ENTROPY SOLUTIONS TO PARABOLIC EQUATIONS IN MUSIELAK FRAMEWORK INVOLVING NON COERCIVITY TERM IN DIVERGENCE FORM

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Abstract. We prove the existence of solutions to nonlinear parabolic problems of the following type:

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} + A(u) = f + \operatorname{div}(\Theta(x;t;u)) & \text{in } Q, \\ u(x;t) = 0 & \text{on } \partial\Omega \times [0;T], \\ b(u)(t=0) = b(u_0) & \text{on } \Omega, \end{cases}$$

where $b\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing function of class $\mathcal{C}^1$, the term

$$A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$$

is an operator of Leray-Lions type which satisfies the classical Leray-Lions assumptions of Musielak type, $\Theta\colon \Omega \times [0;T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory, noncoercive function which satisfies the following condition: $\sup_{|s|\leqslant k} |\Theta(\cdot,\cdot,s)| \in E_\psi(Q)$ for all $k > 0$, where ψ is the Musielak complementary function of Θ, and the second term f belongs to $L^1(Q)$.

Keywords: inhomogeneous Musielak-Orlicz-Sobolev space; parabolic problems; Galerkin method

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1. Introduction

Our aim is to prove the existence of solutions u to the following nonlinear parabolic problem:

$$
(1.1)\qquad \begin{cases} \dfrac{\partial b(u)}{\partial t} + A(u) = f + \operatorname{div}(\Theta(x,t,u)) & \text{in } Q,\\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T],\\ b(u)(t=0) = b(u_0) & \text{on } \Omega, \end{cases}
$$

where Ω is an open subset $\mathbb{R}^N$ which satisfies the segment property and $Q = \Omega\times[0,T]$, $T > 0$, $b\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing function of class $\mathcal{C}^1$ with $b(0) = 0$ and $\lim\limits_{t\to\pm\infty} b'(t) = l < \infty$, $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray-Lions operator defined on $D(A) \subset W_0^{1,x}L_\varphi(Q)$ into its dual satisfying some conditions in Section 3, φ is Musielak function and $W_0^{1,x}L_\varphi(Q)$ is the Musielak space defined in Section 2, $f \in L^1(Q)$ and $\Theta\colon \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a noncoercive function which satisfies the following condition: $\sup\limits_{|s|\leqslant k} |\Theta(\cdot,\cdot,s)| \in E_\psi(Q)$ for all $k > 0$, where ψ is the complementary function of φ and $E_\psi(Q)$ is a Musielak space defined in Section 2.

Under our assumptions, the above problem does not admit, in general, a weak solution since the field $a(x,t,u,\nabla u)$ does not belong to $(L^1_{\rm loc}(Q))^N$ in general. To overcome this difficulty we use in this paper the framework of entropy solutions. This notion was introduced by Benilan et al. [9] for the study of nonlinear elliptic problems.

In the classical Sobolev spaces, Aberqi et al. in [1] have proved the existence of renormalized solutions (1.1) in the case where $b(u) \equiv b(x,u)$ and Θ satisfies a growth condition (for the definition of this notion of solution see [1], [20]), Redwane in [19] has proved the existence of renormalized solutions of (1.1), where $\Theta(x,t,u) = \Theta(u)$.

In the Sobolev variable exponent setting, Azroul, Benboubker, Redwane, and Yazough [6] have proved the existence result of renormalized solutions to a class of nonlinear parabolic equations without sign condition involving nonstandard growth in the particular case, where $\operatorname{div}(\Theta(x,t,u)) = H(x,t,u,\nabla u)$ and in the elliptic case (see [8]).

In Orlicz framework, Redwane in [20] has proved the existence of renormalized solutions of (1.1), where $b(u) \equiv b(x,u)$ and $\Theta(x,t,u) = \Theta(u)$, Hadj Nassar, Moussa and Rhoudaf in [16] have studied the existence of renormalized solutions of (1.1) in $W^{1,x}L_M(Q)$, where $b(u) \equiv b(x,u)$ and Θ satisfies $|\Theta(x,u)| \leqslant \overline{P}^{-1}P(|u|)$, where P and $\overline{P}$ are two complementary Orlicz functions with $P \ll M$. See also [7], [13], and [14] for related topics. For some existing results for strongly nonlinear elliptic and parablic equations in Musielak-Orlicz-Sobolev spaces see [2], [3], [4], [5], [21].

This research is divided into several parts. In Section 2 we recall some important definitions and results of Musielak-Orlicz-Sobolev spaces. We introduce the assumptions that allow us to demonstrate our result in Section 3. Section 4 contains some important and useful lemmas to prove our main result. In Section 5 we prove the main result of this paper (Theorem 5.1) concerning the existence of solutions.

2. Preliminary

2.1. Musielak-Orlicz-Sobolev spaces. Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$, and satisfiying the following conditions:

(a) $\varphi(x,\cdot)$ is an N-function $\big($convex, increasing, continous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$, $\lim\limits_{t\to 0}\sup\limits_{x\in\Omega}\varphi(x,t)t^{-1}=0$, $\lim\limits_{t\to\infty}\inf\limits_{x\in\Omega}\varphi(x,t)t^{-1}=\infty\big)$.

(b) $\varphi(\cdot,t)$ is a measurable function.

A function φ, which satisfies conditions (a) and (b) is called Musielak-Orlicz function.

For a Musielak-Orlicz function φ we put $\varphi_x(t)=\varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1} with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t))=\varphi(x,\varphi_x^{-1}(t))=t.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if for some $k>0$ and a nonnegative function h integrable in Ω we have

$$\varphi(x,2t)\leqslant k\varphi(x,t)+h(x)\quad \forall\, x\in\Omega \text{ and } t\geqslant 0. \tag{2.1}$$

If (2.1) holds only for $t\geqslant t_0>0$, then φ is said to satisfy Δ_2 near infinity.

Let φ and γ be two Musielak-Orlicz functions. We say that φ dominates γ, and we write $\gamma\prec\varphi$, near infinity (or globally) if there exist two positive constants c and t_0 such that for almost all $x\in\Omega$

$$\gamma(x,t)\leqslant\varphi(x,ct)\quad\forall\, t\geqslant t_0,\quad(\text{or }\forall\, t\geqslant 0,\ \text{i.e. } t_0=0).$$

We say that γ grows essentially less rapidly than φ at 0 (or near infinity), and we write $\gamma\prec\prec\varphi$, if for every positive constant c we have

$$\lim_{t\to 0}\Big(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\Big)=0\quad(\text{or }\lim_{t\to\infty}\Big(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\Big)=0).$$

Remark 2.1 ([11]). If $\gamma\prec\prec\varphi$ near infinity, then for all $\varepsilon>0$ there exists $k(\varepsilon)>0$ such that for almost all $x\in\Omega$ we have

$$\gamma(x,t)\leqslant k(\varepsilon)\varphi(x,\varepsilon t)\quad\forall\, t\geqslant 0. \tag{2.2}$$

We define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\,\mathrm{d}x,$$

where $u\colon\ \Omega \to \mathbb{R}$ is a Lebesgue measurable function. In the following, the measurability of function $u\colon\ \Omega \to \mathbb{R}$ means the Lebesgue measurability. The set

$$K_\varphi(\Omega) = \{u\colon\ \Omega \to \mathbb{R} \text{ measurable: } \varrho_{\varphi,\Omega}(u) < \infty\},$$

is called the generalized Orlicz class.

The Musielak-Orlicz space (or the generalized Orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently,

$$L_\varphi(\Omega) = \Big\{u\colon\ \Omega \to \mathbb{R} \quad \text{measurable: } \varrho_{\varphi,\Omega}\Big(\frac{|u(x)|}{\lambda}\Big) < \infty \text{ for some } \lambda > 0\Big\}.$$

We define the Musielak-Orlicz function complementary to φ in the sense of Young with respect to the variable s as

$$\psi(x, s) = \sup_{t \geqslant 0}\{st - \varphi(x, t)\}.$$

We define in the space $L_\varphi(\Omega)$ the two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0\colon\ \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big)\,\mathrm{d}x \leqslant 1\right\},$$

which is called the Luxemburg norm and the so called Orlicz norm defined as

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_{\psi,\Omega} \leqslant 1} \int_\Omega |u(x)v(x)|\,\mathrm{d}x,$$

where ψ is the Musielak-Orlicz function complementary to φ and $\|v\|_{\psi,\Omega}$ is the Luxemburg norm of v associate to the Musielak function ψ. These two norms are equivalent (see [18]).

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space.

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega)\colon\ \forall\,|\alpha| \leqslant m,\ D^\alpha u \in L_\varphi(\Omega)\}$$

and

$$W^m E_\varphi(\Omega) = \{u \in E_\varphi(\Omega)\colon\ \forall\,|\alpha| \leqslant m,\ D^\alpha u \in E_\varphi(\Omega)\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + \ldots + |\alpha_n|$ and $D^\alpha u$ denotes the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha| \leqslant m} \varrho_{\varphi,\Omega}(D^\alpha u) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0\colon\ \overline{\varrho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \leqslant 1\Big\}.$$

For $u \in W^m L_\varphi(\Omega)$, these functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $(W^m L_\varphi(\Omega), \|\cdot\|^m_{\varphi,\Omega})$ is a Banach space if φ satisfies the following condition (see [18]):

$$\exists\, c > 0\colon\ \inf_{x \in \Omega} \varphi(x, 1) \geqslant c. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha| \leqslant m} L_\varphi(\Omega) = \Pi L_\varphi$; this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

We denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega})$ the restriction of $\mathcal{D}(\mathbb{R}^N)$ on Ω.

Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W^m E_\varphi(\Omega)$ be the space of functions u such that u and its distributional derivatives up to order m lie in $E_\varphi(\Omega)$, and $W^m_0 E_\varphi(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \Big\{f \in \mathcal{D}'(\Omega)\colon\ f = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega)\Big\}$$

and

$$W^{-m} E_\psi(\Omega) = \Big\{f \in \mathcal{D}'(\Omega)\colon\ f = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega)\Big\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n \to \infty} \overline{\varrho}_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0.$$

For φ and its complementary function ψ the following inequality is called the Young inequality (see [18]):

$$ts \leqslant \varphi(x,t) + \psi(x,s) \quad \forall\, t,s \geqslant 0,\ x \in \Omega. \tag{2.4}$$

This inequality implies that

$$|||u|||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}(u) + 1. \tag{2.5}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular:

$$\|u\|_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}(u) \quad \text{if } \|u\|_{\varphi,\Omega} > 1, \tag{2.6}$$

$$\|u\|_{\varphi,\Omega} \geqslant \varrho_{\varphi,\Omega}(u) \quad \text{if } \|u\|_{\varphi,\Omega} \leqslant 1. \tag{2.7}$$

For two complementary Musielak-Orlicz functions φ and ψ let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$. Then we have the Hölder inequality (see [18])

$$\left| \int_\Omega u(x)v(x)\,\mathrm{d}x \right| \leqslant \|u\|_{\varphi,\Omega} |||v|||_{\psi,\Omega}. \tag{2.8}$$

Definition 2.1. We say that $\Omega \subset \mathbb{R}^N$ satisfies the segment propriety if there exists a locally finite open covering $\{\mathcal{O}\}$ of $\partial\Omega$ and corresponding vectors $\{y_i\}$ such that for $x \in \overline{\Omega} \cap \mathcal{O}$ and $0 < t < 1$ one has $x + ty_i \in \Omega$.

2.2. Inhomogeneous Musielak-Orlicz-Sobolev spaces. Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega \times [0,T]$. Let $m \geqslant 1$ be an integer and let φ and ψ be two complementary Musielak-Orlicz functions. For each $\alpha \in \mathbb{N}^N$ denote by D_x^α the distributional derivative on Q of order α with respect to $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as

$$W^{m,x}L_\varphi(Q) = \{u \in L_\varphi(Q)\colon\ D_x^\alpha u \in L_\varphi(Q)\ \forall |\alpha| \leqslant m\}$$

and

$$W^{m,x}E_\varphi(Q) = \{u \in E_\varphi(Q)\colon\ D_x^\alpha u \in E_\varphi(Q)\ \forall |\alpha| \leqslant m\}.$$

This second space is a subspace of the first one, and both are Banach spaces with the norm

$$\|u\|_{m,x} = \sum_{|\alpha| \leqslant m} \|D_x^\alpha u\|_{\varphi,Q}.$$

These spaces constitute a complementary system since Ω satisfies the segment property. These spaces are considered subspaces of the product space $\Pi L_\varphi(Q)$, which have as many copies as there is α order derivatives, $|\alpha| \leqslant m$. We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$.

If $u \in W^{m,x}L_\varphi(Q)$, then the function $t \to u(t) = u(\cdot, t)$ is defined on $[0, T]$ with values in $W^m L_\varphi(\Omega)$. If $u \in W^{m,x}E_\varphi(Q)$, then $u \in W^m E_\varphi(\Omega)$ and it is strongly measurable.

Furthermore, the imbedding $W^{m,x}E_\varphi(Q) \subset L^1(0, T, W^m E_\varphi(\Omega))$ holds. The space $W^{m,x}L_\varphi(Q)$ is not in general separable, for $u \in W^{m,x}L_\varphi(Q)$ we cannot conclude that the function $u(t)$ is measurable on $[0, T]$.

However, the scalar function $t \to \|u(t)\|_{\varphi,\Omega} \in L^1(0, T)$. The space $W_0^{m,x}E_\varphi(Q)$ is defined as the norm closure of $\mathcal{D}(Q)$ in $W^{m,x}E_\varphi(Q)$. We can easily show as in [15] that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q)$ with respect to the weak* topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is a limit in $W^{m,x}L_\varphi(Q)$ of some subsequence $(v_j) \in \mathcal{D}(Q)$ for the modular convergence, i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leqslant m$

$$\int_Q \varphi\Big(x, \frac{D_x^\alpha v_j - D_x^\alpha u}{\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t \to 0, \quad \text{as } j \to \infty,$$

which gives that (v_j) converges to u in $W^{m,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_\psi)$.

Consequently,

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi L_\psi)}.$$

The space of functions satisfying such a property will be denoted by $W_0^{m,x}L_\varphi(Q)$. Furthermore, $W_0^{m,x}E_\varphi(Q) = W_0^{m,x}L_\varphi(Q) \cap \Pi E_\varphi(Q)$. Thus, both sides of the last inequality are equivalent norms on $W_0^{m,x}L_\varphi(Q)$. We then have the following complementary system:

$$\begin{pmatrix} W_0^{m,x}L_\varphi(Q) & F \\ W_0^{m,x}E_\varphi(Q) & F_0, \end{pmatrix},$$

where F states for the dual space of $W_0^{m,x}E_\varphi(Q)$ and can be defined, except for an isomorphism, as the quotient of ΠL_ψ by the polar set $W_0^{m,x}E_\varphi(Q)^\perp$. It will be denoted by $F = W_0^{-m,x}L_\psi(Q)$, where

$$W^{-m,x}L_\psi(Q) = \Big\{ f = \sum_{|\alpha| \leqslant m} D_x^\alpha f_\alpha \quad \text{with} \quad f_\alpha \in L_\psi(Q) \Big\}.$$

This space will be equipped with the usual quotient norm

$$\|u\|_F = \inf \sum_{|\alpha| \leqslant m} \|f_\alpha\|_{\psi,Q},$$

where the infimum is taken over all possible decompositions

$$f = \sum_{|\alpha| \leqslant m} D_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q).$$

The space F_0 is then given by

$$F_0 = \Bigl\{ f \colon\ f = \sum_{|\alpha| \leqslant m} D_x^\alpha f_\alpha,\ f_\alpha \in E_\psi(Q) \Bigr\},$$

and is denoted by $W^{-m,x} E_\psi(Q)$, see [4].

3. Essential assumptions

Let φ be a Musielak-Orlicz function which decreases with respect to one of the coordinates of x. We denote by ψ the Musielak complementary function of φ. Throughout this paper, we assume that the following assumptions hold true:

$$b \colon\ \mathbb{R} \mapsto \mathbb{R} \text{ is strictly increasing } \mathcal{C}^1 \text{ function} \\ \text{with } b(0) = 0 \text{ and } \lim_{t \to \pm\infty} b'(t) = l < \infty, \tag{3.1}$$

$a \colon\ \Omega \times \,]0, T[\, \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions:

for almost every $(x,t) \in \Omega \times \,]0, T[$ and all $s \in \mathbb{R}$, $\xi \neq \xi^* \in \mathbb{R}^N$,

$$|a(x,t,s,\xi)| \leqslant \beta(h_1(x,t) + \psi_x^{-1} \gamma(x, \nu|s|) + \psi_x^{-1} \varphi(x, \nu|\xi|)), \tag{3.2}$$

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi - \xi^*) > 0, \tag{3.3}$$

$$a(x,t,s,\xi)\xi \geqslant \alpha\varphi\Bigl(x, \frac{|\xi|}{\lambda}\Bigr) \tag{3.4}$$

with $h_1(x,t) \in E_\Psi(Q)$, $h_1 \geqslant 0$, α, β and $\nu > 0$.

Furthermore, let $\Theta \colon\ \Omega \times [0,T] \times \mathbb{R} \mapsto \mathbb{R}^N$ be a Carathéodory function such that

$$\sup_{|s| \leqslant k} |\Theta(\cdot,\cdot,s)| \in E_\psi(Q) \quad \forall\, k > 0 \tag{3.5}$$

and

$$f \in L^1(Q). \tag{3.6}$$

We consider the following parabolic initial-boundary problem:

$$
(P) \qquad \begin{cases} \dfrac{\partial b(u)}{\partial t} + A(u) = f + \operatorname{div}(\Theta(x,t,u)) & \text{in } Q, \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}
$$

where u_0 is a given function in $L^1(\Omega)$.

4. Some technical lemmas

Lemma 4.1 ([10]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

(i) *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \geqslant c$.*

(ii) *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leqslant \frac{1}{2}$ we have*

$$
\frac{\varphi(x,t)}{\varphi(y,t)} \leqslant t^{A/(-\log|x-y|)} \quad \forall\, t \geqslant 1. \tag{4.1}
$$

(iii)

$$
\textit{If } D \subset \Omega \textit{ is a bounded measurable set, then } \int_D \varphi(x,1)\,\mathrm{d}x < \infty. \tag{4.2}
$$

(iv) *There exists a constant $C > 0$ such that $\psi(x,1) \leqslant C$ a.e. in Ω. Under these assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence, and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1 L_\varphi(\Omega)$ for the modular convergence.*

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1 L_\varphi(\Omega)$ is well defined. It will be denoted by $\langle S, u\rangle$.

Truncation operator. *For $k > 0$ we define the truncation at height k as*

$$
T_k(s) = \begin{cases} s & \text{if } |s| \leqslant k, \\ k\dfrac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{4.3}
$$

In the following lemma we give the modular Poincaré's inequality in Musielak-Orlicz spaces.

Lemma 4.2 ([12]). *Under the assumptions of Lemma 4.1 and by assuming that $\varphi(x,t)$ decreases with respect to one of the coordinates of x, there exists a constant $c>0$, which depends only on Ω, such that*

$$\int_\Omega \varphi(x,|u(x)|)\,\mathrm{d}x \leqslant \int_\Omega \varphi(x,c|\nabla u(x)|)\,\mathrm{d}x \quad \forall\, u \in W_0^1 L_\varphi(\Omega). \tag{4.4}$$

Remark 4.1. The following function is an example of a function that satisfies the previous lemma:

$$\varphi(x,t) = t^{\|x\|_2^2 - x_1^2}\log(1+t).$$

Lemma 4.3 (The Nemytskii operator [5]). *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak-Orlicz functions. Let $f\colon \Omega\times\mathbb{R}^p \to \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x\in\Omega$ and all $s\in\mathbb{R}^p$*

$$|f(x,s)| \leqslant c(x) + k_1\psi_x^{-1}\varphi(x,k_2|s|), \tag{4.5}$$

where k_1 and k_2 are real positive constants and $c(\cdot)\in E_\psi(\Omega)$. Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x,u(x))$ is continuous from

$$\Big(\mathcal{P}\Big(E_\varphi(\Omega),\frac{1}{k_2}\Big)\Big)^p = \prod\Big\{u\in L_\varphi(\Omega)\colon\ d(u,E_\varphi(\Omega)) < \frac{1}{k_2}\Big\}$$

into $(L_\psi(\Omega))^q$ for the modular convergence.

Furthermore, if $c(\cdot)\in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$, then N_f is strongly continuous from $(\mathcal{P}(E_\varphi(\Omega),k_2^{-1}))^p$ to $(E_\gamma(\Omega))^q$.

Lemma 4.4 ([12]). *Assume that (3.2)–(3.4) are satisfied and let $(z_n)_n$ be a sequence in $W_0^{1,x}L_\varphi(\Omega)$ such that*

(i) *$z_n \rightharpoonup z$ in $W_0^{1,x}L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$,*

(ii) *$(a(\cdot,t,z_n,\nabla z_n))_n$ is bounded in $(L_\psi(\Omega))^N$,*

(iii) *$\int_\Omega (a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s))(\nabla z_n - \nabla z\chi_s)\,\mathrm{d}x \to 0$ as $n,s\to\infty$, where χ_s is the characteristic function of $\Omega_s = \{x\in\Omega\colon\ |\nabla z|\leqslant s\}$.*

Then we have

$$z_n \to z \text{ for the modular convergence in } W_0^1 L_\varphi(\Omega).$$

5. Main result

We shall prove the following existence theorem.

Theorem 5.1. *Let φ and ψ be two complementary Musielak-Orlicz functions satisfying the assumptions of Lemma 4.2, we assume that (3.1)–(3.6) hold true. Then problem (P) has at least one entropy solution $u \in D(A) \cap W_0^{1,x}L_\varphi(Q) \cap \mathcal{C}([0,T], L^2(\Omega))$ in the following sense:*

$$
(5.1)\qquad \begin{cases} T_k(u) \in W_0^{1,x}L_\varphi(Q) \quad \forall\, k > 0, \\ \Big\langle \dfrac{\partial b(u)}{\partial t}, T_k(u-v) \Big\rangle + \displaystyle\int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t \\ \quad \leqslant \displaystyle\int_Q f T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t + \int_Q \Theta(x,t,u)\nabla T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t \\ \qquad \forall\, v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q) \text{ such that } \dfrac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q). \end{cases}
$$

P r o o f. We will use the Galerkin method due to Landes and Mustonen (see [17]), we choose a sequence $\{w_1, w_2, \ldots\}$ in $D(\Omega)$ such that $\bigcup_{p=0}^{\infty} V_p$ with $V_p = \{w_1, \ldots, w_p\}$ is dense in $H_0^m(\Omega)$ with m large enough so that $H_0^m(\Omega)$ is continuously embedded in $\mathcal{C}^1(\overline{\Omega})$. For every $v \in H_0^m(\Omega)$ there exists a sequence $(v_j) \subset \bigcup_{p=0} V_p$ such that $v_n \to v$ in $H_0^m(\Omega)$ and in $\mathcal{C}^1(\overline{\Omega})$.

We denote further $\mathcal{V}_p = \mathcal{C}([0,T], V_p)$. It is easy to see that the closure of $\bigcup_{p=0}^{\infty} \mathcal{V}_p$ with respect to the norm

$$
\|v\|_{\mathcal{C}^{1,0}(Q)} = \sup_{|\alpha|\leqslant 1} \{|D_x^\alpha v(x,t)| \colon (x,t) \in Q\}
$$

contains $D(Q)$. This implies that for any $f \in W^{-1,x}E_\psi(Q)$ there exists a sequence $(f_n) \subset \bigcup_{p=0}^{\infty} \mathcal{V}_p$ such that $f_n \to f$ strongly in $W^{-1,x}E_\psi(Q)$.

Indeed, let $\varepsilon > 0$ be given. Write $f = \sum_{|\alpha|\leqslant 1} D_x^\alpha f_\alpha$. There exists $g_\alpha \in \mathcal{D}(Q)$ such that $\|f_\alpha - g_\alpha\|_{\psi,Q} \leqslant \varepsilon(2N+2)^{-1}$. Moreover, by setting $g = \sum_{|\alpha|\leqslant 1} D_x^\alpha g_\alpha$, we see that $g \in \mathcal{D}(Q)$, and so there exists $v \in \bigcup_{p=0}^{\infty} \mathcal{V}_p$ such that $\|g - v\|_{\infty,Q} \leqslant \varepsilon(2\text{meas}(Q))^{-1}$. We deduce that

$$
\|f - v\|_{W^{-1,x}L_\psi(Q)} \leqslant \sum_{|\alpha|\leqslant 1} \|f_\alpha - g_\alpha\|_{\psi,Q} + \|g - v\|_{\psi,Q} \leqslant \varepsilon.
$$

We devide the proof into six steps.

Step 1: Approximate problem. For $n \in \mathbb{N}$ we define the following approximations:

$$b_n(r) = T_n(b(r)) + \frac{r}{n} \quad \forall\, r \in \mathbb{R}, \tag{5.2}$$

$$\Theta_n(x,t,s) = \Theta(x,t,T_n(s)), \tag{5.3}$$

$(f_n)_n$ is a sequence in $W^{-1}E_\psi(Q) \cap L^1(Q)$ such that

$$f_n \to f \text{ in } L^1(Q) \text{ with } \|f_n\|_{L^1(Q)} \leqslant \|f\|_{L^1(Q)}, \tag{5.4}$$

and u_{0n} is a sequence of $D(\Omega)$ such that

$$b_n(u_{0n}) \to b(u_0) \text{ strongly in } L^1(\Omega) \text{ with } \|b_n(u_{0n})\|_{L^1(\Omega)} \leqslant \|b(u_0)\|_{L^1(\Omega)}. \tag{5.5}$$

We consider the approximate problem

$$(\mathcal{P}_n) \qquad \begin{cases} u_n \in \mathcal{V}_n, \quad \dfrac{\partial b(u_n)}{\partial t} \in L^1(0,T,V_n), \quad u_n(\cdot,0) = u_{0n} \quad \text{a.e. in } \Omega, \\ \dfrac{\partial b_n(u_n)}{\partial t} - \operatorname{div}(a(x,t,u_n,\nabla u_n)) = f_n + \operatorname{div}(\Theta_n(x,t,u_n)). \end{cases}$$

There exists at least one solution u_n of $(\mathcal{P}_n)$ (this solution u_n can be obtained from Galerkin solution (see [17]).

Step 2: A priori estimates. In this section we denote by c_i, $i = 1,2,\ldots$ constants not depending on k and n.

For $\tau \in [0,T]$, taking $T_k(u_n)\chi_{[0,\tau]}$ as test function in $(\mathcal{P}_n)$, we obtain

$$\int_{Q_\tau} \frac{\partial b_n(u_n)}{\partial t} T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_\tau} f_n T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\tau} \Theta_n(x,t,u_n)\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t.$$

We set

$$S_n^k(\sigma) = \int_0^\sigma b_n'(r)T_k(r)\,\mathrm{d}r.$$

Then we have

$$\int_{Q_\tau} \frac{\partial b_n(u_n)}{\partial t} T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_\tau} \frac{\partial u_n}{\partial t} b_n'(u_n)T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega S_n^k(u_n(\tau))\,\mathrm{d}x - \int_\Omega S_n^k(u_{0n})\,\mathrm{d}x.$$

Hence, we have

$$\int_\Omega S_n^k(u_n(\tau))\,\mathrm{d}x - \int_\Omega S_n^k(u_{0n})\,\mathrm{d}x + \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_Q f_n T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\tau} \Theta_n(x,t,u_n)\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t.$$

Due to the definition of S_n^k, (3.1) and (5.5), one has

$$\int_\Omega S_n^k(u_{0n})\,\mathrm{d}x \leqslant k\int_\Omega |b_n(u_{0n})|\,\mathrm{d}x \leqslant \|b(u_0)\|_{L^1(\Omega)}. \tag{5.6}$$

Using (5.4) and (5.6), we obtain

$$\begin{aligned}\int_\Omega S_n^k(u_n(\tau))\,\mathrm{d}x &+ \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant k(\|f\|_{L^1(Q)} + \|b(u_0)\|_{L^1(\Omega)}) + \int_{Q_\tau} \Theta_n(x,t,u_n)\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant c_1 k + \int_{Q_\tau} \Theta_n(x,t,u_n)\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{5.7}$$

For $n \geqslant k$, condition (3.5) and Young's inequality gives

$$\begin{aligned}\int_{Q_\tau} \Theta_n(x,t,u_n)\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t &\leqslant \int_{Q_\tau} |\Theta_n(x,t,u_n)||\nabla T_k(u_n)|\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_\tau} |\Theta_n(x,t,T_k(u_n))||\nabla T_k(u_n)|\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_\tau} |\Theta(x,t,T_k(u_n))||\nabla T_k(u_n)|\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_{Q_\tau} \sup_{|s|\leqslant k} |\Theta(x,t,s)||\nabla T_k(u_n)|\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_{Q_\tau} \psi\Big(x, c_\alpha \sup_{|s|\leqslant k}|\Theta(x,t,s)|\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{\alpha}{2(\alpha+1)}\int_{Q_\tau} \varphi(x,|\nabla T_k(u_n)|)\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant r(k) + \frac{\alpha}{2(\alpha+1)}\int_{Q_\tau} \varphi(x,|\nabla T_k(u_n)|)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{5.8}$$

where $r(k) = \int_{Q_\tau} \psi\big(x, c_\alpha \sup_{|s|\leqslant k}|\Theta(x,t,s)|\big)\,\mathrm{d}x\,\mathrm{d}t$. Then by condition (3.4) and by combining (5.7) and (5.8), we get

$$\int_\Omega S_n^k(u_n(\tau))\,\mathrm{d}x + \frac{2\alpha+1}{2(\alpha+1)}\int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \leqslant c_1 k + r(k). \tag{5.9}$$

Now, using the fact that $S_n^k(u_n(\tau)) \geqslant 0$, one has

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{2(\alpha+1)}{2\alpha+1}(c_1k+r(k)). \tag{5.10}$$

Then using (3.4), we have

$$\int_Q \varphi\Big(x,\frac{|\nabla T_k(u_n)|}{\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{2(\alpha+1)(c_1k+r(k))}{\alpha(2\alpha+1)}. \tag{5.11}$$

Using Lemma 4.2, we have that $(T_k(u_n))$ is bounded in $W_0^{1,x}L_\varphi(Q)$, then there exists v_k such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k & \text{in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi,\Pi E_\psi),\\ T_k(u_n)\to v_k & \text{strongly in } E_\varphi(Q). \end{cases} \tag{5.12}$$

Therefore, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω. Then for all $k>0$ and $\delta,\varepsilon>0$ there exists $n_0=n_0(k,\delta,\varepsilon)$ such that

$$\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\} \leqslant \frac{\varepsilon}{3} \quad \forall\, m,n\geqslant n_0. \tag{5.13}$$

It is easy to show that

$$\begin{aligned}\inf_{x\in\Omega}\varphi\Big(x,\frac{k}{\lambda c}\Big)\operatorname{meas}\{|u_n|>k\} &= \int_{\{|u_n|>k\}}\inf_{x\in\Omega}\varphi\Big(x,\frac{k}{\lambda c}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \int_Q \varphi\Big(x,\frac{|T_k(u_n)|}{\lambda c}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \int_Q \varphi\Big(x,\frac{|\nabla T_k(u_n)|}{\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t \quad \text{(using Lemma 4.2)}\\ &\leqslant \frac{2(\alpha+1)(c_1k+r(k))}{\alpha(2\alpha+1)} \quad \text{(using (5.11))},\end{aligned}$$

where this c is the constant of Lemma 4.2. Then, by using the definition of φ,

$$\operatorname{meas}\{|u_n|>k\} \leqslant \frac{2(\alpha+1)(c_1k+r(k))}{\alpha(2\alpha+1)\inf\limits_{x\in\Omega}\varphi(x,k/\lambda c)} \to 0, \quad \text{as} \quad k\to\infty. \tag{5.14}$$

Since for all $\delta>0$,

$$\begin{aligned}\operatorname{meas}\{|u_n-u_m|>\delta\} \leqslant{}& \operatorname{meas}\{|u_n|>k\}+\operatorname{meas}\{|u_m|>k\}\\ &+\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}.\end{aligned} \tag{5.15}$$

Using (5.14), we get for all $\varepsilon > 0$ there exists $k_0 > 0$ such that

$$\text{(5.16)} \qquad \operatorname{meas}\{|u_n| > k\} \leqslant \frac{\varepsilon}{3}, \quad \operatorname{meas}\{|u_m| > k\} \leqslant \frac{\varepsilon}{3} \quad \forall\, k \geqslant k_0(\varepsilon).$$

Combining (5.13), (5.15) and (5.16), we obtain that for all $\delta, \varepsilon > 0$ there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\operatorname{meas}\{|u_m - u_m| > \delta\} \leqslant \varepsilon \quad \forall\, n, m \geqslant n_0.$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure. Then the there exists a function u such that

$$\text{(5.17)} \qquad \begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi), \\ T_k(u_n) \to T_k(u) & \text{strongly in } E_\varphi(\Omega). \end{cases}$$

Step 3: Boundness of $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ *in* $(L_\psi(Q))^N$. Let $w \in (E_\varphi(Q))^N$ be arbitrary such that $\|w\|_{\varphi,Q} = 1$. By (3.3) we have

$$\Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a\Big(x,t,T_k(u_n),\frac{w}{\nu}\Big)\Big)\Big(\nabla T_k(u_n) - \frac{w}{\nu}\Big) > 0.$$

Hence,

$$\text{(5.18)} \qquad \begin{aligned} &\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\frac{w}{\nu}\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \leqslant \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\quad - \int_Q a\Big(x,t,T_k(u_n),\frac{w}{\nu}\Big)\Big(\nabla T_k(u_n) - \frac{w}{\nu}\Big)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

and hence, using (5.10),

$$\text{(5.19)} \qquad \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{2(\alpha+1)(c_1 k + r(k))}{\alpha(2\alpha+1)}.$$

For μ large enough ($\mu > \beta$), using (3.2) we have

$$\begin{aligned} &\int_Q \psi_x\Big(\frac{a(x,t,T_k(u_n),w\nu^{-1})}{3\mu}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \int_Q \psi_x\Big(\frac{\beta(h_1(x,t) + \psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|)) + \psi_x^{-1}(\varphi(x,|w|)))}{3\mu}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \frac{\beta}{\mu}\int_Q \psi_x\Big(\frac{h_1(x,t) + \psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|)) + \psi_x^{-1}(\varphi(x,|w|))}{3}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \frac{\beta}{3\mu}\Big(\int_Q \psi_x(h_1(x,t))\,\mathrm{d}x\,\mathrm{d}t + \int_Q \gamma(x,\nu|T_k(u_n)|)\,\mathrm{d}x\,\mathrm{d}t + \int_Q \varphi(x,|w|)\,\mathrm{d}x\,\mathrm{d}t\Big) \\ &\quad \leqslant c_2(k). \end{aligned}$$

Now, since γ grows essentially less rapidly than φ near infinity and by using Remark 2.1, there exists $r'(k) > 0$ such that $\gamma(x, \nu k) \leqslant r'(k)\varphi(x, 1)$ and so we have

$$\int_Q \psi_x\Big(\frac{a(x,t,T_k(u_n), w\nu^{-1})}{3\mu}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \frac{\beta}{3\mu}\Big(\int_Q \psi_x(h_1(x,t))\,\mathrm{d}x\,\mathrm{d}t + r'(k)\int_Q \varphi(x,1)\,\mathrm{d}x\,\mathrm{d}t + \int_Q \varphi(x,|w|)\,\mathrm{d}x\,\mathrm{d}t\Big).$$

Hence $a(x, t, T_k(u_n), w\nu^{-1})$ is bounded in $(L_\psi(Q))^N$. This implies that the second term of the right-hand side of (5.18) is bounded, consequently, we obtain

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))w\,\mathrm{d}x\,\mathrm{d}t \leqslant c_2(k) \quad \forall\, w \in (L^\varphi(Q))^N \text{ with } \|w\|_{\varphi,Q} \leqslant 1.$$

Hence, by the theorem of Banach Steinhaus, the sequence $(a(x, t, T_k(u_n), \nabla T_k(u_n)))_n$ remains bounded in $(L_\psi(Q))^N$, which implies that for all $k > 0$ there exists a function $l_k \in (L_\psi(Q))^N$ such that

$$(5.20) \qquad a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup l_k \text{ weak star in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi, \Pi E\varphi).$$

Step 4: Modular convergence of the truncations. Since $T_k(u) \in W^{1,x}L_\varphi(Q)$, there exists a sequence $(v_j^k) \subset D(\Omega)$ such that $v_j^k \to T_k(u)$. For the sake of simplicity, we denote by $\varepsilon(n, j, \mu, s)$ any quantity (possible different) such that

$$\lim_{s\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\varepsilon(n,j,\mu,s) = 0.$$

If the quantity we consider does not depend on one of the parameters n, j, μ and s, we will omit the dependence on the corresponding parameter: as an example, $\varepsilon(n, j)$ is any quantity such that

$$\lim_{j\to\infty}\lim_{n\to\infty}\varepsilon(n,j) = 0.$$

We denote also by $\chi_{j,s}$ (or χ_s) the characteristic functions of the set

$$Q_{j,s} = \{(x,t) \in Q\colon\ |\nabla T_k(v_j^k)| \leqslant s\} \quad \text{or} \quad Q_s = \{(x,t) \in Q\colon\ |\nabla T_k(u)| \leqslant s\}.$$

For $k > 0$, taking $T_k(u_n) - T_k(v_j^k)_\mu$ as a test function in $(\mathcal{P}_n)$, we get

$$(5.21) \qquad \int_Q \frac{\partial b_n(u_n)}{\partial t}(T_k(u_n) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_Q a(x,t,u_n,\nabla u_n)\nabla(T_k(u_n) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_Q f_n(T_k(u_n) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_Q \Theta_n(x,t,u_n)\nabla(T_k(u_n) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t.$$

Firstly, for the first term of the left-hand side of (5.21) we get

$$\int_Q \frac{\partial b_n(u_n)}{\partial t}(T_k(u_n) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_Q \frac{\partial b_n(u_n)}{\partial t} T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t - \int_Q \frac{\partial b_n(u_n)}{\partial t} T_k(v_j^k)_\mu\,\mathrm{d}x\,\mathrm{d}t = I_1 + I_2.$$

For I_1 we have

$$I_1 = \int_\Omega B_n^k(u_n(T))\,\mathrm{d}x - \int_\Omega B_n^k(u_{0n})\,\mathrm{d}x,$$

where $B_n^k(s) = \int_0^s b_n'(r)T_k(r)\,\mathrm{d}r$. Then, by passing to the limit as $n \to \infty$, we get

$$I_1 = \int_\Omega B^k(u(T))\,\mathrm{d}x - \int_\Omega B^k(u_0)\,\mathrm{d}x + \varepsilon(n), \tag{5.22}$$

where $B^k(s) = \int_0^s b'(r)T_k(r)\,\mathrm{d}r$. For I_2, by integration by parts with respect to t, we find

$$I_2 = \int_\Omega b_n(u_{0n})T_k(v_j^k)_\mu(0)\,\mathrm{d}x - \int_\Omega b_n(u_n(T))T_k(v_j^k)_\mu(T)\,\mathrm{d}x$$
$$+ \mu \int_Q (T_k(v_j^k) - T_k(v_j^k)_\mu) b_n(u_n)\,\mathrm{d}x\,\mathrm{d}t.$$

Passing to the limit as $n, j \to \infty$ and since $u_n \to u$ a.e. in Q and by Lebesgue dominated convergence theorem, we get

$$\begin{aligned} I_2 = &\int_\Omega b(u_0)T_k(u)_\mu(0)\,\mathrm{d}x - \int_\Omega b(u(T))T_k(u)_\mu(T)\,\mathrm{d}x \\ &+ \mu \int_Q (T_k(u) - T_k(u)_\mu) b(u)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n,j) \\ = &J_1 + J_2 + \varepsilon(n,j). \end{aligned} \tag{5.23}$$

For J_2 we have

$$\begin{aligned} J_2 &= \mu \int_Q (T_k(u) - T_k(u)_\mu) b(u)\,\mathrm{d}x\,\mathrm{d}t \\ &= \mu \int_Q (T_k(u) - T_k(u)_\mu)(b(u) - b(T_k(u)))\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \mu \int_Q (T_k(u) - T_k(u)_\mu)(b(T_k(u)) - b(T_k(u)_\mu))\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \mu \int_Q (T_k(u) - T_k(u)_\mu) b(T_k(u)_\mu)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Since b is increasing, we get

$$
\begin{aligned}
J_2 &\geqslant \mu \int_Q (T_k(u) - T_k(u)_\mu)(b(u) - b(T_k(u)))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \mu \int_Q (T_k(u) - T_k(u)_\mu) b(T_k(u)_\mu)\,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant \mu \int_{u>k} (k - T_k(u)_\mu)(b(u) - b(k))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \mu \int_{u<-k} (-k - T_k(u)_\mu)(b(u) - b(-k))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_Q \frac{\partial T_k(u)_\mu}{\partial t} b(T_k(u)_\mu)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Since b is increasing and $-k \leqslant T_k(u)_\mu \leqslant k$, we get

$$
J_2 \geqslant \int_\Omega \overline{B}(T_k(u(T))_\mu)\,\mathrm{d}x - \int_\Omega \overline{B}(T_k(u_0)_\mu)\,\mathrm{d}x, \tag{5.24}
$$

where $\overline{B}(s) = \int_0^s b(\tau)\,\mathrm{d}\tau$.

Combining (5.22), (5.23) and (5.24), we get

$$
\begin{aligned}
&\int_Q \frac{\partial b_n(u_n)}{\partial t}(T_k(u_n) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\geqslant \int_\Omega B^k(u(T))\,\mathrm{d}x - \int_\Omega B^k(u_0)\,\mathrm{d}x + \int_\Omega b(u_0) T_k(u)_\mu(0)\,\mathrm{d}x \\
&\qquad - \int_\Omega b(u(T)) T_k(u)_\mu(T)\,\mathrm{d}x + \int_\Omega \overline{B}(T_k(u(T))_\mu)\,\mathrm{d}x \\
&\qquad - \int_\Omega \overline{B}(T_k(u_0)_\mu)\,\mathrm{d}x + \varepsilon(n,j).
\end{aligned} \tag{5.25}
$$

Passing now to the limit for $\mu \to \infty$, we obtain

$$
\begin{aligned}
&\int_Q \frac{\partial b_n(u_n)}{\partial t}(T_k(u_n) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\geqslant \int_\Omega B^k(u(T))\,\mathrm{d}x - \int_\Omega B^k(u_0)\,\mathrm{d}x + \int_\Omega b(u_0) T_k(u_0)\,\mathrm{d}x \\
&\qquad - \int_\Omega b(u(T)) T_k(u(T))\,\mathrm{d}x + \int_\Omega \overline{B}(T_k(u(T)))\,\mathrm{d}x \\
&\qquad - \int_\Omega \overline{B}(T_k(u_0))\,\mathrm{d}x + \varepsilon(n,j,\mu).
\end{aligned} \tag{5.26}
$$

Observe that for all $z \in \mathbb{R}$ we have

$$
\overline{B}(T_k(z)) = b(z) T_k(z) - B^k(z).
$$

Then, we deduce that

$$\int_Q \frac{\partial b_n(u_n)}{\partial t}(T_k(u_n) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \geqslant \varepsilon(n,j,\mu). \tag{5.27}$$

Secondly, since $f_n \to f$ strongly in $L^1(Q)$ and $T_k(u_n) - T_k(v_j^k)_\mu$ converges to $T_k(u) - T_k(v_j^k)_\mu$ weakly star in $L^\infty(Q)$, the first term of the right-hand side can be written as

$$\int_Q f_n(T_k(u_n) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t = \int_Q f(T_k(u) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n).$$

Hence, by letting j and μ to infinity, one has

$$\int_Q f_n(T_k(u_n) - T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t = \varepsilon(n,j,\mu). \tag{5.28}$$

Thirdly, for the last term of the right-hand side, one has for $n \geqslant 2k$

$$\begin{aligned}\int_Q \Theta_n(x,t,u_n)(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q \Theta_n(x,t,T_{2k}(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q \Theta(x,t,T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

and as $\Theta(x,t,T_{2k}(u_n))$ converges strongly to $\Theta(x,t,T_{2k}(u))$ in $E_\psi(Q)$ and $\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu$ converges weakly to $\nabla T_k(u) - \nabla T_k(v_j^k)_\mu$ in $(L_\varphi(Q))^N$, we get

$$\begin{aligned}\int_Q \Theta_n(x,t,u_n)(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q \Theta(x,t,T_{2k}(u))(\nabla T_k(u) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n).\end{aligned}$$

Then by letting j and μ to infinity, we get

$$\int_Q \Theta_n(x,t,u_n)(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t = \varepsilon(n,j,\mu). \tag{5.29}$$

Thus, by combining (5.21), (5.27), (5.28) and (5.29), we obtain

$$\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \leqslant \varepsilon(n,j,\mu). \tag{5.30}$$

Splitting the first term of the last inequality on $\{|u_n| \leqslant k\}$ and $\{|u_n| > k\}$ and observing that $\nabla(T_k(u_n) - T_k(v_j^k)_\mu) = 0$ on $\{|u_n| > 2k\}$, we get

$$(5.31)\quad \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{\{|u_n|>k\}} a(x,t,T_{2k}(u_n),\nabla T_{2k}(u_n))\nabla T_k(v_j^k)_\mu\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n,j,\mu).$$

For the first term of the right-hand side of the last inequality we have

$$\int_{\{|u_n|>k\}} a(x,t,T_{2k}(u_n),\nabla T_{2k}(u_n))\nabla T_k(v_j^k)_\mu\,\mathrm{d}x\,\mathrm{d}t = \int_{\{|u|>k\}} l_{2k}\nabla T_k(v_j^k)_\mu\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n).$$

Then by letting j and μ to infinity, we get

$$\int_{\{|u_n|>k\}} a(x,t,T_{2k}(u_n),\nabla T_{2k}(u_n))\nabla T_k(v_j^k)_\mu\,\mathrm{d}x\,\mathrm{d}t = \varepsilon(n,j,\mu).$$

Then (5.31) becomes

$$(5.32)\qquad \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \leqslant \varepsilon(n,j,\mu).$$

By a simple calculus, we get

$$\begin{aligned}
&\int_Q (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)) \\
&\qquad \times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_Q (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)) \\
&\qquad\qquad \times (\nabla T_k(u)\chi_s - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_Q a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant -\int_Q (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)) \\
&\qquad\qquad \times (\nabla T_k(u)\chi_s - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_Q a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n,j,\mu) \\
&\quad = L_1 + L_2 + \varepsilon(n,j,\mu).
\end{aligned}$$

For L_1, since $a(x,t,T_k(u_n),\nabla T_k(u_n))$ weakly star converges to l_k in $(L_\psi(Q))^N$ and $a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)$ strongly converges to $a(x,t,T_k(u),\nabla T_k(u)\chi_s)$ in $(L_\psi(Q))^N$, we get

$$L_1 = -\int_Q (l_k - a(x,t,T_k(u),\nabla T_k(u)\chi_s))(\nabla T_k(u)\chi_s - \nabla T_k(v_j^k)_\mu)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n).$$

Then by letting j and μ to infinity, we obtain

$$L_1 = \varepsilon(n,j,\mu,s).$$

Similarly,

$$L_2 = \varepsilon(n,j,\mu).$$

Consequently, we deduce that

$$\int_Q (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)) \times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,\mathrm{d}x\,\mathrm{d}t \to 0, \quad \text{as } n\to\infty. \tag{5.33}$$

Using Lemma 4.4, we get

$$T_k(u_n) \to T_k(u) \text{ for the modular convergence in } W_0^{1,x}L_\varphi(Q). \tag{5.34}$$

Step 5: Passage to the limit. Since the sequence $T_k(u_n)$ converges for the modular convergence in $W_0^{1,x}L_\varphi(Q)$, there exists a subsequence, which is also denoted by $(u_n)_n$, such that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } Q. \tag{5.35}$$

Let $v \in W_0^1 L_\varphi(\Omega)\cap L^\infty(\Omega)$ and $\lambda = k + \|v\|_\infty$ with $k>0$. Taking $T_k(u_n - v)$ as a test function in $(\mathcal{P}_n)$, we get

$$\begin{aligned}\int_Q \frac{\partial b_n(u_n)}{\partial t} T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t &\\ + \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t &\\ = \int_Q f_n T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t + \int_Q \Theta_n(x,t,u_n)\nabla T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t.&\end{aligned} \tag{5.36}$$

For the first term of the left-hand side of (5.36), by using the fact that $b_n(u_n) \rightharpoonup b(u)$ weakly in $L_\varphi(Q)$, we get

$$
(5.37)\qquad \int_Q \frac{\partial b_n(u_n)}{\partial t} T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t = \left[\int_\Omega B_n^k(u_n)\,\mathrm{d}t\right]_0^T = \left[\int_\Omega B^k(u)\,\mathrm{d}t\right]_0^T + \varepsilon(n)
$$
$$
= \int_Q \frac{\partial b(u)}{\partial t} T_k(u - v)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n),
$$

where $B_n^k(s) = \int_0^s b_n'(\tau)T_k(\tau - v)\,\mathrm{d}\tau$ and $B^k(s) = \int_0^s b'(\tau)T_k(\tau - v)\,\mathrm{d}\tau$.

For the second term of the left-hand side of (5.36) we have

$$
\liminf_{n\to\infty} \int_Q a(x, u_n, \nabla u_n)\nabla T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_Q a(x, u, \nabla u)\nabla T_k(u - v)\,\mathrm{d}x\,\mathrm{d}t.
$$

Indeed, if $|u_n| > \lambda$, then $|u_n - v| \geqslant |u_n| - \|v\|_\infty > k$. Let $D_n = \{|u_n - v| \leqslant k\}$, therefore $D_n \subseteq \{|u_n| \leqslant \lambda\}$, which implies that

$$
(5.38)\qquad a(x, t, u_n, \nabla u_n)\nabla T_k(u_n - v)
$$
$$
= a(x, t, u_n, \nabla u_n)\nabla(u_n - v)\chi_{D_n}
$$
$$
= a(x, t, T_\lambda(u_n), \nabla T_\lambda(u_n))(\nabla T_\lambda(u_n) - \nabla v)\chi_{D_n}.
$$

Then

$$
(5.39)\qquad \int_Q a(x, t, u_n, \nabla u_n)\nabla T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t
$$
$$
= \int_Q a(x, t, T_\lambda(u_n)\nabla T_\lambda(u_n))(\nabla T_\lambda(u_n) - \nabla v)\chi_{D_n}\,\mathrm{d}x\,\mathrm{d}t
$$
$$
= \int_Q (a(x, t, T_\lambda(u_n), \nabla T_\lambda(u_n)) - a(x, t, T_\lambda(u_n), \nabla v))
$$
$$
\times (\nabla T_\lambda(u_n) - \nabla v)\chi_{D_n}\,\mathrm{d}x\,\mathrm{d}t
$$
$$
+ \int_Q a(x, t, T_\lambda(u_n), \nabla v)(\nabla T_\lambda(u_n) - \nabla v)\chi_{D_n}\,\mathrm{d}x\,\mathrm{d}t.
$$

Let $D = \{|u - v| \leqslant k\}$, then we obtain

$$
(5.40)\qquad \liminf_{n\to\infty} \int_Q a(x, t, u_n, \nabla u_n)\nabla T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t
$$
$$
\geqslant \int_Q (a(x, t, T_\lambda(u), \nabla T_\lambda(u)) - a(x, t, T_\lambda(u), \nabla v))
$$
$$
\times (\nabla T_\lambda(u) - \nabla v)\chi_D\,\mathrm{d}x\,\mathrm{d}t
$$
$$
+ \lim_{n\to\infty} \int_Q a(x, t, T_\lambda(u_n), \nabla v)(\nabla T_\lambda(u_n) - \nabla v)\chi_{D_n}\,\mathrm{d}x\,\mathrm{d}t.
$$

The second term on the right-hand side of (5.40) is equal to

$$\int_Q a(x, T_\lambda(u), \nabla v)(\nabla T_\lambda(u) - \nabla v)\chi_D \,\mathrm{d}x\,\mathrm{d}t.$$

Finally, we get

$$\begin{aligned}
\liminf_{n\to\infty} \int_Q & a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant \int_Q a(x,t,T_\lambda(u),\nabla T_\lambda(u))(\nabla T_\lambda(u) - \nabla v)\chi_D\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q a(x,t,u,\nabla u)(\nabla u - \nabla v)\chi_D\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q a(x,t,u,\nabla u)\nabla T_k(u - v)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.41}$$

For the first term on the right-hand side of (5.36), using the strong convergence of $(f_n)_n$, we get

$$\int_Q f_n T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t = \int_Q f T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n). \tag{5.42}$$

For the second term on the right-hand side of (5.36), for $n \geqslant \lambda = k + \|v\|_\infty$, we have

$$\begin{aligned}
\int_Q \Theta_n(x,t,u_n)\nabla T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t &= \int_Q \Theta(x,t,T_\lambda(u_n))\nabla T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \Theta(x,t,u)\nabla T_k(u - v)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n).
\end{aligned} \tag{5.43}$$

Combining (5.36)–(5.43), one has

$$\begin{aligned}
\int_Q \frac{\partial b(u)}{\partial t} T_k(u - v)\,\mathrm{d}x\,\mathrm{d}t &+ \int_Q a(x,t,u,\nabla u)\nabla T_k(u - v)\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \int_Q f T_k(u - v)\,\mathrm{d}x\,\mathrm{d}t + \int_Q \Theta(x,t,u)\nabla T_k(u - v)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Consequently, via all steps, the proof of Theorem 5.1 is completed. □

References

[1] *A. Aberqi, J. Bennouna, M. Mekkour, H. Redwane*: Existence results for a nonlinear parabolic problems with lower order terms. Int. J. Math. Anal., Ruse *7* (2013), 1323–1340. zbl MR doi

[2] *M. L. Ahmed Oubeid, A. Benkirane, M. Sidi El Vally*: Nonlinear elliptic equations involving measure data in Musielak-Orlicz-Sobolev spaces. J. Abstr. Differ. Equ. Appl. *4* (2013), 43–57. zbl MR

[3] *M. L. Ahmed Oubeid, A. Benkirane, M. Sidi El Vally*: Parabolic equations in Musielak-Orlicz-Sobolev spaces. Int. J. Anal. Appl. *4* (2014), 174–191. zbl

[4] *M. L. Ahmed Oubeid, A. Benkirane, M. Sidi El Vally*: Strongly nonlinear parabolic problems in Musielak-Orlicz-Sobolev spaces. Bol. Soc. Paran. Mat. (3) *33* (2015), 193–223. MR

[5] *M. Ait Khellou, A. Benkirane, S. M. Douiri*: Existance of solutions for elliptic equations having naturel growth terms in Musielak-Orlicz spaces. J. Math. Comput. Sci. *4* (2014), 665–688.

[6] *E. Azroul, M. B. Benboubker, H. Redwane, C. Yazough*: Renormalized solutions for a class of nonlinear parabolic equations without sign condition involving nonstandard growth. An. Univ. Craiova, Ser. Mat. Inf. *41* (2014), 69–87. zbl MR

[7] *E. Azroul, M. El Lekhlifi, H. Redwane, A. Touzani*: Entropy solutions of nonlinear parabolic equations in Orlicz-Sobolev spaces, without sign condition and L^1 data. J. Nonlinear Evol. Equ. Appl. *2014* (2014), 101–130. zbl MR

[8] *E. Azroul, H. Hjiaj, A. Touzani*: Existence and regularity of entropy solutions for strongly nonlinear $p(x)$-elliptic equations. Electron. J. Differ. Equ. *2013* (2013), Paper No. 68, 27 pages. zbl MR

[9] *P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vazquez*: An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. *22* (1995), 241–273. zbl MR

[10] *A. Benkirane, M. Ould Mohamedhen Val*: Some approximation properties in Musielak-Orlicz-Sobolev spaces. Thai. J. Math. *10* (2012), 371–381. zbl MR

[11] *A. Benkirane, M. Sidi El Vally*: Variational inequalities in Musielak-Orlicz-Sobolev spaces. Bull. Belg. Math. Soc.-Simon Stevin *21* (2014), 787–811. zbl MR

[12] *M. S. B. Elemine Vall, A. Ahmed, A. Touzani, A. Benkirane*: Existence of entropy solutions for nonlinear elliptic equations in Musielak framework with L^1 data. Bol. Soc. Paran. Math. (3) *36* (2018), 125–150. MR doi

[13] *A. Elmahi, D. Meskine*: Parabolic equations in Orlicz spaces. J. Lond. Math. Soc., II. Ser. *72* (2005), 410–428. zbl MR doi

[14] *A. Elmahi, D. Meskine*: Strongly nonlinear parabolic equations with natural growth terms in Orlicz spaces. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *60* (2005), 1–35. zbl MR doi

[15] *J. P. Gossez*: Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. Trans. Am. Math. Soc. *190* (1974), 163–205. zbl MR doi

[16] *S. Hadj Nassar, H. Moussa, M. Rhoudaf*: Renormalized solution for a nonlinear parabolic problems with noncoercivity in divergence form in Orlicz spaces. Appl. Math. Comput. *249* (2014), 253–264. zbl MR doi

[17] *R. Landes, V. Mustonen*: A strongly nonlinear parabolic initial boundary value problem. Ark. Mat. *25* (1987), 29–40. zbl MR doi

[18] *J. Musielak*: Orlicz Spaces and Modular Spaces. Lecture Notes in Math. 1034. Springer, Berlin, 1983. zbl MR doi

[19] *H. Redwane*: Existence of a solution for a class of parabolic equations with three unbounded nonlinearities. Adv. Dyn. Syst. Appl. *2* (2007), 241–264. MR

[20] *H. Redwane*: Existence results for a class of nonlinear parabolic equations in Orlicz spaces. Electron. J. Qual. Theory Differ. Equ. *2010* (2010), Paper No. 2, 19 pages. zbl MR doi
[21] *M. Sidi El Vally*: Strongly nonlinear elliptic problems in Musielak-Orlicz-Sobolev spaces. Adv. Dyn. Syst. Appl. *8* (2013), 115–124. MR

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