A NOTE ON SOME RESULTS OF LI AND LI

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Abstract. The purpose of the paper is to study the uniqueness problems of linear differential polynomials of entire functions sharing a small function and obtain some results which improve and generalize the related results due to J. T. Li and P. Li (2015). Basically we pay our attention to the condition $\lambda(f) \neq 1$ in Theorems 1.3, 1.4 from J. T. Li and P. Li (2015). Some examples have been exhibited to show that conditions used in the paper are sharp.

Keywords: entire function; linear differential polynomial; uniqueness

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1. Introduction, definitions and results

In this paper, by a meromorphic or entire function we shall always mean meromorphic or entire, respectively, function in the complex plane $\mathbb{C}$. We denote by $n(r,\infty;f)$ the number of poles of f lying in $|z|<r$; the poles are counted according to their multiplicities. The quantity

$$N(r,\infty;f)=\int_0^r \frac{n(t,\infty;f)-n(0,\infty;f)}{t}\,\mathrm{d}t+n(0,\infty;f)\log r$$

is called the integrated counting function or simply the counting function of poles of f.

Also $m(r,\infty;f)=\frac{1}{2}\pi^{-1}\int_0^{2\pi}\log^+|f(re^{\mathrm{i}\theta})|\,\mathrm{d}\theta$ is called the proximity function of poles of f, where $\log^+ x=\log x$ for $x\geqslant 1$ and $\log^+ x=0$ for $0\leqslant x<1$.

The sum $T(r,f)=m(r,\infty;f)+N(r,\infty;f)$ is called the Nevanlinna characteristic function of f. We denote by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ except possibly a set of finite linear measure. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying

$S(r) = o(T(r))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

For $a \in \mathbb{C}$, we put $N(r, a; f) = N(r, \infty; (f - a)^{-1})$ and $m(r, a; f) = m(r, \infty; (f - a)^{-1})$.

Let us denote by $\overline{n}(r, a; f)$ the number of distinct a-points of f lying in $|z| < r$, where $a \in \mathbb{C} \cup \{\infty\}$. The quantity

$$\overline{N}(r, a; f) = \int_0^r \frac{\overline{n}(t, a; f) - \overline{n}(0, a; f)}{t} \, \mathrm{d}t + \overline{n}(0, a; f) \log r$$

denotes the reduced counting function of a-points of f (see, e.g. [2], [13]).

The order of f is defined by

$$\lambda(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

Let k be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We use $N_{k)}(r, a; f)$ to denote the counting function of a-points of f with multiplicity not greater than k, $N_{(k+1}(r, a; f)$ to denote the counting function of a-points of f with multiplicity greater than k. Similarly, $\overline{N}_{k)}(r, a; f)$ and $\overline{N}_{(k+1}(r, a; f)$ are their reduced functions, respectively.

For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum

$$\overline{N}_{(1}(r, a; f) + \overline{N}_{(2}(r, a; f) + \ldots + \overline{N}_{(p}(r, a; f).$$

For $a \in \mathbb{C} \cup \{\infty\}$ and $p \in \mathbb{N}$ we put

$$\delta_p(a; f) = 1 - \limsup_{r \to \infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

Clearly,

$$0 \leqslant \delta(a; f) \leqslant \delta_p(a; f) \leqslant \delta_{p-1}(a; f) \leqslant \ldots \leqslant \delta_2(a; f) \leqslant \delta_1(a; f) = \Theta(a; f).$$

A meromorphic function a is said to be a small function of f provided that $T(r, a) = S(r, f)$, that is $T(r, a) = o(T(r, f))$ as $r \to \infty$ except possibly a set of finite linear measure.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

In 1976, Yang [11] posed the following question:

What can be said about the relationship between two non-constant entire functions f and g if f and g share the value 0 CM and f' and g' share the value 1 CM?

The above problem has been studied by Shibazaki [10], Yi [15], [14], Yang and Yi [12], Hua [4], Muse-Reinders [9] and Lahiri [5]. And Yi [14] proved the following theorem.

Theorem A ([14]). *Let f and g be two non-constant entire functions and let k be a non-negative integer. If f and g share the value 0 CM, $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and $\delta(0;f) > \frac{1}{2}$, then $f \equiv g$ unless $f^{(k)}g^{(k)} \equiv 1$.*

Remark 1.1. The following example shows that in Theorem A the condition $\delta(0;f) > \frac{1}{2}$ is sharp.

Example 1.2 ([14]). Let

$$f(z) = -\frac{1}{2^k}\mathrm{e}^{2z} + \frac{(-1)^{k+1}}{2^k}\mathrm{e}^{z} \quad \text{and} \quad g(z) = \frac{(-1)^{k+1}}{2^k}\mathrm{e}^{-2z} - \frac{1}{2^k}\mathrm{e}^{-z},$$

where k is a non-negative integer. Then f and g share the value 0 CM, $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and $\delta(0;f) = \frac{1}{2}$, but $f \not\equiv g$ and $f^{(k)}g^{(k)} \not\equiv 1$.

Let h be a non-constant meromorphic function. We denote by

$$P(h) = h^{(k)} + a_1 h^{(k-1)} + a_2 h^{(k-2)} + \ldots + a_{k-1}h' + a_k h \tag{1.1}$$

the differential polynomial of h, where $a_1, a_2, \ldots, a_k$ are finite complex numbers and k is a positive integer.

Remark 1.3. The following example shows that in Theorem A the functions $f^{(k)}$ and $g^{(k)}$ cannot be replaced by $P(f)$ and $P(g)$.

Example 1.4 ([8]). Let $f(z) = \frac{1}{2}\mathrm{e}^{-2z}$ and $g(z) = \mathrm{e}^{-2z}$. Then f and g share the value 0 CM, $f'' + 2f'$ and $g'' + 2g'$ share the value 1 CM and $\delta(0;f) > \frac{1}{2}$, but $f \not\equiv g$ and $(f'' + 2f')(g'' + 2g') \not\equiv 1$.

In 2015, Li and Li proved the following results.

Theorem B ([8]). *Let f and g be two non-constant entire functions. Suppose that f and g share the value 0 CM, $P(f)$ and $P(g)$ share the value 1 CM and $\delta(0;f) > \frac{1}{2}$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $P(f)P(g) \equiv 1$.*

Theorem C ([8]). *Let f and g be two non-constant entire functions. Suppose that f and g share the value* 0 CM, *$P(f)$ and $P(g)$ share the value* 1 IM *and* $\delta(0; f) > \frac{4}{5}$. *If $\lambda(f) \neq 1$, then $f \equiv g$ unless $P(f)P(g) \equiv 1$.*

Now observing the above results the following questions are inevitable.

Question 1.5. Is the condition "$\lambda(f) \neq 1$" sharp in Theorems B, C?

Question 1.6. Is the condition "$\delta(0; f) > \frac{1}{2}$" sharp in Theorem B?

Question 1.7. What can be said if the sharing value in Theorems B, C is replaced by a small function of f and g?

Question 1.8. Is it really possible in any way to relax the nature of sharing the 1-point in Theorem B (Theorem C)?

In this paper we pay our attention to the nature of the differential polynomial $P(h)$ of h defined as in (1.1). Actually, we want to show that when $a_k \neq 0$ in (1.1), the condition $\lambda(f) \neq 1$ is not necessary. On the other hand, when $a_k = 0$ in (1.1), the condition $\lambda(f) \neq 1$ is necessary.

We now explain the notation of weighted sharing as introduced in [6].

Definition 1.9 ([6]). Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with the weight k.

We write f, g share (a, k) to mean that f, g share the value a with the weight k. Clearly, if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leqslant p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Let h be a non-constant meromorphic function. We denote by

$$P_1(h) = h^{(k)} + a_1 h^{(k-1)} + a_2 h^{(k-2)} + \ldots + a_{k-1} h' + a_k h \tag{1.2}$$

and

$$P_2(h) = h^{(k)} + b_1 h^{(k-1)} + b_2 h^{(k-2)} + \ldots + b_{k-1} h' \tag{1.3}$$

the differential polynomials of h, where $a_1, a_2, \ldots, a_k$ ($\neq 0$), $b_1, b_2, \ldots, b_{k-1}$ are finite complex numbers with $(b_1, b_2, \ldots, b_{k-1}) \neq (0, 0, \ldots, 0)$ and k is a positive integer.

Now taking the possible answers of the above questions into background we obtain the following results.

Theorem 1.10. *Let f and g be two non-constant entire functions and let $\alpha(z)$ $(\not\equiv 0,\infty)$ be a small function with respect to f and g. Suppose that f and g share $(0,\infty)$, $P_1(f)-\alpha$ and $P_1(g)-\alpha$ share $(0,2)$. If $\delta_{k+2}(0;f)>\frac{1}{2}$, then $f\equiv g$ unless $P_1(f)P_1(g)\equiv\alpha^2$.*

Theorem 1.11. *Let f and g be two non-constant entire functions and let $\alpha(z)$ $(\not\equiv 0,\infty)$ be a small function with respect to f and g. Suppose that f and g share $(0,\infty)$, $P_1(f)-\alpha$ and $P_1(g)-\alpha$ share $(0,1)$. If $\delta_{k+2}(0;f)>\frac{3}{5}$, then $f\equiv g$ unless $P_1(f)P_1(g)\equiv\alpha^2$.*

Theorem 1.12. *Let f and g be two non-constant entire functions and let $\alpha(z)$ $(\not\equiv 0,\infty)$ be a small function with respect to f and g. Suppose that f and g share $(0,\infty)$, $P_1(f)-\alpha$ and $P_1(g)-\alpha$ share $(0,0)$. If $\delta_{k+2}(0;f)>\frac{4}{5}$, then $f\equiv g$ unless $P_1(f)P_1(g)\equiv\alpha^2$.*

Theorem 1.13. *Let f and g be two non-constant entire functions and let $\alpha(z)$ $(\not\equiv 0,\infty)$ be a small function with respect to f and g. Suppose that f and g share $(0,\infty)$, $P_2(f)-\alpha$ and $P_2(g)-\alpha$ share $(0,2)$. If $\lambda(f)\neq 1$ and $\delta_{k+2}(0;f)>\frac{1}{2}$, then $f\equiv g$ unless $P_2(f)P_2(g)\equiv\alpha^2$.*

Theorem 1.14. *Let f and g be two non-constant entire functions and let $\alpha(z)$ $(\not\equiv 0,\infty)$ be a small function with respect to f and g. Suppose that f and g share $(0,\infty)$, $P_2(f)-\alpha$ and $P_2(g)-\alpha$ share $(0,1)$. If $\lambda(f)\neq 1$ and $\delta_{k+2}(0;f)>\frac{3}{5}$, then $f\equiv g$ unless $P_2(f)P_2(g)\equiv\alpha^2$.*

Theorem 1.15. *Let f and g be two non-constant entire functions and let $\alpha(z)$ $(\not\equiv 0,\infty)$ be a small function with respect to f and g. Suppose that f and g share $(0,\infty)$, $P_2(f)-\alpha$ and $P_2(g)-\alpha$ share $(0,0)$. If $\lambda(f)\neq 1$ and $\delta_{k+2}(0;f)>\frac{4}{5}$, then $f\equiv g$ unless $P_2(f)P_2(g)\equiv\alpha^2$.*

Remark 1.16. From the following example it is easy to see that the condition

$$\delta_{k+2}(0;f)>\frac{1}{2}$$

in Theorem 1.10 is sharp.

Example 1.17. Let

$$f(z)=\mathrm{e}^z\Big(1-\frac{1}{2}\mathrm{e}^z\Big)\quad\text{and}\quad g(z)=\mathrm{e}^{-z}\Big(\frac{1}{2}-\mathrm{e}^{-z}\Big).$$

Then

$$P_1(f) = -\frac{3}{8}\Big(f^{(\mathrm{iv})} + \frac{2}{3}f''' - 5f'' - 2f' + 8f\Big) = \mathrm{e}^z(1-\mathrm{e}^z)$$

and

$$P_1(g) = -\frac{3}{8}\Big(g^{(\mathrm{iv})} + \frac{2}{3}g''' - 5g'' - 2g' + 8f\Big) = \mathrm{e}^{-z}(1-\mathrm{e}^{-z}).$$

Clearly $P_1(f)$ and $P_1(g)$ share $(1,\infty)$, f, g share $(0,\infty)$ and $\delta_{k+2}(0;f) = \frac{1}{2}$, but neither $f \equiv g$ nor $P_1(f)P_1(g) \equiv 1$.

R e m a r k 1.18. From the following example it is easy to see that the conditions

$$\delta_{k+2}(0;f) > \frac{1}{2} \quad \text{and} \quad \lambda(f) \neq 1$$

in Theorem 1.13 are sharp.

E x a m p l e 1.19. Let

$$f(z) = \mathrm{e}^z\Big(1 - \frac{1}{2}\mathrm{e}^z\Big) \quad \text{and} \quad g(z) = \mathrm{e}^{-z}\Big(\frac{1}{2} - \mathrm{e}^{-z}\Big).$$

Then

$$P_2(f) = -\frac{3}{8}\Big(f^{(\mathrm{iv})} - \frac{2}{3}f''' - 5f'' + 2f'\Big) = \mathrm{e}^z(1-\mathrm{e}^z)$$

and

$$P_2(g) = -\frac{3}{8}\Big(g^{(\mathrm{iv})} - \frac{2}{3}g''' - 5g'' + 2g'\Big) = \mathrm{e}^{-z}(1-\mathrm{e}^{-z}).$$

Clearly $P_2(f)$ and $P_2(g)$ share $(1,\infty)$, f, g share $(0,\infty)$, $\delta_{k+2}(0;f) = \frac{1}{2}$ and $\lambda(f) = 1$, but neither $f \equiv g$ nor $P_2(f)P_2(g) \equiv 1$.

R e m a r k 1.20. From the following example it is easy to see that the condition "f and g share $(0,\infty)$" in Theorem 1.10 is necessary.

E x a m p l e 1.21. Let

$$f(z) = \mathrm{e}^{3z} - \mathrm{e}^{2z} \quad \text{and} \quad g(z) = \mathrm{e}^{z} - \mathrm{e}^{-2z}.$$

Then

$$P_1(f) = \frac{1}{24}(f^{(\mathrm{iv})} + 6f''' + 23f'' + 42f' + 48f) = \mathrm{e}^{3z} - \mathrm{e}^{2z}$$

and

$$P_1(g) = \frac{1}{24}(g^{(\mathrm{iv})} + 6g''' + 23g'' + 42g' + 48g) = \mathrm{e}^{z} - \mathrm{e}^{-2z}.$$

Clearly $P_1(f)$ and $P_1(g)$ share $(1,\infty)$, f, g do not share $(0,\infty)$ and $\delta_{k+2}(0;f) = \frac{2}{3} > \frac{1}{2}$, but neither $f \equiv g$ nor $P_1(f)P_1(g) \equiv 1$.

2. Lemmas

Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function:

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big). \tag{2.1}$$

Lemma 2.1 ([17]). *Let f be a non-constant meromorphic function, $P(f)$ be defined by (1.1) and p, k be positive integers. If $P(f)\not\equiv 0$, we have*

$$N_p(r,0;P(f))\leqslant T(r,P(f))-T(r,f)+N_{p+k}(r,0;f)+S(r,f),$$
$$N_p(r,0;P(f))\leqslant k\overline{N}(r,\infty;f)+N_{p+k}(r,0;f)+S(r,f).$$

Lemma 2.2. *Let f and g be two non-constant entire functions. Suppose $P_2(f)\equiv P_2(g)$, where $P_2(f)$ is defined by (1.3). If $\lambda(f)\neq 1$, then $f\equiv g$.*

Proof. Proof of the lemma follows from the proof of Theorem 1.4 in [8]. □

Lemma 2.3 ([13]). *Suppose f_j, $j=1,2,\dots,m+1$ and g_j, $j=1,2,\dots,m$ are entire functions satisfying the following conditions:*

(i) $\sum\limits_{j=1}^{m} f_j(z)\mathrm{e}^{g_j(z)}\equiv f_{m+1}$;

(ii) *The order of $f_j(z)$ is less than the order of $\mathrm{e}^{g_k(z)}$ for $1\leqslant j\leqslant m+1$, $1\leqslant k\leqslant m$; and furthermore, the order of $f_j(z)$ is less than the order of $\mathrm{e}^{g_l-g_k}$ for $m\leqslant 2$ and $1\leqslant j\leqslant m+1$, $1\leqslant l,k\leqslant m$, $l\neq k$.*

Then $f_j\equiv 0$, $j=1,2,\dots,m+1$.

Lemma 2.4. *Let us consider the linear differential equations*

$$a_n(z)f^{(n)}(z)+a_{n-1}(z)f^{(n-1)}(z)+\ldots+a_0(z)f(z)=0 \tag{2.2}$$

with entire coefficients $a_0(z)$ $(\not\equiv 0), a_1(z),\dots,a_n(z)$ $(\not\equiv 0)$. Then all solutions of (2.2) are entire functions of finite order if and only if the coefficients $a_0,a_1,\dots,a_n$ of (2.2) are polynomials.

Proof. Proof of the lemma follows from the proof of Theorem 4.1 (see [7]) and Remark 1 (see [7], page 58). □

Lemma 2.5 ([6]). *Let f and g be two non-constant meromorphic functions sharing $(1,2)$. Then one of the following holds:*

(i) $T(r,f) \leqslant N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r,f) + S(r,g)$,
(ii) $fg \equiv 1$,
(iii) $f \equiv g$.

Lemma 2.6 ([1]). *Let F and G be two non-constant meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$. Then*

$$T(r,F) \leqslant N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) + S(r,F) + S(r,G).$$

Lemma 2.7 ([1]). *Let F and G be two non-constant meromorphic functions sharing $(1,0)$ and $H \not\equiv 0$. Then*

$$T(r,F) \leqslant N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,F) + S(r,G).$$

Lemma 2.8 ([16]). *Let H be defined as in (2.1). If $H \equiv 0$ and*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)}{T(r)} < 1, \quad r \in I,$$

where I is a set of infinite linear measures. Then $F \equiv G$ or $FG \equiv 1$.

3. Proofs of the theorems

Proof of Theorem 1.10. Let $F(z) = P_1(f)/\alpha(z)$ and $G(z) = P_1(g)/\alpha(z)$. Then F and G share $(1,2)$ except for the zeros and poles of $\alpha(z)$. Now applying Lemma 2.5 we see that one of the following three cases holds.

Case 1. Suppose

$$T(r,F) \leqslant N_2(r,0;F) + N_2(r,0;G) + S(r,F) + S(r,G).$$

Now applying Lemma 2.1 we have

$$
\begin{aligned}
T(r,f) &\leqslant T(r,F)+N_{k+2}(r,0;f)-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leqslant N_{k+2}(r,0;f)+N_2(r,0;G)+S(r,f)+S(r,g)\\
&\leqslant N_{k+2}(r,0;f)+N_{k+2}(r,0;g)+S(r,f)+S(r,g)\\
&\leqslant 2N_{k+2}(r,0;f)+S(r,f)+S(r,g)\\
&\leqslant (2-2\delta_{k+2}(0;f)+\varepsilon)T(r,f)+S(r,f)+S(r,g)\\
&\leqslant (2-2\delta_{k+2}(0;f)+\varepsilon)T(r)+S(r),
\end{aligned}
$$

i.e.

$$T(r,f)\leqslant (2-2\delta_{k+2}(0;f)+\varepsilon)T(r)+S(r). \tag{3.1}$$

Similarly we have

$$T(r,g)\leqslant (2-2\delta_{k+2}(0;f)+\varepsilon)T(r)+S(r). \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$(-1+2\delta_{k+2}(0;f)-\varepsilon)T(r)\leqslant S(r). \tag{3.3}$$

Since $\varepsilon>0$ is arbitrary, we see that (3.3) leads to a contradiction.

Case 2. $F\equiv G$. Then we have

$$P_1(f)\equiv P_1(g). \tag{3.4}$$

Let

$$\frac{f}{g}=h=\mathrm{e}^{\alpha}, \tag{3.5}$$

where α is an entire function.

We now consider the following subcases.

Subcase 2.1. Suppose α is a constant. Let $\mathrm{e}^{\alpha}=c_0$, where c_0 is a finite complex constant. We obtain $f\equiv c_0 g$ and so $P_1(f)\equiv c_0P_1(g)$. Now by (3.4) we find that $c_0=1$ and so $f\equiv g$.

Subcase 2.2. Suppose α is a non-constant entire function.

Now from (3.4) we have $P_1(f-g)\equiv 0$. Solving this equation (see [3], [7]) we get

$$f(z)-g(z)=\sum_{j=1}^{m}p_j(z)\mathrm{e}^{\beta_j z}, \tag{3.6}$$

where m ($\leqslant k$) is a positive integer, β_j, $j = 1, 2, \ldots, m$ are distinct complex constants and $p_j(z)$, $j = 1, 2, \ldots, m$ are polynomials.

We deduce from (3.5) that

$$\begin{aligned} f' &= (g' + \alpha' g)\mathrm{e}^{\alpha} \\ f'' &= (g'' + 2\alpha' g' + (\alpha'' + (\alpha')^2)g)\mathrm{e}^{\alpha} \\ f''' &= (g''' + 3\alpha' g'' + 3(\alpha'' + (\alpha')^2)g' + n^3(\alpha')^3 + (\alpha'' + 3\alpha'\alpha'' + (\alpha')^3)g)\mathrm{e}^{\alpha} \\ &\vdots \\ f^{(k)} &= (g^{(k)} + Q^k_{k-1} g^{(k-1)} + Q^k_{k-2} g^{(k-2)} + \ldots + Q^k_0 g)\mathrm{e}^{\alpha}, \end{aligned}$$

where $Q^k_i(\alpha', \alpha'', \ldots, \alpha^{(k)})$, $i = 0, 1, 2, \ldots, k-1$ are differential polynomials in $\alpha', \alpha'', \ldots, \alpha^{(k)}$. Next we suppose

$$\begin{aligned} P_1(f) &= f^{(k)} + a_1 f^{(k-1)} + a_2 f^{(k-1)} + \ldots + a_{k-1} f' + a_k f \\ &= (g^{(k)} + Q_{k-1} g^{(k-1)} + \ldots + Q_1 g' + Q_0 g)\mathrm{e}^{\alpha}, \end{aligned}$$

where $Q_i(\alpha', \alpha'', \ldots, \alpha^{(k)})$, $i = 0, 1, 2, \ldots, k-1$ are differential polynomials in $\alpha', \alpha'', \ldots, \alpha^{(k)}$. Since α is an entire function, we obtain $T(r, \alpha^{(j)}) = S(r, h)$ for $j = 1, 2, \ldots, k$. Hence $T(r, Q_i) = S(r, h)$ for $i = 0, 1, 2, \ldots, k-1$. Now from (3.4) we have

$$(\mathrm{e}^{\alpha} - 1)g^{(k)} + (\mathrm{e}^{\alpha} Q_{k-1} - a_1)g^{(k-1)} + \ldots + (\mathrm{e}^{\alpha} Q_1 - a_{k-1})g' + (\mathrm{e}^{\alpha} Q_0 - a_k)g \equiv 0.$$

Clearly $\mathrm{e}^{\alpha} - 1 \not\equiv 0$ and $\mathrm{e}^{\alpha} Q_0 - a_k \not\equiv 0$. Now by Lemma 2.4 one can easily conclude that both f and g are of infinite order. By the Weierstrass's factorization theorem we have

$$f(z) = \gamma(z)\mathrm{e}^{\alpha_1(z)}, \quad g(z) = \gamma(z)\mathrm{e}^{\alpha_2(z)},$$

where $\gamma(z)$ is canonical product formed with common zeros of f and g and $\alpha_1(z)$, $\alpha_2(z)$ are non-constant entire functions.

Clearly $\alpha_1(z) \not\equiv \alpha_2(z)$. Since $\alpha(z)$ is a non-constant entire function, from (3.5) it follows that $\alpha_1(z) - \alpha_2(z)$ is a non-constant entire function. Since $\lambda(\gamma)$ is equal to $\tau(f)$ which is the exponent of convergence of zeros of $f(z)$ and $\tau(f) \leqslant \tau(f-g) \leqslant \lambda(f-g)$, by (3.6) we have

$$\lambda(\gamma) \leqslant \lambda(f-g) = \lambda\biggl(\sum_{j=1}^{m} p_j(z)\mathrm{e}^{\beta_j z}\biggr) \leqslant 1.$$

Note that $\lambda(\mathrm{e}^{\alpha_1}) = \lambda(f/\gamma)$ and $\lambda(\mathrm{e}^{\alpha_2}) = \lambda(g/\gamma)$. Since $\lambda(f) > 1$, $\lambda(g) > 1$ and $\lambda(\gamma) \leqslant 1$, it follows that $\lambda(\mathrm{e}^{\alpha_1}) > 1$ and $\lambda(\mathrm{e}^{\alpha_2}) > 1$. Also we see that

$$f - g = (\mathrm{e}^{\alpha_1 - \alpha_2} - 1)g.$$

Clearly,

$$\lambda(\mathrm{e}^{\alpha_1-\alpha_2}) = \lambda(\mathrm{e}^{\alpha_1-\alpha_2} - 1) = \lambda\Big(\frac{f-g}{g}\Big).$$

Since $\lambda(g) > 1$ and $\lambda(f-g) \leqslant 1$, it follows that $\lambda(\mathrm{e}^{\alpha_1-\alpha_2}) > 1$. From (3.6) we see that

$$\gamma(z)\mathrm{e}^{\alpha_1(z)-\alpha_2(z)} + \sum_{j=1}^{m}(-p_j(z))\mathrm{e}^{\beta_j z-\alpha_2(z)} = \gamma(z),$$

where $\lambda(\mathrm{e}^{\beta_j z-\alpha_2(z)}) > 1$ for $j = 1, 2, \ldots, m$. Now by Lemma 2.3, we see that $\gamma(z) \equiv 0$. Therefore $f(z) \equiv 0$, which is a contradiction.

Case 3. $FG \equiv 1$. Then we have $P_1(f)P_1(g) \equiv \alpha^2(z)$. This completes the proof. □

P r o o f of Theorem 1.11. Let $F(z) = P_1(f)/\alpha(z)$ and $G(z) = P_1(g)/\alpha(z)$. Then F and G share $(1, 1)$ except for the zeros and poles of $\alpha(z)$. We now consider the following two cases.

Case 1. $H \not\equiv 0$. Applying Lemmas 2.1 and 2.6 we have

$$\begin{aligned} T(r,f) &\leqslant T(r,F) + N_{k+2}(r,0;f) - N_2(r,0;F) + S(r,f) + S(r,g) \\ &\leqslant N_2(r,0;F) + N_2(r,0;G) + \frac{1}{2}\overline{N}(r,0;F) \\ &\quad + N_{k+2}(r,0;f) - N_2(r,0;F) + S(r,f) + S(r,g) \\ &\leqslant N_{k+2}(r,0;g) + \frac{1}{2}N_{k+1}(r,0;f) + N_{k+2}(r,0;f) + S(r,f) + S(r,g) \\ &\leqslant \frac{5}{2}N_{k+2}(r,0;f) + S(r,f) + S(r,g) \\ &\leqslant \Big(\frac{5}{2} - \frac{5}{2}\delta_{k+2}(0;f) + \varepsilon\Big)T(r) + S(r), \end{aligned}$$

i.e.

$$T(r,f) \leqslant \Big(\frac{5}{2} - \frac{5}{2}\delta_{k+2}(0;f) + \varepsilon\Big)T(r) + S(r). \tag{3.7}$$

Similarly we have

$$T(r,g) \leqslant \Big(\frac{5}{2} - \frac{5}{2}\delta_{k+2}(0;f) + \varepsilon\Big)T(r) + S(r). \tag{3.8}$$

Combining (3.7) and (3.8) we get

$$\Big(-\frac{3}{2} + \frac{5}{2}\delta_{k+2}(0;f) - \varepsilon\Big)T(r) \leqslant S(r). \tag{3.9}$$

Since $\varepsilon > 0$ is arbitrary, we see that (3.9) leads to a contradiction.

Case 2. $H \equiv 0$. In view of Lemma 2.4 we get

$$\begin{aligned}
&\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
&\qquad\leqslant N_{k+2}(r,0;f)+N_{k+2}(r,0;g)+S(r,f)+S(r,g)\\
&\qquad\leqslant 2N_{k+2}(r,0;f)+S(r,f)+S(r,g)\\
&\qquad\leqslant (2-2\delta_{k+2}(0;f)+\varepsilon)T(r)+S(r).
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary and $\delta_{k+2}(0;f)>\frac{3}{5}$, we must have

$$\limsup_{r\to\infty}\frac{\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)}{T(r)}<1$$

and so by Lemma 2.8 we have either $F\equiv G$ or $FG\equiv 1$. So the theorem follows from the proof of Theorem 1.10. This completes the proof. □

P r o o f of Theorem 1.12. Let $F(z)=P_1(f)/\alpha(z)$ and $G(z)=P_1(g)/\alpha(z)$. Then F and G share $(1,0)$ except for the zeros and poles of $\alpha(z)$. We now consider the following two cases.

Case 1. $H\not\equiv 0$. Applying Lemmas 2.1 and 2.7 we have

$$\begin{aligned}
T(r,f)&\leqslant T(r,F)+N_{k+2}(r,0;f)-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leqslant N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)\\
&\quad+N_{k+2}(r,0;f)-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leqslant 3N_{k+2}(r,0;f)+2N_{k+2}(r,0;g)+S(r,f)+S(r,g)\\
&\leqslant 5N_{k+2}(r,0;f)+S(r,f)+S(r,g)\\
&\leqslant (5-5\delta_{k+2}(0;f)+\varepsilon)T(r)+S(r),
\end{aligned}$$

i.e.

$$T(r,f)\leqslant (5-5\delta_{k+2}(0;f)+\varepsilon)T(r)+S(r). \tag{3.10}$$

Similarly we have

$$T(r,g)\leqslant (5-5\delta_{k+2}(0;f)+\varepsilon)T(r)+S(r). \tag{3.11}$$

Combining (3.10) and (3.11) we get

$$(-4+5\delta_{k+2}(0;f)-\varepsilon)T(r)\leqslant S(r). \tag{3.12}$$

Since $\varepsilon>0$ is arbitrary, we see that (3.12) leads to a contradiction.

Case 2. $H\equiv 0$. The remaining part of the theorem follows from the proof of Theorem 1.10. This completes the proof. □

P r o o f of Theorems 1.13–1.15. The proofs of theorems follow from the proof of Theorem 1.10, Theorem 1.11, Theorem 1.12, respectively, and Lemma 2.2. So we omit the detailed proofs. □

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