A \in \mathcal{M}$ such that $d^{[3]}_{(a,0)}(A,0) \to 0$. Then $\|a|A|\| \to 0$. Then we have

$$\|F_3(A) - F_3(0)\| = \|\widetilde{A}_a - \widetilde{0}\| = \|a|A|(I + S_a(A))^{-1} - 0\| \leqslant \|a|A|\| \to 0.$$

Let A and B be two nonzero elements of $\mathcal{M}$ and $d^{[3]}_{(a,b)}(A,B) \to 0$. Then

$$\|a|A| - b|B|\| \to 0.$$

Again by Theorem 2.7 and definition of $d^{[3]}$ we have

$$\begin{aligned}\|F_3(A) - F_3(B)\| &= \|\widetilde{A}_a - \widetilde{B}_b\| = \|a|A|(I + S_a(A))^{-1} - b|B|(I + S_b(B))^{-1}\| \\ &\leqslant \|a|A| - b|B|\|\|(I + S_a(A))^{-1}\| \\ &\quad + \|b|B|\|\|(I + S_a(A))^{-1}\|\|S_a(A) - S_b(B)\|\|(I + S_b(B))^{-1}\| \\ &\leqslant \|a|A| - b|B|\| + \|b|B|\|\sqrt{\|a^2|A|^2 - b^2|B|^2\|} \\ &\leqslant \|a|A| - b|B|\| + \|b|B|\|\sqrt{\|a|A|\| + \|b|B|\|}\sqrt{\|a|A| - b|B|\|} \to 0.\end{aligned}$$

Thus, F_3 is also continuous.

Let $A \in \mathcal{M}$ and $d^{[4]}_{(a,0)}(A,0) \to 0$. Then $\|a|A|\| \to 0$. Then

$$\begin{aligned}\|F_4(A) - F_4(0)\| &= \|\widetilde{A}_a - \widetilde{0}\| = \|a|A|(I + S_a(A))^{-1} - 0\| \\ &\leqslant \|a|A|\|\|(I + S_a(A))^{-1}\| \leqslant \|a|A|\| \to 0.\end{aligned}$$

Let $A, B \in \mathcal{M}$ such that $d^{[4]}_{(a,b)}(A,B) \to 0$. Then $\|a|A| - b|B|\| \to 0$ and $\|S_a(A) - S_b(B)\| \to 0$. Then we have

$$\begin{aligned}\|F_4(A) - F_4(B)\| &= \|\widetilde{A}_a - \widetilde{B}_b\| = \|a|A|(I + S_a(A))^{-1} - b|B|(I + S_b(B))^{-1}\| \\ &\leqslant \|a|A| - b|B|\|\|(I + S_a(A))^{-1}\| \\ &\quad + \|b|B|\|\|(I + S_a(A))^{-1}\|\|S_a(A) - S_b(B)\|\|(I + S_b(B))^{-1}\| \\ &\leqslant \|a|A| - b|B|\| + \|b|B|\|\|S_a(A) - S_b(B)\| \to 0.\end{aligned}$$

Consequently, $\|F_4(A) - F_4(B)\| \to 0$ as $d^{[4]}_{(a,b)}(A,B) \to 0$. □

Definition 3.4. If $A, B \in \mathcal{M}$, $0 < a < \|A\|^{-1}$ and $0 < b < \|B\|^{-1}$. The Cordes-Labrousse transform with respect to the pair (A, B) is the operator $V^{(a,b)}_{A,B}$ given by

$$V^{(a,b)}_{A,B} = S_a(A)S_b(B) + (a|A|)(b|B|).$$

We will write $V^{(a,b)}_{A,B}$ simply as $V_{A,B}$ for fixed elements A and B when no confusion can arise. Since A and B are normal operators then $V^*_{A,B} = V_{B,A}$. Also, $V_{A,A} = R_a(A) + a^2|A|^2 = R_a(A) + I - R_a(A) = I$.

The proof of the following proposition is similar in spirit to [2], Lemma 5.3.

Lemma 3.5. *Let $A, B \in \mathcal{M}$ and let $x \in H$. Then the following assertions hold.*

(a) $\big|\|V_{A,B}(x)\|^2 - \|x\|^2\big| \leqslant \|x\|^2 d^{[2]}_{(a,b)}(A,B)$;

(b) $\|V_{A,B}(x)\|^2 \geqslant (1 - (d^{[2]}_{(a,b)}(A,B))^2)\|x\|^2$;

(c) *If* $d^{[2]}_{(a,b)}(A,B) < 1$, *then* $V_{A,B}$ *is invertible.*

Example 3.6. Let (X, Σ, μ) be a complete σ-finite measure space. Let $\varphi\colon X \to X$ be a non-singular measurable point transformation, which means the measure $\mu \circ \varphi^{-1}$, defined by $\mu \circ \varphi^{-1}(B) = \mu(\varphi^{-1}(B))$ for all $B \in \Sigma$, is absolutely continuous with respect to μ (we write $\mu \circ \varphi^{-1} \ll \mu$). It follows that $\mu \circ \varphi^{-n} \ll \mu$ for every $n \in \mathbb{N}$. Then by Radon-Nikodym theorem there exists a unique non-negative Σ-measurable function h_n on X with $h_n = \mathrm{d}\mu \circ \varphi^{-n}/\mathrm{d}\mu$. Put $h_1 = h$. Now, let C_φ defined by $C_\varphi(f) = f \circ \varphi$ be a composition operator on $L^2(\Sigma)$. Note that $C_\varphi \in B(L^2(\Sigma))$ if and only if $h \in L^\infty(\Sigma)$ and in this case $\|C_\varphi\| = \|h\|_\infty^{1/2}$. Also it is a classical fact that $C_\varphi \in B(L^2(\Sigma))$ is normal if and only if $\varphi^{-1}(\Sigma) = \Sigma$ and $h \circ \varphi = h$ (see [10]). Let $\mathcal{M} = \{C_\varphi \in B(L^2(\Sigma))\colon C_\varphi \text{ is normal}\}$. Let $C_\varphi \in B(L^2(\Sigma))$ and $f \in L^2(\Sigma)$. Then we have

$$\begin{aligned}\langle C_\varphi^{*^n} C_\varphi^n f, f\rangle &= \langle C_\varphi^n f, C_\varphi^n f\rangle = \|C_\varphi^n f\|^2 = \|C_{\varphi^n} f\|^2 \\ &= \|M_{\sqrt{h_n}} f\|^2 = \langle M_{\sqrt{h_n}} f, M_{\sqrt{h_n}} f\rangle = \langle M_{h_n} f, f\rangle,\end{aligned}$$

where M_{h_n} is the multiplication operator. So, $C_\varphi^{*^n} C_\varphi^n = M_{h_n}$. In particular, if $C_\varphi \in \mathcal{M}$, then $C_\varphi^{*^n} C_\varphi^n = (C_\varphi^* C_\varphi)^n = (M_h)^n = M_{h^n}$, and so $h_n = h^n$ for each $n \in \mathbb{N}$. Let $0 < a < \|h\|_\infty^{-1/2} = \|C_\varphi\|^{-1} = r(C_\varphi)^{-1}$. Then

$$K_a(C_\varphi) = \sum_{n=0}^\infty a^{2n} C_\varphi^{*^n} C_\varphi^n = \sum_{n=0}^\infty M_{a^{2n}h^n} = (I - M_{a^2h})^{-1}.$$

Hence

$$\begin{gathered}R_a(C_\varphi) = K_a(C_\varphi)^{-1} = I - M_{a^2h}, \quad S_a(C_\varphi) = R_a\sqrt{C_\varphi} = M_{\sqrt{1-a^2h}}, \\ (\widetilde{C}_\varphi)_a = a|C_\varphi|(I + S_a(C_\varphi))^{-1} = M_{\sqrt{a^2h}/(1+\sqrt{1-a^2h})}.\end{gathered}$$

Now, for $i = 1, 2$ let $C_{\varphi_i} \in \mathcal{M}$ and $h_i = (\mathrm{d}\mu \circ \varphi_i^{-1})/\mathrm{d}\mu$. Then we have

$$V_{C_{\varphi_1}, C_{\varphi_2}} = S_a(C_{\varphi_1}) S_b(C_{\varphi_2}) + (a|C_{\varphi_1}|)(b|C_{\varphi_2}|) = M_{\sqrt{(1-a^2h_1)(1-b^2h_2)} + \sqrt{a^2b^2h_1h_2}}.$$

Acknowledgments. The authors thank the referee of this paper for her/his kindness and interest concerning this article.

References

[1] *M. Benharrat, B. Messirdi*: Strong metrizability for closed operators and the semi-Fredholm operators between two Hilbert spaces. Int. J. Anal. Appl. *8* (2015), 110–122.

[2] *H. O. Cordes, J. P. Labrousse*: The invariance of the index in the metric space of closed operators. J. Math. Mech. *12* (1963), 693–719. zbl MR

[3] *F. Gilfeather*: Norm conditions on resolvents of similarities of Hilbert space operators and applications to direct sums and integrals of operators. Proc. Am. Math. Soc. *68* (1978), 44–48. zbl MR doi

[4] *T. Kato*: Perturbation Theory for Linear Operators. Grundlehren der mathematischen Wissenschaften 132. Springer, Berlin, 1976. zbl MR doi

[5] *W. E. Kaufman*: A stronger metric for closed operators in Hilbert space. Proc. Am. Math. Soc. *90* (1984), 83–87. zbl MR doi

[6] *F. Kittaneh*: On some equivalent metrics for bounded operators on Hilbert space. Proc. Am. Math. Soc. *110* (1990), 789–798. zbl MR doi

[7] *J. P. Labrousse*: On a metric space of closed operators on a Hilbert space. Univ. Nac. Tucumán, Rev., Ser. A *16* (1966), 45–77. zbl MR

[8] *J. P. Labrousse*: Quelques topologies sur des espaces d'opérateurs dans des espaces de Hilbert et leurs applications. Faculté des Sciences de Nice (Math.) *1* (1970), 47 pages.

[9] *A. Lambert, S. Petrovic*: Beyond hyperinvariance for compact operators. J. Funct. Anal. *219* (2005), 93–108. zbl MR doi

[10] *R. K. Singh, J. S. Manhas*: Composition Operators on Function Spaces. North-Holland Mathematics Studies 179. North-Holland, Amsterdam, 1993. zbl MR

Authors' address: *Mohammad Reza Jabbarzadeh, Rana Hajipouri*, Faculty of Mathematical Sciences, University of Tabriz, P. O. Box: 5166616471, Tabriz, Iran, e-mail: `mjabbar@tabrizu.ac.ir`, `r.hajipouri@tabrizu.ac.ir`.