FURTHER INVESTIGATION "ON AN OPEN PROBLEM OF ZHANG AND XU"

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Abstract. The purpose of the paper is to study the uniqueness of meromorphic functions sharing a nonzero polynomial. The result of the paper improves and generalizes the recent results due to X. B. Zhang and J. F. Xu (2011). We also solve an open problem posed in the last section of X. B. Zhang and J. F. Xu (2011).

Keywords: uniqueness; meromorphic function; small function; nonlinear differential polynomial

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1. Introduction, definitions and results

In this paper, by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [6]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\to\infty$, outside of a possible exceptional set of finite linear measure.

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A finite value z_0 is said to be a fixed point of $f(z)$ if $f(z_0) = z_0$. Throughout this paper, we need the following definition:

$$\Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where a is a value in the extended complex plane.

In 1959, Hayman (see [5], Corollary of Theorem 9) proved the following theorem.

Theorem A. *Let f be a transcendental meromorphic function and $n \in \mathbb{N}$ with $n \geqslant 3$. Then $f^n f' = 1$ has infinitely many solutions.*

In 1997, Yang and Hua obtained the following uniqueness result corresponding to Theorem A.

Theorem B ([16]). *Let f and g be two non-constant meromorphic functions, $n \in \mathbb{N}$ with $n \geqslant 11$. If $f^n f'$ and $g^n g'$ share* 1 CM, *then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2002, using the idea of sharing fixed points, Fang and Qiu further generalized and improved Theorem B in the following manner.

Theorem C ([3]). *Let f and g be two non-constant meromorphic functions and let $n \in \mathbb{N}$ with $n \geqslant 11$. If $f^n f' - z$ and $g^n g' - z$ share* 0 CM, *then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

During the last couple of years a handful number of astonishing results have been obtained regarding the value sharing of nonlinear differential polynomials which are mainly the kth derivative of some linear expression of f and g.

In 2010, Xu, Lü and Yi studied the analogous problem corresponding to Theorem C, where in addition to the fixed point sharing problem, sharing of poles are also taken under supposition. Thus, the research has somehow been shifted towards the following direction.

Theorem D ([13]). *Let f and g be two non-constant meromorphic functions and let $n, k \in \mathbb{N}$ with $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z* CM, *f and g share ∞* IM, *then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^n c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem E ([13]). *Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > 2/n$ and let $n, k \in \mathbb{N}$ with $n \geqslant 3k+12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, f and g share ∞ IM, then $f \equiv g$.*

Recently Zhang and Xu (see [20]) further generalized as well as improved the results of [13] as follows.

Theorem F ([20]). *Let f and g be two transcendental meromorphic functions, let $p(z)$ be a nonzero polynomial with $\deg(p) = l \leqslant 5$, $n, k, m \in \mathbb{N}$ with $n > 3k + m + 7$. Let $P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$ be a nonzero polynomial. If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share p CM, f and g share ∞ IM, then one of the following three cases holds:*

(1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = \mathrm{GCD}(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$;*
(2) *f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$;*
(3) *$P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$;*
if $p(z)$ is not a constant, then $f(z) = c_1 \mathrm{e}^{cQ(z)}$, $g(z) = c_2 \mathrm{e}^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)\,\mathrm{d}t$, c_1, c_2 and c are constants such that $a_i^2(c_1c_2)^{n+i}((n+i)c)^2 = -1$,
if $p(z)$ is a nonzero constant b, then $f(z) = c_3\mathrm{e}^{cz}$, $g(z) = c_4\mathrm{e}^{-cz}$, where c_3, c_4 and c are constants such that $(-1)^k a_i^2(c_3c_4)^{n+i}((n+i)c)^{2k} = b^2$.

Zhang and Xu made the following comment in Remark 1.2 in [20]:

"From the proof of Theorem 1.3, we can see that the computation will be very complicated when $\deg(p)$ becomes large, so we are not sure whether Theorem 1.3 holds for the general polynomial $p(z)$."

Also at the end of the paper, the following open problem was posed by the authors in [20].

Open problem. What happens to Theorem 1.3 (see [20]) if the condition "$l \leqslant 5$" is removed?

One of our objectives is to solve this open problem. Now observing the above results, the following question is inevitable.

Question 1.1. Can the lower bound of n be further reduced in Theorem F?

Before going to our main result we explain the following definition and notation which is used in the paper.

Definition 1.1 ([8], [9]). Let $k \in \mathbb{N} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted

m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity m ($\leqslant k$) if and only if it is an a-point of g with multiplicity m ($\leqslant k$) and z_0 is an a-point of f with multiplicity m ($> k$) if and only if it is an a-point of g with multiplicity n ($> k$), where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leqslant p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

In this paper, taking the possible answer of the above question into consideration we obtain the following result.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions and let $n, k \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ such that $n > 3k + m + 6$. Let $p(z)$ be a nonzero polynomial such that* $\deg(p) \neq (n+i)s$, *where $s \in \mathbb{N}$, $i \in \{0, 1, \ldots, m\}$ and $P(w)$ be defined as in Theorem* F. *If $(f^n P(f))^{(k)}$, $(g^n P(g))^{(k)}$ share (p, k_1), where $k_1 = (3+k)(n+m-k-1)^{-1} + 3$ and f, g share $(\infty, 0)$, then the conclusion of Theorem* F *holds.*

We now further explain the following definitions and notations, which are used in the paper.

Definition 1.2 ([7]). Let $a \in \mathbb{C} \cup \{\infty\}$. For $p \in \mathbb{N}$ we denote by $N(r, a; f \mid\leqslant p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid\leqslant p)$ we denote the corresponding reduced counting function.

In an analogous manner we can define $N(r, a; f \mid\geqslant p)$ and $\overline{N}(r, a; f \mid\geqslant p)$.

Definition 1.3 ([9]). Let $k \in \mathbb{N} \cup \{\infty\}$. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geqslant 2) + \ldots + \overline{N}(r, a; f \mid\geqslant k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.4 ([2]). Let f and g be two non-constant meromorphic functions such that f and g share the value a IM for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p and also an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ the reduced counting function of those a-points of f and g, respectively, where $p > q \geqslant 1$ ($q > p \geqslant 1$). Also we denote by $\overline{N}_E^{(1}(r,a;f)$ the reduced counting function of those a-points of f and g, where $p = q \geqslant 1$.

Definition 1.5 ([8], [9]). Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

Definition 1.6 ([10]). Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;$ $f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H and V the functions as:

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big) \tag{2.1}$$

and

$$V = \Big(\frac{F'}{F-1} - \frac{F'}{F}\Big) - \Big(\frac{G'}{G-1} - \frac{G'}{G}\Big). \tag{2.2}$$

Lemma 2.1 ([15]). *Let f be a non-constant meromorphic function and let $a_n(z)$ ($\not\equiv 0$), $a_{n-1}(z)$, ..., $a_0(z)$ be meromorphic functions such that $T(r,a_i(z)) = S(r,f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2.2 ([19]). *Let f be a non-constant meromorphic function and $p, k \in \mathbb{N}$. Then*

$$N_p(r,0;f^{(k)}) \leqslant T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{2.3}$$

$$N_p(r,0;f^{(k)}) \leqslant k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{2.4}$$

Lemma 2.3 ([11]). *If $N(r,0;f^{(k)}\mid f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}\mid f\neq 0)\leqslant k\overline{N}(r,\infty;f)+N(r,0;f\mid< k)+k\overline{N}(r,0;f\mid\geqslant k)+S(r,f).$$

Lemma 2.4 ([6], Theorem 3.10). *Suppose that f is a non-constant meromorphic function, $k\in\mathbb{N}\setminus\{1\}$. If*

$$N(r,\infty;f)+N(r,0;f)+N(r,0;f^{(k)})=S\Big(r,\frac{f'}{f}\Big),$$

then $f(z)=\mathrm{e}^{az+b}$, where $a\neq 0$, b are constants.

Lemma 2.5 ([4]). *Let $f(z)$ be a non-constant entire function and let $k\in\mathbb{N}\setminus\{1\}$. If $f(z)f^{(k)}(z)\neq 0$, then $f(z)=\mathrm{e}^{az+b}$, where $a\neq 0,b$ are constant.*

Lemma 2.6 ([17], Theorem 1.24). *Let f be a non-constant meromorphic function and let $k\in\mathbb{N}$. Suppose that $f^{(k)}\not\equiv 0$. Then*

$$N(r,0;f^{(k)})\leqslant N(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.7 ([20]). *Let f and g be two non-constant meromorphic functions, let $P(w)$ be defined as in Theorem* F *and $n,k\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$ with $n>2k+m+1$. If $(f^nP(f))^{(k)}=(g^nP(g))^{(k)}$, then $f^nP(f)=g^nP(g)$.*

Lemma 2.8 ([18], Lemma 6). *If $H=0$, then F, G share* 1 CM. *If further F, G share ∞* IM, *then F, G share ∞* CM.

Lemma 2.9 ([20]). *Let f, g be non-constant meromorphic functions, let $n,k\in\mathbb{N}$ with $n>k+2$, and let $P(w)$ be defined as in* Theorem F. *Let $\alpha(z)$ $(\not\equiv 0,\infty)$ be a small function with respect to f with finitely many zeros and poles. If $(f^nP(f))^{(k)}(g^nP(g))^{(k)}=\alpha^2$, f and g share ∞* IM, *then $P(w)$ is reduced to a nonzero monomial, namely $P(w)=a_iw^i\not\equiv 0$ for some $i\in\{0,1,\ldots,m\}$.*

Lemma 2.10 ([17]). *Let f_j, $j=1,2,3$ be meromorphic and f_1 be non-constant. Suppose that*

$$\sum_{j=1}^{3}f_j=1\quad\text{and}\quad\sum_{j=1}^{3}N(r,0;f_j)+2\sum_{j=1}^{3}\overline{N}(r,\infty;f_j)<(\lambda+o(1))T(r),$$

as $r\to\infty$, $r\in I$, $\lambda<1$ and $T(r)=\max\limits_{1\leqslant j\leqslant 3}T(r,f_j)$. Then $f_2=1$ or $f_3=1$.

Lemma 2.11. *Let f, g be two transcendental meromorphic functions and let $P(w)$ be defined as in Theorem* F. *Let $F=(f^nP(f))^{(k)}p^{-1}$, $G=(g^nP(g))^{(k)}p^{-1}$, where $p(z)$ is a nonzero polynomial and $n,k\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$ such that $n>3k+m+3$. If f,g share ∞* IM *and $H=0$, then either $(f^nP(f))^{(k)}(g^nP(f))^{(k)}=p^2$, where $(f^nP(f))^{(k)}$ and $(g^nP(f))^{(k)}$ share p* CM, *or $f^nP(f)=g^nP(g)$.*

P r o o f. Since $H\equiv 0$, by Lemma 2.8 we get F and G share 1 CM. On integration we get

$$\frac{1}{F-1}=\frac{bG+a-b}{G-1}, \tag{2.5}$$

where a,b are constants and $a\neq 0$. We now consider the following cases:

Case 1. Let $b\neq 0$ and $a\neq b$. If $b=-1$, then from (2.5) we have

$$F=\frac{-a}{G-a-1}.$$

Therefore

$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+S(r,f).$$

So in view of Lemmas 2.1 and 2.2 for $p=1$ and using the second fundamental theorem we get

$$\begin{aligned}(n+m)T(r,g)&\leqslant T(r,G)+N_{k+1}(r,0;g^nP(g))-\overline{N}(r,0;G)\\&\leqslant \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,a+1;G)\\&\quad+N_{k+1}(r,0;g^nP(g))-\overline{N}(r,0;G)+S(r,g)\\&\leqslant \overline{N}(r,\infty;g)+N_{k+1}(r,0;g^nP(g))+\overline{N}(r,\infty;f)+S(r,g)\\&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+N_{k+1}(r,0;g^n)\\&\quad+N_{k+1}(r,0;P(g))+S(r,g)\\&\leqslant 2\overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,P(g))+S(r,g)\\&\leqslant (k+3+m)T(r,g)+S(r,g),\end{aligned}$$

which is a contradiction since $n>k+3$. If $b\neq -1$, from (2.5) we obtain that

$$F-\Big(1+\frac{1}{b}\Big)=\frac{-a}{b^2(G+(a-b)/b)}.$$

So

$$\overline{N}\Big(r,\frac{b-a}{b};G\Big)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+S(r,f).$$

Using Lemmas 2.1, 2.2 and the same argument as in the case when $b=-1$ we can get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$. If $b = -1$, then from (2.5) we have

$$FG = 1,$$

i.e.

$$(f^n P(f))^{(k)} (g^n P(g))^{(k)} = p^2,$$

where $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share p CM.

If $b \neq -1$, from (2.5) we have

$$\frac{1}{F} = \frac{bG}{(1+b)G - 1}.$$

Therefore

$$\overline{N}\Big(r, \frac{1}{1+b}; G\Big) = \overline{N}(r, 0; F).$$

So in view of Lemmas 2.1 and 2.2 for $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned}
(n+m)T(r,g) &\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\Big(r, \frac{1}{1+b}; G\Big) \\
&\quad + N_{k+1}(r,0;g^n P(g)) - \overline{N}(r,0;G) + S(r,g) \\
&\leqslant \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + \overline{N}(r,0;F) + S(r,g) \\
&\leqslant \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + (k+1)\overline{N}(r,0;f) \\
&\quad + T(r,P(f)) + k\overline{N}(r,\infty;f) + S(r,f) + S(r,g) \\
&\leqslant (k+2+m)T(r,g) + (2k+1+m)T(r,f) + S(r,f) + S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leqslant T(r,g)$ for $r \in I$. So for $r \in I$ we have

$$(n - 3k - 3 - m)T(r,g) \leqslant S(r,g),$$

which is a contradiction since $n > 3k + 3 + m$.

Case 3. Let $b = 0$. From (2.5) we obtain

$$F = \frac{G + a - 1}{a}. \tag{2.6}$$

If $a \neq 1$, then from (2.6) we obtain

$$\overline{N}(r, 1-a; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in Case 2. Therefore $a = 1$ and from (2.6) we obtain $F = G$, i.e. $(f^n P(f))^{(k)} = (g^n P(g))^{(k)}$. Then by Lemma 2.7 we have $f^n P(f) = g^n P(g)$. This completes the proof. $\square$

Lemma 2.12. *Let f and g be two transcendental entire functions such that $f(z) = h_1(z)e^{\alpha(z)}$ and $g(z) = h_2(z)e^{\beta(z)}$, where h_1, h_2 are nonzero polynomials and α, β are two non-constant polynomials such that $\alpha(z) + \beta(z) = C_0$, $C_0 \in \mathbb{C}$. Let $n, k \in \mathbb{N}$ such that $n > k$ and $p(z)$ be a non-constant polynomial such that $\deg(p) \neq ns$, where $s \in \mathbb{N}$. If $(f^n)^{(k)} = pe^{\gamma}$ and $(g^n)^{(k)} = pe^{-\gamma}$, where γ is a non-constant entire function, then both h_1 and h_2 must be nonzero constants.*

P r o o f. By the given condition either both h_1 and h_2 are non-constant polynomials or both are nonzero constants.

First we suppose both h_1 and h_2 are non-constant polynomials. Also we have $\alpha + \beta = C_0$, i.e. $\alpha' \equiv -\beta'$. This shows that $\deg(\alpha) = \deg(\beta)$ and $\deg(\alpha^{(i)}) = \deg(\beta^{(i)})$, where $i \in \mathbb{N}$. We claim that for all $l, t \in \mathbb{N}$ with $t > l$,

$$(f^t)^{(l)} = h_1^{t-l}\Big(t^l h_1^l(\alpha')^l + lt^l h_1^{l-1}h_1'(\alpha')^{l-1} + \frac{l(l-1)}{2}t^{l-1}h_1^l(\alpha')^{l-2}\alpha'' + P_{l-1}(h_1, h_1', \alpha')\Big)e^{t\alpha}, \tag{2.7}$$

where $P_{l-1}(h_1, h_1', \alpha')$ is a differential polynomial in h_1, h_1' and α'. Also we define $P_0 = 0$. We will use the mathematical induction to prove the claim. Since $f(z) = h_1(z)e^{\alpha(z)}$, we deduce that

$$(f^t)' = h_1^{t-1}(th_1\alpha' + th_1')e^{t\alpha},$$
$$(f^t)'' = h_1^{t-2}(t^2h_1^2(\alpha')^2 + 2t^2h_1h_1'\alpha' + th_1^2(\alpha')^{2-2}\alpha'' + t(t-1)(h_1')^2 + th_1h_1'')e^{t\alpha}$$

and

$$(f^t)''' = h_1^{t-3}(t^3h_1^3(\alpha')^3 + 3t^3h_1^{3-1}h_1'(\alpha')^{3-1} + 3t^2h_1^3(\alpha')^{3-2}\alpha'' + 3t^2(t-1)h_1(h_1')^2\alpha' + \ldots)e^{t\alpha}.$$

Therefore the claim is true for $l = 1$ with $t > 1$, $l = 2$ with $t > 2$ and $l = 3$ with $t > 3$, respectively. We assume that the claim is true for $l = l^*$ with $t > l^*$, i.e.

$$(f^t)^{(l^*)} = h_1^{t-l^*}(t^{l^*}h_1^{l^*}(\alpha')^{l^*} + l^*t^{l^*}h_1^{l^*-1}h_1'(\alpha')^{l^*-1} + \frac{l^*(l^*-1)}{2}t^{l^*-1}h_1^{l^*}(\alpha')^{l^*-2}\alpha'' + P_{l^*-1}(h_1, h_1', \alpha'))e^{t\alpha}.$$

Now we prove that the claim is also true for $l = l^* + 1$ with $t > l^* + 1$. By differentiation we have

$$(f^t)^{(l^*+1)} = \Big(t^{l^*+1}h_1^t(\alpha')^{l^*+1} + t^{l^*+1}h_1^{t-1}h_1'(\alpha')^{l^*} + l^*t^{l^*}h_1^t(\alpha')^{l^*-1}\alpha'' + l^*t^{l^*+1}h_1^{t-1}h_1'(\alpha')^{l^*} + \ldots + \frac{l^*(l^*-1)}{2}t^{l^*}h_1^t(\alpha')^{l^*-1}\alpha'' + \ldots + h_1^{t-l^*-1}h_1'P_{l^*-1}(h_1, h_1', \alpha') + h_1^{t-l^*}P_{l^*-1}'(h_1, h_1', \alpha')\Big)e^{t\alpha}$$

$$
\begin{aligned}
&= h_1^{t-(l^*+1)}\Big(t^{l^*+1}h_1^{l^*+1}(\alpha')^{l^*+1} + (l^*+1)t^{l^*+1}h_1^{l^*}h_1'(\alpha')^{l^*} \\
&\quad + \frac{l^*(l^*+1)}{2}t^{l^*}h_1^{l^*+1}(\alpha')^{l^*-1}\alpha'' + P_{l^*}(h_1, h_1', \alpha')\Big)\mathrm{e}^{t\alpha}.
\end{aligned}
$$

So we complete the proof of the claim. Since $n > k$, we have

$$
\begin{aligned}
(f^n)^{(k)} &= h_1^{n-k}\Big(n^k h_1^k(\alpha')^k + kn^k h_1^{k-1}h_1'(\alpha')^{k-1} \\
&\quad + \frac{k(k-1)}{2}n^{k-1}h_1^k(\alpha')^{k-2}\alpha'' + P_{k-1}(h_1, h_1', \alpha')\Big)\mathrm{e}^{n\alpha}.
\end{aligned}
\tag{2.8}
$$

Similarly we get

$$
\begin{aligned}
(g^n)^{(k)} &= h_2^{n-k}\Big(n^k h_2^k(\beta')^k + kn^k h_2^{k-1}h_2'(\beta')^{k-1} + \frac{k(k-1)}{2}n^{k-1}h_2^k(\beta')^{k-2}\beta'' \\
&\quad + P_{k-1}(h_2, h_2', \beta')\Big)\mathrm{e}^{n\beta} \\
&= h_2^{n-k}\Big((-1)^k n^k h_2^k(\alpha')^k + (-1)^{k-1}kn^k h_2^{k-1}h_2'(\alpha')^{k-1} \\
&\quad + (-1)^{k-1}\frac{k(k-1)}{2}n^{k-1}h_2^k(\alpha')^{k-2}\alpha'' + P_{k-1}(h_2, h_2', -\alpha')\Big)\mathrm{e}^{n\beta}.
\end{aligned}
\tag{2.9}
$$

Here we see that every term of $P_{k-1}(h_1, h_1', \alpha')$ has the form

$$
Kh_1^{l_0}(h_1')^{l_1}\dots(h_1^{(k)})^{l_k}(\alpha')^{m_0}(\alpha'')^{m_1}\dots(\alpha^{(k)})^{m_k},
$$

where $l_0, \dots, l_k, m_0, \dots, m_k \in \mathbb{N} \cup \{0\}$ and K is a suitably positive integer.

Note that $\deg(h_1^k(\alpha')^k) > \deg(h_1^{k-1}h_1'(\alpha')^{k-1}) = \deg(h_1^k(\alpha')^{k-2}\alpha'')$. Also

$$
\deg(h_1^{l_0}(h_1')^{l_1}\dots(h_1^{(k)})^{l_k}(\alpha')^{m_0}(\alpha'')^{m_1}\dots(\alpha^{(k)})^{m_k}) < \deg(h_1^k(\alpha')^{k-2}\alpha'').
$$

Let

$$
\begin{aligned}
h_1(z) &= a_{1p}z^p + a_{1p-1}z^{p-1} + \dots + a_{10}, \\
h_2(z) &= b_{1q}z^q + b_{1q-1}z^{q-1} + \dots + b_{10}
\end{aligned}
$$

and

$$
(\alpha(z))' = c_{1r}z^r + c_{1r-1}z^{r-1} + \dots + c_{10},
$$

where $a_{1p}, b_{1q}, c_{1r} \in \mathbb{C} \setminus \{0\}$. Then we have

$$
\begin{aligned}
(h_1(z))^i &= a_{1p}^i z^{ip} + ia_{1p}^{i-1}a_{1p-1}z^{ip-1} + \dots, \\
(h_2(z))^i &= b_{1q}^i z^{iq} + ib_{1q}^{i-1}b_{1q-1}z^{iq-1} + \dots,
\end{aligned}
$$

and

$$
((\alpha(z))')^i = c_{1r}^i z^{ir} + ic_{1r}^{i-1}c_{1r-1}z^{ir-1} + \dots,
$$

where $i \in \mathbb{N}$. Then

$$(f^n)^{(k)} = \Big(n^k a_{1p}^n c_{1r}^k z^{np+kr} + n^k a_{1p}^{n-1} c_{1r}^{k-1}(k a_{1p} c_{1r-1} + n a_{1p-1} c_{1r}) z^{np+kr-1} \\ + \Big(pkn^k a_{1p}^n c_{1r}^{k-1} + \frac{k(k-1)}{2} n^{k-1} a_{1p}^n c_{1r}^{k-1} r\Big) z^{np+r(k-1)-1} + \ldots\Big) \mathrm{e}^{n\alpha}$$

and

$$(g^n)^{(k)} = \Big((-1)^k n^k b_{1q}^n c_{1r}^k z^{nq+kr} + (-1)^k n^k b_{1q}^{n-1} c_{1r}^{k-1}(k b_{1q} c_{1r-1} + n b_{1q-1} c_{1r}) z^{nq+kr-1} \\ + (-1)^{k-1}\Big(qkn^k b_{1q}^n c_{1r}^{k-1} + \frac{k(k-1)}{2} n^{k-1} b_{1q}^n c_{1r}^{k-1} r\Big) z^{nq+r(k-1)-1} + \ldots\Big) \mathrm{e}^{n\beta}.$$

Since $(f^n)^{(k)} = p\mathrm{e}^{\gamma}$ and $(g^n)^{(k)} = p\mathrm{e}^{-\gamma}$, it follows that

$$\begin{aligned} &n^k a_{1p}^n c_{1r}^k z^{np+kr} + n^k a_{1p}^{n-1} c_{1r}^{k-1}(k a_{1p} c_{1r-1} + n a_{1p-1} c_{1r}) z^{np+kr-1} \\ &\quad + \Big(pkn^k a_{1p}^n c_{1r}^{k-1} + \frac{k(k-1)}{2} n^{k-1} a_{1p}^n c_{1r}^{k-1} r\Big) z^{np+r(k-1)-1} + \ldots \qquad (2.10) \\ &\equiv d_1 p(z) \qquad (2.11) \end{aligned}$$

and

$$\begin{aligned} &(-1)^k n^k b_{1q}^n c_{1r}^k z^{nq+kr} + (-1)^k n^k b_{1q}^{n-1} c_{1r}^{k-1}(k b_{1q} c_{1r-1} + n b_{1q-1} c_{1r}) z^{nq+kr-1} \\ &\quad + (-1)^{k-1}\Big(qkn^k b_{1q}^n c_{1r}^{k-1} + \frac{k(k-1)}{2} n^{k-1} b_{1q}^n c_{1r}^{k-1} r\Big) z^{nq+r(k-1)-1} + \ldots \qquad (2.12) \\ &\equiv d_2 p(z), \qquad (2.13) \end{aligned}$$

where $d_1, d_2 \in \mathbb{C} \setminus \{0\}$. From (2.10) and (2.12) it is clear that $p = q$.

Now we consider the following two cases.

Case 1. Let $\deg(\alpha') = r \in \mathbb{N}$. If $np + kr = ns$, where $s \in \mathbb{N}$, then we arrive at a contradiction from (2.10) and (2.12). Next we suppose $np + kr \neq ns$. Then from (2.10) and (2.12) we get

$$\begin{aligned} &n^k a_{1p}^n c_{1r}^k z^{np+kr} + n^k a_{1p}^{n-1} c_{1r}^{k-1}(k a_{1p} c_{1r-1} + n a_{1p-1} c_{1r}) z^{np+kr-1} \qquad (2.14) \\ &\quad + \Big(pkn^k a_{1p}^n c_{1r}^{k-1} + \frac{k(k-1)}{2} n^{k-1} a_{1p}^n c_{1r}^{k-1} r\Big) z^{np+r(k-1)-1} + \ldots \\ &\equiv d\Big((-1)^k n^k b_{1q}^n c_{1r}^k z^{nq+kr} \\ &\quad + (-1)^k n^k b_{1q}^{n-1} c_{1r}^{k-1}(k b_{1q} c_{1r-1} + n b_{1q-1} c_{1r}) z^{nq+kr-1} \\ &\quad + (-1)^{k-1}\Big(qkn^k b_{1q}^n c_{1r}^{k-1} + \frac{k(k-1)}{2} n^{k-1} b_{1q}^n c_{1r}^{k-1} r\Big) z^{nq+r(k-1)-1} + \ldots\Big), \end{aligned}$$

where $d \in \mathbb{C} \setminus \{0\}$. Note that

$$pkn^k a_{1p}^n c_{1r}^{k-1} + \frac{k(k-1)}{2} n^{k-1} a_{1p}^n c_{1r}^{k-1} r = \frac{2np + (k-1)r}{2} kn^{k-1} a_{1p}^n c_{1r}^{k-1} \neq 0$$

and

$$qkn^k b_{1q}^n c_{1r}^{k-1} + \frac{k(k-1)}{2} n^{k-1} b_{1q}^n c_{1r}^{k-1} r = \frac{2nq + (k-1)r}{2} kn^{k-1} b_{1q}^n c_{1r}^{k-1} \neq 0.$$

Since $p = q$, from (2.14) we get

$$a_{1p}^n = (-1)^k d b_{1p}^n, \tag{2.15}$$

$$a_{1p}^{n-1}(ka_{1p}c_{1r-1} + na_{1p-1}c_{1r}) = (-1)^k d b_{1p}^{n-1}(kb_{1p}c_{1r-1} + nb_{1p-1}c_{1r}) \tag{2.16}$$

and

$$\begin{aligned} pkn^k a_{1p}^n c_{1r}^{k-1} &+ \frac{k(k-1)}{2} n^{k-1} a_{1p}^n c_{1r}^{k-1} r \\ &= (-1)^{k-1} d\Big(pkn^k b_{1p}^n c_{1r}^{k-1} + \frac{k(k-1)}{2} n^{k-1} b_{1p}^n c_{1r}^{k-1} r\Big). \end{aligned} \tag{2.17}$$

Then (2.15) and (2.17) yield

$$pn + \frac{k-1}{2} r = 0,$$

which is impossible.

Case 2. Let $\deg(\alpha') = r = 0$. Now from (2.10) we get $\deg(p) = np$, which is a contradiction. Hence, both h_1 and h_2 must be nonzero constants. This completes the proof. □

Lemma 2.13. *Let f, g be two transcendental meromorphic functions and $n, k \in \mathbb{N}$ such that $n > k$. Suppose $p(z)$ be a nonzero polynomial such that $\deg(p) \neq ns$, where $s \in \mathbb{N}$. Suppose $(f^n)^{(k)}$, $(g^n)^{(k)}$ share p CM and f, g share ∞ IM. Now when $(f^n)^{(k)}(g^n)^{(k)} = p^2$,*

(i) *if $p(z)$ is not a constant, then $f(z) = c_1 \mathrm{e}^{cQ(z)}$, $g(z) = c_2 \mathrm{e}^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)\,\mathrm{d}t$, c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$;*

(ii) *if $p(z)$ is a nonzero constant b, then $f(z) = c_3 \mathrm{e}^{dz}$, $g(z) = c_4 \mathrm{e}^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3c_4)^n (nd)^{2k} = b^2$.*

Proof. Suppose

$$(f^n)^{(k)}(g^n)^{(k)} = p^2. \tag{2.18}$$

Since f and g share ∞ IM, from (2.18) one can easily say that f and g are transcendental entire functions. We consider the following cases.

Case 1. Let $\deg(p(z)) = l \in \mathbb{N}$. Since $n > k$, it follows that $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$.

Let

$$F_1 = \frac{(f^n)^{(k)}}{p} \quad \text{and} \quad G_1 = \frac{(g^n)^{(k)}}{p}. \tag{2.19}$$

From (2.18) we get

$$F_1 G_1 = 1. \tag{2.20}$$

If $F_1 = c_1^* G_1$, where $c_1^* \in \mathbb{C}\backslash\{0\}$, then by (2.20) F_1 is constant and so f is polynomial, which contradicts our assumption. Hence $F_1 \neq c_1^* G_1$.

Let

$$\Phi = \frac{(f^n)^{(k)} - p}{(g^n)^{(k)} - p}. \tag{2.21}$$

We deduce from (2.21) that

$$\Phi = \mathrm{e}^{\gamma_1}, \tag{2.22}$$

where γ_1 is an entire function.

Let $f_1 = F_1$, $f_2 = -\mathrm{e}^{\gamma_1} G_1$ and $f_3 = \mathrm{e}^{\gamma_1}$. Here f_1 is transcendental. Now from (2.22) we have

$$f_1 + f_2 + f_3 = 1.$$

Hence, by Lemma 2.6 we get

$$\begin{aligned}\sum_{j=1}^{3} N(r,0;f_j) + 2\sum_{j=1}^{3} \overline{N}(r,\infty;f_j) &\leqslant N(r,0;F_1) + N(r,0;\mathrm{e}^{\gamma_1} G_1) + O(\log r)\\ &\leqslant (\lambda + o(1))T(r),\end{aligned}$$

as $r \to \infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max\limits_{1 \leqslant j \leqslant 3} T(r, f_j)$.

So by Lemma 2.10 we get either $\mathrm{e}^{\gamma_1} G_1 = -1$ or $\mathrm{e}^{\gamma_1} = 1$. But here the only possibility is that $\mathrm{e}^{\gamma_1} G_1 = -1$, i.e. $(g^n)^{(k)} = -\mathrm{e}^{-\gamma_1} p(z)$ and so from (2.18) we get

$$(f^n)^{(k)} = c_2^* \mathrm{e}^{\gamma_1} p, \quad (g^n)^{(k)} = c_2^* \mathrm{e}^{-\gamma_1} p, \tag{2.23}$$

where $c_2^* = \pm 1$. Since $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$, we can take

$$f(z) = h_1(z)\mathrm{e}^{\alpha(z)}, \quad g(z) = h_2(z)\mathrm{e}^{\beta(z)}, \tag{2.24}$$

where h_1 and h_2 are nonzero polynomials and α, β are two non-constant entire functions.

We deduce from (2.18) and (2.24) that either both α and β are transcendental entire functions or both are polynomials. We consider the following subcases.

Subcase 1.1. Let $k \in \mathbb{N} \setminus \{1\}$. First we suppose both α and β are transcendental entire functions. Let $\alpha_1 = \alpha' + h_1'/h_1$ and $\beta_1 = \beta' + h_2'/h_2$. Clearly both α_1 and β_1 are transcendental functions.

Note that

$$S(r, n\alpha_1) = S\Bigl(r, \frac{(f^n)'}{f^n}\Bigr), \quad S(r, n\beta_1) = S\Bigl(r, \frac{(g^n)'}{g^n}\Bigr).$$

Moreover, we see that

$$N(r,0;(f^n)^{(k)}) \leqslant N(r,0;p^2) = O(\log r),$$
$$N(r,0;(g^n)^{(k)}) \leqslant N(r,0;p^2) = O(\log r).$$

From these and using (2.24) we have

$$N(r,\infty;f^n) + N(r,0;f^n) + N(r,0;(f^n)^{(k)}) = S(r,n\alpha_1) = S\Bigl(r, \frac{(f^n)'}{f^n}\Bigr) \tag{2.25}$$

and

$$N(r,\infty;g^n) + N(r,0;g^n) + N(r,0;(g^n)^{(k)}) = S(r,n\beta_1) = S\Bigl(r, \frac{(g^n)'}{g^n}\Bigr). \tag{2.26}$$

Then from (2.25), (2.26) and Lemma 2.4 we must have

$$f(z) = \mathrm{e}^{a_3^* z + b_3^*}, \quad g(z) = \mathrm{e}^{c_3^* z + d_3^*}, \tag{2.27}$$

where $a_3^* \neq 0$, b_3^*, $c_3^* \neq 0$ and d_3^* are constants. But these types of f and g do not agree with relation (2.18).

Next we suppose α and β are both non-constant polynomials, since otherwise f, g reduce to polynomials contradicting that they are transcendental. Also from (2.18) we get $\alpha + \beta = C_1$, i.e. $\alpha' = -\beta'$. Therefore $\deg(\alpha) = \deg(\beta)$. By Lemma 2.12 we conclude that both h_1 and h_2 are nonzero constants. So we can rewrite f and g as:

$$f(z) = \mathrm{e}^{\gamma(z)}, \quad g(z) = \mathrm{e}^{\delta(z)}, \tag{2.28}$$

where $\gamma(z)+\delta(z)=C_2$ and $\deg(\gamma)=\deg(\delta)$. Clearly $\gamma'=-\delta'$. If $\deg(\gamma)=\deg(\delta)=1$, then we again get a contradiction from (2.18). Next we suppose $\deg(\gamma)=\deg(\delta)\geqslant 2$. Now using (2.8) and (2.9) one can easily deduce from (2.28) that

$$(f^n)^{(k)}=\Big(n^k(\gamma')^k+\frac{k(k-1)}{2}n^{k-1}(\gamma')^{k-2}\gamma''+P_{k-1}(\gamma')\Big)\mathrm{e}^{n\gamma}$$

and

$$\begin{aligned}(g^n)^{(k)}&=\Big(n^k(\delta')^k+\frac{k(k-1)}{2}n^{k-1}(\delta')^{k-2}\delta''+P_{k-1}(\delta')\Big)\mathrm{e}^{n\delta}\\&=\Big((-1)^kn^k(\gamma')^k-\frac{k(k-1)}{2}n^{k-1}(-1)^{k-2}(\gamma')^{k-2}\gamma''+P_{k-1}(-\gamma')\Big)\mathrm{e}^{n\delta}.\end{aligned}$$

Since $\deg(\gamma)\geqslant 2$, we observe that $\deg((\gamma')^k)\geqslant k\deg(\gamma')$ and so $(\gamma')^{k-2}\gamma''$ is either a nonzero constant or $\deg((\gamma')^{k-2}\gamma'')\geqslant(k-1)\deg(\gamma')-1$. Also we see that

$$\deg((\gamma')^k)>\deg((\gamma')^{k-2}\gamma'')>\deg(P_{k-2}(\gamma'))\quad(\text{or }\deg(P_{k-2}(-\gamma'))).$$

Let

$$(\gamma(z))'=e_tz^t+e_{t-1}z^{t-1}+\ldots+e_0,$$

where $e_t\in\mathbb{C}\setminus\{0\}$. Then we have

$$((\gamma(z))')^i=e_t^iz^{it}+ie_t^{i-1}e_{t-1}z^{it-1}+\ldots,$$

where $i\in\mathbb{N}$. Therefore we have

$$(f^n)^{(k)}=(n^ke_t^kz^{kt}+kn^ke_t^{k-1}e_{t-1}z^{kt-1}+\ldots+(D_1+D_2)z^{kt-t-1}+\ldots)\mathrm{e}^{n\gamma}$$

and

$$\begin{aligned}(g^n)^{(k)}&=((-1)^kn^ke_t^kz^{kt}+(-1)^kkn^ke_t^{k-1}e_{t-1}z^{kt-1}+\ldots\\&\quad+((-1)^kD_1+(-1)^{k-1}D_2)z^{kt-t-1}+\ldots)\mathrm{e}^{n\delta},\end{aligned}$$

where $D_1,D_2\in\mathbb{C}$ such that $D_2=\frac{1}{2}k(k-1)tn^{k-1}e_t^{k-1}$. Now from (2.23) we see that

$$n^ke_t^kz^{kt}+kn^ke_t^{k-1}e_{t-1}z^{kt-1}+\ldots+(D_1+D_2)z^{kt-t-1}+\ldots=d_4^*p(z)\tag{2.29}$$

and

$$\begin{aligned}&(-1)^kn^ke_t^kz^{kt}+(-1)^kkn^ke_t^{k-1}e_{t-1}z^{kt-1}\\&\quad+\ldots+((-1)^kD_1+(-1)^{k-1}D_2)z^{kt-t-1}+\ldots=d_5^*p(z),\end{aligned}\tag{2.30}$$

where $d_4^*, d_5^* \in \mathbb{C} \setminus \{0\}$. Now from (2.29) and (2.30) it is clear that

$$\begin{aligned} n^k e_t^k z^{kt} + k n^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots + (D_1 + D_2) z^{kt-t-1} + \ldots \\ = d_6^*((-1)^k n^k e_t^k z^{kt} + (-1)^k k n^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots \\ + ((-1)^k D_1 + (-1)^{k-1} D_2) z^{kt-t-1} + \ldots), \end{aligned} \tag{2.31}$$

where $d_6^* \in \mathbb{C} \setminus \{0\}$. From (2.31) we get $D_2 = 0$, i.e.

$$\frac{k(k-1)}{2} t n^{k-1} e_t^{k-1} = 0,$$

which is impossible for $k \geqslant 2$.

Subcase 1.2. Let $k = 1$. Now from (2.18) we get

$$f^{n-1} f' g^{n-1} g' = p_1^2, \tag{2.32}$$

where $p_1^2 = n^{-2} p^2$. First we suppose both α and β are transcendental entire functions.

Let $h = fg$ and we consider the following subcases.

Subcase 1.2.1. Suppose that h is a nonzero polynomial. Then from (2.24) it is clear that $h = A h_1 h_2$, where $A = \mathrm{e}^{C_1}$ and $\alpha + \beta = C_1$. Therefore $\alpha' = -\beta'$. Now from (2.32) we see that

$$A\alpha'(-h_1' h_2 + h_1 h_2' - h_1 h_2 \alpha') = \mathrm{e}^{-(n-1)C} \frac{p_1^2}{(h_1 h_2)^{n-1}} - A h_1' h_2',$$

where $p_1^2 (h_1 h_2)^{1-n}$ is a polynomial. From this it is clear that

$$N(r, 0; \alpha') = O(\log r), \quad N(r, 0; -h_1' h_2 + h_1 h_2' - h_1 h_2 \alpha') = O(\log r).$$

By the second fundamental theorem for small functions (see [14]) we have

$$\begin{aligned} T(r, \alpha') &\leqslant \overline{N}(r, \infty; \alpha') + \overline{N}(r, 0; \alpha') + \overline{N}(r, 0; -h_1' h_2 + h_1 h_2' - h_1 h_2 \alpha') \\ &\quad + (\varepsilon + o(1)) T(r, \alpha') \\ &\leqslant O(\log r) + (\varepsilon + o(1)) T(r, \alpha') \end{aligned}$$

for all $\varepsilon > 0$. This shows that α' is a polynomial and so is α, which is a contradiction.

Subcase 1.2.2. Suppose that h is a transcendental entire function. Now from (2.32) we get

$$\left(\frac{g'}{g} - \frac{1}{2}\frac{h'}{h}\right)^2 = \frac{1}{4}\left(\frac{h'}{h}\right)^2 - h^{-n} p_1^2. \tag{2.33}$$

Let

$$\alpha_2 = \frac{g'}{g} - \frac{1}{2}\frac{h'}{h}.$$

From (2.33) we get

$$\alpha_2^2 = \frac{1}{4}\Big(\frac{h'}{h}\Big)^2 - h^{-n}p_1^2. \tag{2.34}$$

First we suppose $\alpha_2 = 0$. Then we get $h^{-n}p_1^2 = \frac{1}{4}(h'/h)^2$ and so $T(r,h) = S(r,h)$, which is impossible. Next we suppose that $\alpha_2 \neq 0$. Differentiating (2.34) we get

$$2\alpha_2\alpha_2' = \frac{1}{2}\frac{h'}{h}\Big(\frac{h'}{h}\Big)' + nh'h^{-n-1}p_1^2 - 2h^{-n}p_1p_1'.$$

Applying (2.34) we obtain

$$h^{-n}\Big(-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2\Big) = \frac{1}{2}\frac{h'}{h}\Big(\Big(\frac{h'}{h}\Big)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\Big). \tag{2.35}$$

First we suppose

$$-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2 = 0.$$

Then there exists a nonzero constant c_8 such that $\alpha_2^2 = c_8h^{-n}p_1^2$ and so from (2.34) we get

$$(c_8+1)h^{-n}p_1^2 = \frac{1}{4}\Big(\frac{h'}{h}\Big)^2.$$

If $c_8 = -1$, then h will be a constant. If $c_8 \neq -1$, then we have $T(r,h) = S(r,h)$, which is impossible. Next we suppose that

$$-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2 \neq 0.$$

Then by (2.35) we have

$$\begin{aligned} nT(r,h) &= nm(r,h) \\ &\leqslant m\Big(r, h^n\frac{1}{2}\frac{h'}{h}\Big(\Big(\frac{h'}{h}\Big)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\Big)\Big) \\ &\quad + m\Big(r, \Big(\frac{1}{2}\frac{h'}{h}\Big(\Big(\frac{h'}{h}\Big)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\Big)\Big)^{-1}\Big) + O(1) \\ &\leqslant T\Big(r, \frac{1}{2}\frac{h'}{h}\Big(\Big(\frac{h'}{h}\Big)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\Big)\Big) \\ &\quad + m\Big(r, n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2\Big) \\ &\leqslant \overline{N}(r,0;\alpha_2) + S(r,h) + S(r,\alpha_2). \end{aligned} \tag{2.36}$$

From (2.34) we get
$$T(r,\alpha_2) \leqslant \frac{1}{2}nT(r,h) + S(r,h).$$
Now from (2.36) we get
$$\frac{1}{2}nT(r,h) \leqslant S(r,h),$$
which is impossible.

Thus, α and β are both polynomials. Also from (2.18) we can conclude that $\alpha(z)+\beta(z)=C_1$ and so $\alpha'(z)+\beta'(z)=0$. By Lemma 2.12 we conclude that both h_1 and h_2 are nonzero constants. So we can rewrite f and g as:

$$f(z) = \mathrm{e}^{\gamma_2(z)}, \quad g(z) = \mathrm{e}^{\delta_2(z)}. \tag{2.37}$$

Now from (2.18) we get

$$n^2\gamma_2'\delta_2'\mathrm{e}^{n(\gamma_2+\delta_2)} = p^2. \tag{2.38}$$

Also from (2.38) we can conclude that $\gamma_2(z)+\delta_2(z)=C_3$ for a constant C_3 and so $\gamma_2'(z)+\delta_2'(z)=0$. Thus, from (2.38) we get $n^2\mathrm{e}^{nC_3}\gamma_2'(z)\delta_2'(z)=p^2(z)$. By computation we get

$$\gamma_2'(z) = cp(z), \quad \delta_2'(z) = -cp(z). \tag{2.39}$$

Hence,

$$\gamma_2(z) = cQ(z)+b_1, \quad \delta_2(z) = -cQ(z)+b_2, \tag{2.40}$$

where $Q(z)=\int_0^z p(t)\,\mathrm{d}t$ and b_1, b_2 are constants. Finally we take f and g as
$$f(z) = c_1\mathrm{e}^{cQ(z)}, \quad g(z) = c_2\mathrm{e}^{-cQ(z)},$$
where c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n=-1$.

Case 2. Let $p(z)$ be a nonzero constant b. In this case we see that f and g have no zeros and so we can take f and g as:

$$f(z) = \mathrm{e}^{\alpha(z)}, \quad g(z) = \mathrm{e}^{\beta(z)}, \tag{2.41}$$

where $\alpha(z)$, $\beta(z)$ are two non-constant entire functions. We now consider the following subcases.

Subcase 2.1. Let $k \geqslant 2$. We see that
$$N(r,0;(f^n)^{(k)}) = 0.$$

From this and using (2.41) we have

$$f^n(z)(f^n(z))^{(k)} \neq 0 \quad \text{and} \quad g^n(z)(g^n(z))^{(k)} \neq 0. \tag{2.42}$$

Then from (2.42) and Lemma 2.5 we must have

$$f(z) = \mathrm{e}^{a_9^* z + b_9^*}, \quad g(z) = \mathrm{e}^{c_9^* z + d_9^*}, \tag{2.43}$$

where $a_9^* \neq 0$, b_9^*, $c_9^* \neq 0$ and d_9^* are constants.

Subcase 2.1. Let $k = 1$. Considering Subcase 1.2 one can easily get

$$f(z) = \mathrm{e}^{a_{10}^* z + b_{10}^*}, \quad g(z) = \mathrm{e}^{c_{10}^* z + d_{10}^*},$$

where $a_{10}^* \neq 0$, b_{10}^*, $c_{10}^* \neq 0$ and d_{10}^* are constants. Finally, we can take f and g as

$$f(z) = c_3 \mathrm{e}^{dz}, \quad g(z) = c_4 \mathrm{e}^{-dz},$$

where c_3, c_4 and d are nonzero constants such that $(-1)^k (c_3 c_4)^n (nd)^{2k} = b^2$. This completes the proof. □

Lemma 2.14. *Let f and g be two transcendental meromorphic functions and $n, k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ with $n > k + 2$. Let $p(z)$ be a nonzero polynomial such that $\deg(p) \neq (n+i)s$, where $s \in \mathbb{N}$, $i \in \{0, 1, \ldots, m\}$. Let $P(w)$ be defined as in Theorem* F *and $(f^n P(f))^{(k)}$, $(g^n P(g))^{(k)}$ share p* CM *and also f, g share ∞* IM. *Suppose $(f^n P(f))^{(k)} (g^n P(g))^{(k)} = p^2$. Then $P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$;*

if $p(z)$ is not a constant, then $f(z) = c_1 \mathrm{e}^{cQ(z)}$, $g(z) = c_2 \mathrm{e}^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)\,\mathrm{d}t$, c_1, c_2 and c are constants such that $a_i^2 (c_1 c_2)^{n+i} ((n+i)c)^2 = -1$,

if $p(z)$ is a nonzero constant b, then $f(z) = c_3 \mathrm{e}^{cz}$, $g(z) = c_4 \mathrm{e}^{-cz}$, where c_3, c_4 and c are constants such that $(-1)^k a_i^2 (c_3 c_4)^{n+i} ((n+i)c)^{2k} = b^2$.

P r o o f. The proof follows from Lemmas 2.9 and 2.13. □

Lemma 2.15 ([1])**.** *Let f and g be two non-constant meromorphic functions sharing $(1, k_1)$, where $2 \leqslant k_1 \leqslant \infty$. Then*

$$\begin{aligned} &\overline{N}(r,1; f \mid= 2) + 2\overline{N}(r,1; f \mid= 3) + \ldots + (k_1 - 1)\overline{N}(r,1; f \mid= k_1) + k_1 \overline{N}_L(r,1; f) \\ &\quad + (k_1+1)\overline{N}_L(r,1; g) + k_1 \overline{N}_E^{(k_1+1}(r,1; g) \leqslant N(r,1; g) - \overline{N}(r,1; g). \end{aligned}$$

Lemma 2.16. *Suppose that f and g are two non-constant meromorphic functions. Let $F=(f^nP(f))^{(k)}$, $G=(g^nP(g))^{(k)}$, where $n,k\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$ and $P(w)$ be defined as in Theorem* F. *If f, g share ∞* IM *and $V=0$, then $F=G$.*

P r o o f. Suppose $V=0$. Then by integration we obtain

$$1-\frac{1}{F}=A\Bigl(1-\frac{1}{G}\Bigr).$$

It is that if z_0 is a pole of f, then it is a pole of g. Hence, from the definition of F and G we have $1/F(z_0)=0$ and $1/G(z_0)=0$. So $A=1$ and hence $F=G$. $\square$

Lemma 2.17. *Suppose that f and g are two non-constant meromorphic functions. Let F, G be defined as in Lemma* 2.16 *and $H\not\equiv 0$. If f, g share $(\infty,0)$ and F, G share $(1,k_1)$, where $0\leqslant k_1\leqslant\infty$, then*

$$\begin{aligned}\bigl(n+m-k-1\bigr)\overline{N}(r,\infty;f)\leqslant (k+m+1)(T(r,f)+T(r,g))+\overline{N}_*(r,1;F,G)\\ +S(r,f)+S(r,g).\end{aligned}$$

P r o o f. Suppose ∞ is an e.v.P of f and g. Then the result follows immediately.

Next suppose ∞ is not an e.v.P of f and g. Since $H\not\equiv 0$, from Lemma 2.16 we have $V\not\equiv 0$. We suppose that z_0 is a pole of f with multiplicity q and a pole of g with multiplicity r. Clearly z_0 is a pole of F with multiplicity $(n+m)q+k$ and a pole of G with multiplicity $(n+m)r+k$. Noting that f, g share $(\infty,0)$, from the definition of V it is clear that z_0 is a zero of V with multiplicity at least $n+m+k-1$. Now using the Milloux theorem [6], page 55, and Lemma 2.1, we obtain from the definition of V that $m(r,V)=S(r,f)+S(r,g)$. Thus, using Lemma 2.1 and (2.4) we get

$$\begin{aligned}&\bigl(n+m+k-1\bigr)\overline{N}(r,\infty;f)\\
&\qquad\leqslant N(r,0;V)\leqslant T(r,V)+O(1)\\
&\qquad\leqslant N(r,\infty;V)+m(r,V)+O(1)\\
&\qquad\leqslant \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\qquad\leqslant N_{k+1}(r,0;f^nP(f))+N_{k+1}(r,0;g^nP(g))+k\overline{N}(r,\infty;f)\\
&\qquad\quad+k\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\qquad\leqslant N_{k+1}(r,0;f^n)+N_{k+1}(r,0;P(f))+N_{K+1}(r,0;g^n)\\
&\qquad\quad+N_{k+1}(r,0;P(g))+2k\overline{N}(r,\infty;f)+\overline{N}_*(r,1;F,G)\\
&\qquad\quad+S(r,f)+S(r,g)\end{aligned}$$

$$
\begin{aligned}
&\leqslant (k+1)\overline{N}(r,0;f) + N(r,0;P(f)) + (k+1)\overline{N}(r,0;g)\\
&\quad + N(r,0;P(g)) + 2k\overline{N}(r,\infty;f) + \overline{N}_*(r,1;F,G)\\
&\quad + S(r,f) + S(r,g).
\end{aligned}
$$

This gives

$$
\begin{aligned}
\bigl(n+m-k-1\bigr)\overline{N}(r,\infty;f) \leqslant (k+m+1)(T(r,f)+T(r,g)) + \overline{N}_*(r,1;F,G)\\
+ S(r,f) + S(r,g).
\end{aligned}
$$

This completes the proof of the lemma. □

3. Proof of the theorem

Proof of Theorem 1.1. Let $F = (f^nP(f))^{(k)}/p(z)$ and $G = (g^nP(g))^{(k)}/p(z)$. Note that since f and g are transcendental meromorphic functions, $p(z)$ is a small function with respect to both $(f^nP(f))^{(k)}$ and $(g^nP(g))^{(k)}$. Also F, G share $(1,k_1)$ except the zeros of $p(z)$ and f, g share $(\infty,0)$.

Case 1. Let $H \not\equiv 0$. From (2.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different, (iv) zeros of F' (or G') which are not the zeros of $F(F-1)$ (or $G(G-1)$).

Since H has only simple poles, we get

$$
\begin{aligned}
(3.1)\ N(r,\infty;H) \leqslant \overline{N}_*(r,\infty;F,G) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F\,|\geqslant 2)\\
+ \overline{N}(r,0;G\,|\geqslant 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g),
\end{aligned}
$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$, and $\overline{N}_0(r,0;G')$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$
(3.2)\qquad N(r,1;F\,|=1) \leqslant N(r,0;H) \leqslant N(r,\infty;H) + S(r,f) + S(r,g).
$$

Using (3.1) and (3.2) we get

$$
\begin{aligned}
(3.3)\quad &\overline{N}(r,1;F)\\
&\leqslant N(r,1;F\,|=1) + \overline{N}(r,1;F\,|\geqslant 2)\\
&\leqslant \overline{N}_*(r,\infty;f,g) + \overline{N}(r,0;F\,|\geqslant 2) + \overline{N}(r,0;G\,|\geqslant 2) + \overline{N}_*(r,1;F,G)\\
&\quad + \overline{N}(r,1;F\,|\geqslant 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g)\\
&\leqslant \overline{N}(r,\infty;f) + \overline{N}(r,0;F\,|\geqslant 2) + \overline{N}(r,0;G\,|\geqslant 2) + \overline{N}_*(r,1;F,G)\\
&\quad + \overline{N}(r,1;F\,|\geqslant 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g).
\end{aligned}
$$

Now in view of Lemmas 2.6 and 2.3 we get

$$
\begin{aligned}
(3.4)\quad & \overline{N}_0(r,0;G') + \overline{N}(r,1;F\,|\geqslant 2) + \overline{N}_*(r,1;F,G)\\
&\leqslant \overline{N}_0(r,0;G') + \overline{N}(r,1;F\,|=2)\\
&\quad + \overline{N}(r,1;F\,|=3) + \ldots + \overline{N}(r,1;F\,|=k_1) + \overline{N}_E^{(k_1+1}(r,1;F)\\
&\quad + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_*(r,1;F,G)\\
&\leqslant \overline{N}_0(r,0;G') - \overline{N}(r,1;F\,|=3) - \ldots - (k_1-2)\overline{N}(r,1;F\,|=k_1)\\
&\quad - (k_1-1)\overline{N}_L(r,1;F) - k_1\overline{N}_L(r,1;G) - (k_1-1)\overline{N}_E^{(k_1+1}(r,1;F)\\
&\quad + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_*(r,1;F,G)\\
&\leqslant \overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}(r,1;G) - (k_1-2)\overline{N}_L(r,1;F)\\
&\quad - (k_1-1)\overline{N}_L(r,1;G)\\
&\leqslant N(r,0;G' \mid G\neq 0) - (k_1-2)\overline{N}_L(r,1;F) - (k_1-1)\overline{N}_L(r,1;G)\\
&\leqslant \overline{N}(r,0;G) + \overline{N}(r,\infty;g) - (k_1-2)\overline{N}_*(r,1;F,G) - \overline{N}_L(r,1;G).
\end{aligned}
$$

Hence, using (3.3), (3.4), Lemmas 2.2 and 2.17 we get from the second fundamental theorem that

$$
\begin{aligned}
(3.5)\quad (n+m)T(r,f) &\leqslant T(r,F) + N_{k+2}(r,0;f^nP(f)) - N_2(r,0;F) + S(r,f)\\
&\leqslant \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}(r,0;f^nP(f))\\
&\quad - N_2(r,0;F) - N_0(r,0;F') + S(r,f)\\
&\leqslant \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F) + N_{k+2}(r,0;f^nP(f))\\
&\quad + \overline{N}(r,0;F\,|\geqslant 2) + \overline{N}(r,0;G\,|\geqslant 2) + \overline{N}(r,1;F\,|\geqslant 2) + \overline{N}_*(r,1;F,G)\\
&\quad + \overline{N}_0(r,0;G') - N_2(r,0;F) + S(r,f) + S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^nP(f)) + N_2(r,0;G)\\
&\quad - (k_1-2)\overline{N}_*(r,1;F,G) - \overline{N}_L(r,1;G) + S(r,f) + S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^nP(f)) + k\overline{N}(r,\infty;g)\\
&\quad + N_{k+2}(r,0;g^nP(g)) - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant (3+k)\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,0;f) + T(r,P(f)) + (k+2)\overline{N}(r,0;g)\\
&\quad + T(r,P(g)) - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant (k+m+2)(T(r,f)+T(r,g)) + (3+k)\overline{N}(r,\infty;f)\\
&\quad - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant (k+m+2)(T(r,f)+T(r,g))\\
&\quad + \frac{(3+k)(k+m+1)}{n+m-k-1}(T(r,f)+T(r,g)) + \frac{3+k}{n+m-k-1}\overline{N}_*(r,1;F,G)\\
&\quad - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)
\end{aligned}
$$

$$\leqslant \Big(k+m+2+\frac{(3+k)(k+m+1)}{n+m-k-1}\Big)(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

In a similar way we can obtain

$$(3.6)\quad (n+m)T(r,g)\leqslant \Big(k+m+2+\frac{(3+k)(k+m+1)}{n+m-k-1}\Big)(T(r,f)+T(r,g)) \\ +S(r,f)+S(r,g).$$

Adding (3.5) and (3.6) we get

$$\Big(n-m-2k-4-\frac{(6+2k)(k+m+1)}{n+m-k-1}\Big)(T(r,f)+T(r,g))\leqslant S(r,f)+S(r,g).$$

Since the quantity in the first bracket can be written as

$$(3.7)\qquad \frac{(n+m-k-1)^2-(2m+k+3)(n+m-k-1)-2(k+3)(k+m+1)}{n+m-k-1},$$

by a simple computation one can easily verify that when

$$n+m-k-1>2m+2k+5 \\ >\frac{2m+k+3+\sqrt{(2m+k+3)^2+8(k+3)(k+m+1)}}{2},$$

i.e. when $n>3k+m+6$, we obtain a contradiction from (3.7).

Case 2. Let $H=0$. Then by Lemma 2.11 we have either

$$(3.8)\qquad (f^nP(f))^{(k)}(g^nP(g))^{(k)}=p^2,$$

or

$$(3.9)\qquad f^nP(f)=g^nP(g).$$

From (3.9) we get

$$(3.10)\quad f^n(a_mf^m+a_{m-1}f^{m-1}+\ldots+a_0)=g^n(a_mg^m+a_{m-1}g^{m-1}+\ldots+a_0).$$

Let $h=f/g$. If h is a constant, then substituting $f=gh$ into (3.10) we deduce that

$$a_mg^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+\ldots+a_0g^n(h^n-1)=0,$$

which implies $h^d=1$, where $d=\mathrm{GCD}(n+m,\ldots,n+m-i,\ldots,n)$, $a_{m-i}\neq 0$ for some $i=0,1,\ldots,m$. Thus, $f=tg$ for a constant t such that $t^d=1$, where $d=\mathrm{GCD}(n+m,\ldots,n+m-i,\ldots,n)$, $a_{m-i}\neq 0$ for some $i=0,1,\ldots,m$.

If h is not a constant, then we know by (3.10) that f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0).$$

The remaining part of the theorem follows from (3.8) and Lemma 2.14. This completes the proof of the theorem. □

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