THE SYMMETRY REDUCTION OF VARIATIONAL INTEGRALS

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Abstract. The Routh reduction of cyclic variables in the Lagrange function and the Jacobi-Maupertuis principle of constant energy systems are generalized. The article deals with one-dimensional variational integral subject to differential constraints, the Lagrange variational problem, that admits the Lie group of symmetries. Reduction to the orbit space is investigated in the absolute sense relieved of all accidental structures. In particular, the widest possible coordinate-free approach to the underdetermined systems of ordinary differential equations, Poincaré-Cartan forms, variations and extremals is involved for the preparation of the main task. The self-contained exposition differs from the common actual theories and rests only on the most fundamental tools of classical mathematical analysis, however, they are applied in infinite-dimensional spaces. The article may be of a certain interest for nonspecialists since all concepts of the calculus of variations undergo a deep reconstruction.

Keywords: Routh reduction; Lagrange variational problem; Poincaré-Cartan form; diffiety; standard basis; controllability; variation

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Introduction

If a Lie transformation group $\mathbf{G}$ acts on a space $\mathbf{M}$ and preserves a certain mathematical structure $\mathbf{S}$ in $\mathbf{M}$, then as a rule naturally appears a reduced structure $\mathbf{S}/\mathbf{G}$ on the orbit space $\mathbf{M}/\mathbf{G}$ of the invariants. This is the core of the magnificent Erlangen program and the substance of classical geometry. In particular, we recall the primary Lie's reduction of differential equations (see [18], [24]) and subsequent Cartan's reduction of symplectical structure (see [6], [17], [19]). Together they both appear as

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the components of the Routh reduction of variational integrals (see [12], [3], [1], [2]) treated in full generality here. In more detail, if $\breve{\varphi}$ denotes the Poincaré-Cartan form related to a one-dimensional variational integral $\int \varphi$ where φ is a differential 1-form and if Ω denotes the differential constraints for the admissible curves, then the symplectical structure $\mathrm{d}\breve{\varphi}$ together with the differential equations Ω are closely interconnected in the symmetry reduction problem treated here. Still more explicitly, we are interested in the symmetry reduction to the orbit spaces of one-dimensional Lagrange variational problems.

The article starts with informal *Preface*, where the original Routh achievement together with a scheme of our new approach are outlined. The following brief survey of all *Fundamental concepts* includes the absolute differential equations, the extremals of variational integral, the Poincaré-Cartan forms of the Lagrange variational problem and, marginally, a mention of infinitesimal symmetries. This is illustrated by *Introductory examples* but we also refer to literature. *The standard basis* introduces the main technical tool, a certain "differentiation by parts" under the differential constraints, which is a mere linear algebra. With such modest and in principle self-contained preparation, the proper reduction problem of the calculus of variations to the orbit spaces of one-parameter symmetry group is discussed. Though the substance of *The main result* can be easily understood, the lengthy proof demands a certain patience with delicate details. We therefore conclude with short *Introductory applications* and succinct *Perspectives.*

It should be noted with regret that our approach is inconsistent with the common theories. We believe that this unpleasant fact cannot be regarded as a defect of the article. For the convenience of reader, the *Concluding comments* briefly survey the essence of our unorthodox point of view in intelligible terms.

1. Preface

With only a small short cut, the primary Routh reduction, see [12], concerns the variational integral

$$\int f(t, y, z, y', z')\,\mathrm{d}t, \tag{1.1}$$

where $f_y = 0$, hence $f = f(t, z, y', z')$ is independent of the variable y. Then the Euler-Lagrange system $(f_{y'})' = 0$, $f_z = (f_{z'})'$ implies the conservation law

$$f_{y'} = c \in \mathbb{R}, \quad \text{hence} \quad y' = g(t, z, z', c) \quad \text{(certain function } g) \tag{1.2}$$

in the regular case $f_{y'y'} \neq 0$. The Routh variational integral

$$\int f[c]\,\mathrm{d}t, \quad f[c] = f(t,z,g,z') - cg \tag{1.3}$$

does not depend on the "cyclic variables" y, y' and provides the same z-extremals as above (easy direct verification). Analogous reduction appears for the variational integrals (1.1) with the vanishing derivative $f_t = 0$. If in particular

$$f = \frac{1}{2}({y'}^2 + {z'}^2) - V(y,z)$$

is the Lagrange function of classical mechanics, the important Jacobi-Maupertuis principle with any of the variational integrals

$$\int (c-V)\,\mathrm{d}t, \quad \int ({y'}^2 + {z'}^2)\,\mathrm{d}t, \quad \int \sqrt{(c-V)({y'}^2 + {z'}^2)}\,\mathrm{d}t$$

follows after some effort (see [3]) where the conservation of the energy law

$$\frac{1}{2}({y'}^2 + {z'}^2) + V(y,z) = c \in \mathbb{R}$$

stands for the previous condition (1.2).

The primary Routh reduction concerns the infinitesimal symmetry $\partial/\partial y$ of the integral (1.1). Assuming instead the most general nonvanishing infinitesimal point symmetry

$$S = A(\cdot)\frac{\partial}{\partial t} + B(\cdot)\frac{\partial}{\partial y} + C(\cdot)\frac{\partial}{\partial z} \neq 0, \quad (\cdot) = (t,y,z), \tag{1.4}$$

this vector field S can be transformed into $\partial/\partial y$ and then the primary Routh result (1.3) may be applied to obtain the corresponding generalization, see [1], [2]. This is however a clumsy procedure. An alternative approach is as follows.

Denoting $\varphi = f(t,y,z,y',z')\,\mathrm{d}t$, we recall the Poincaré-Cartan form

$$\breve{\varphi} = f\,\mathrm{d}t + f_{y'}(\mathrm{d}y - y'\,\mathrm{d}t) + f_{z'}(\mathrm{d}z - z'\,\mathrm{d}t). \tag{1.5}$$

Due to the Noether theorem, the assumed infinitesimal symmetry (1.4) provides the conservation law

$$\breve{\varphi}(S) = Af + f_{y'}(B - y'A) + f_{z'}(C - z'A) = c \in \mathbb{R} \tag{1.6}$$

for the extremals. If in particular $S = \partial/\partial y$, we have the above law (1.2) and one can then observe that the restriction

$$\breve{\varphi}|_{y'=g} = f[c]\,\mathrm{d}t + c\,\mathrm{d}y + f_{z'}|_{y'=g}(\mathrm{d}z - z'\,\mathrm{d}t) \tag{1.7}$$

of the form (1.5) naturally leads to the Routh function $f[c]$. Omitting the summand $c\,\mathrm{d}y$ which does not affect the extremals, the restriction

$$f[c]\,\mathrm{d}t + f_{z'}(t, z, g, z')(\mathrm{d}z - z'\,\mathrm{d}t)$$

becomes the Poincaré-Cartan form for the variational integral $\int f[c]\,\mathrm{d}t$ with the z-extremals, which provides the Routh result as well. The restriction concept (1.7) is of coordinate-free nature and can be in principle carried over to the general case (1.4) of the symmetries.

Let us, however, apply this alternative approach to the second-order variational integral

$$\int \varphi, \quad \varphi = f(t, z, y', z', y'', z'')\,\mathrm{d}t$$

with the infinitesimal symmetry $S = \partial/\partial y$. The Poincaré-Cartan form

$$\begin{aligned}\breve{\varphi} = f\,\mathrm{d}t + (f_{y'} - (f_{y''})')(\mathrm{d}y - y'\,\mathrm{d}t) + f_{y''}(\mathrm{d}y' - y''\,\mathrm{d}t)\\ + \{(f_{z'} - (f_{z''})')(\mathrm{d}z - z'\,\mathrm{d}t) + f_{z''}(\mathrm{d}z' - z''\,\mathrm{d}t)\}\end{aligned}$$

provides the conservation law

$$\breve{\varphi}(S) = f_{y'} - (f_{y''})' = c, \quad \text{hence} \quad y''' = g(t, z, y', z', y'', z'', z''') \tag{1.8}$$

in the regular case $f_{y''y''} \neq 0$. We obtain the restriction

$$\breve{\varphi}|_{y'''=g} = (f - cy')\,\mathrm{d}t + c\,\mathrm{d}y + f_{y''}(\mathrm{d}y' - y''\,\mathrm{d}t) + \{\ldots\}$$

and, omitting the summand $c\,\mathrm{d}y$, quite analogous arguments as above make a good sense. The Routh integral at the level set (1.8) appears as above. However, the final result of the reduction is a Lagrange variational problem. In more detail, we have the variational integral $\int (f - cy')\,\mathrm{d}t$ together with the differential constraint (1.8).

We conclude that a self-contained group reduction theory is reasonable only within the framework of the Lagrange variational problems. Let us recall that every such variational problem involves two ingredients: it consists of a certain symplectical structure and of a system of differential constraints. While the pure symplectical structures are available in many textbooks, this is paradoxically not the case for a useful geometrical theory of differential equations and even for the intermediate concept, the Poincaré-Cartan forms of the Lagrange variational problem. The actual geometrical jet theory fails. We instead introduce the "absolute approach" relieved of all accidental structures, the diffieties. In spite of disbelief of most specialists, this is correct and a well established domain of mathematics, but the simple variant (see [7]) is of better use for our aims than the monographs [16], [25].

2. Fundamental concepts

Unless otherwise stated, our reasonings concern the infinite-dimensional manifolds $\mathbf{M}$ modelled on $\mathbb{R}^\infty$ and though the theory is of global and coordinate-free nature, we are mainly interested in the local and algorithmical results.

We suppose the infinite number of (local) coordinates $h^1, h^2, \ldots \colon \mathbf{M} \to \mathbb{R}$ such that the structural ring $\mathcal{F}(\mathbf{M})$ of admissible functions $f\colon \mathbf{M} \to \mathbb{R}$ involves just all functions $f = f(\cdot\cdot, h^j, \cdot\cdot)$ locally expressible in terms of a smooth composition of a finite number of coordinates. In particular, the coordinates can be changed by smooth invertible transformations. Admissible mappings $\mathbf{n}\colon \mathbf{N} \to \mathbf{M}$ between manifolds satisfy the inclusion $\mathbf{n}^*\mathcal{F}(\mathbf{M}) \subset \mathcal{F}(\mathbf{N})$ and we use the formal convention that $\mathbf{n}$ can be defined only on an open subset of $\mathbf{N}$. We speak of an inclusion $\mathbf{n}$ if a part of functions $\mathbf{n}^*h^1, \mathbf{n}^*h^2, \ldots$ can be taken for (local) coordinates on $\mathbf{N}$. (Then $\mathbf{N} \subset \mathbf{M}$ may be identified with a subset.) We speak of a projection $\mathbf{n}$ if the family $\mathbf{n}^*h^1, \mathbf{n}^*h^2, \ldots$ can be completed to the (local) coordinates on $\mathbf{N}$. (Then $\mathbf{M}$ is a factorspace of $\mathbf{N}$ and the functions $f = \mathbf{n}^*f$ are occasionally identified, therefore $\mathbf{n}^*\mathcal{F}(\mathbf{M}) \subset \mathcal{F}(\mathbf{N})$ becomes an $\mathcal{F}(\mathbf{M})$-submodule of the $\mathcal{F}(\mathbf{N})$-module.) Abbreviations like $\mathcal{F} = \mathcal{F}(\mathbf{M})$ occur whenever possible without much confusion.

We regret to say that a thorough exposition of the mathematical analysis on the infinite-dimensional manifolds modelled on $\mathbb{R}^\infty$ does not exist yet.

The structural ring $\mathcal{F} = \mathcal{F}(\mathbf{M})$ uniquely determines the $\mathcal{F}$-module $\Phi = \Phi(\mathbf{M})$ of differential 1-forms φ and, moreover, the $\mathcal{F}$-module $\mathcal{T} = \mathcal{T}(\mathbf{M})$ of vector fields Z. In more detail,

$$\varphi = \sum f^j \,\mathrm{d}g^j \quad \text{(finite sum)}, \quad Z = \sum z^j \frac{\partial}{\partial h^j} \quad \text{(infinite sum)},$$

where $f^j, g^j, z^j \in \mathcal{F}$. The vector fields Z are regarded as $\mathcal{F}$-linear functions on the module Φ, where

$$\varphi(Z) = Z \rfloor \varphi = \sum f^j Z g^j, \quad \mathrm{d}f(Z) = \sum \frac{\partial f}{\partial h^j} z^j = Zf, \quad \varphi \in \Phi,\ f \in \mathcal{F}$$

in the common sense. If $\varphi^1, \varphi^2, \ldots$ is a (local) basis of Φ, then the equivalent formulae

$$Z = \sum z^j \frac{\partial}{\partial \varphi^j} \quad (\text{infinite sum, } z^j = \varphi^j(Z)),$$
$$Zf = \sum z^j f^j, \qquad f \in \mathcal{F},\ \mathrm{d}f = \sum f^j \varphi^j$$

will be frequently employed and we abbreviate $\partial/\partial f = \partial/\partial\,\mathrm{d}f$. The familiar rules of the exterior algebra and the Lie derivatives

$$\mathcal{L}_Z\varphi = Z\rfloor\,\mathrm{d}\varphi + \mathrm{d}Z\rfloor\varphi, \quad \mathcal{L}_Z f = Zf = \mathrm{d}f(Z), \quad \mathcal{L}_Z X = [Z, X]$$

do not need any comment.

We always suppose the existence of (finite or infinite) bases in all $\mathcal{F}$-modules to appear. This is a universal measure which deletes the "singularities". We also always suppose that the $\mathcal{F}$-bases turn into $\mathbb{R}$-bases after taking the values at a fixed point of $\mathbf{M}$. This measure is necessary in order to ensure the existence of effective "pointwise" algorithms.

Passing to the fundamental concepts proper, we start with differential constraints, that is, with differential equations.

Definition 2.1. A submodule $\Omega \subset \Phi(\mathbf{M})$ of codimension one is called a *diffiety* if there exists a filtration $\Omega_*\colon\ \Omega_0 \subset \Omega_1 \subset \ldots \subset \Omega = \bigcup \Omega_l$ with finite-dimensional $\mathcal{F}(\mathbf{M})$-submodules $\Omega_l \subset \Omega$ such that

$$\mathcal{L}_{\mathcal{H}}\Omega_l \subset \Omega_{l+1} \text{ (all } l), \quad \Omega_l + \mathcal{L}_{\mathcal{H}}\Omega_l = \Omega_{l+1} \text{ (} l \text{ large enough)}, \tag{2.1}$$

is the *good filtration*, where $\mathcal{H} = \mathcal{H}(\Omega) \subset \mathcal{T}(\mathbf{M})$ is the submodule of vector fields Z satisfying $\Omega(Z) = 0$.

This is a global definition. In the local theory, condition (2.1) can be replaced with the requirement

$$\mathcal{L}_Z\Omega_l \subset \Omega_{l+1} \text{ (all } l), \quad \Omega_l + \mathcal{L}_Z\Omega_l = \Omega_{l+1} \text{ (} l \text{ large enough)}, \tag{2.2}$$

where $Z \in \mathcal{T}(\mathbf{M})$ is any nonvanishing vector field such that $\Omega(Z) = 0$. Our diffieties provide the "absolute version" of underdetermined systems of ordinary differential equations, see [7], [16], [25], [22]. See also the subsequent examples.

Definition 2.2. An inclusion $\mathbf{n}\colon\ (a \leqslant t \leqslant b) \to \mathbf{M}$ of an interval $(a \leqslant t \leqslant b) \subset \mathbb{R}$ is called a *solution* of diffiety $\Omega \subset \Phi(\mathbf{M})$ if $\mathbf{n}^*\Omega = 0$. A vector field $A \in \mathcal{T}(\mathbf{M})$ is called an (*admissible*) *variation of solution* $\mathbf{n}$ if $\mathbf{n}^*\mathcal{L}_A\Omega = 0$.

All preparations are done to introduce the fundamental concept of the calculus of variations, that is, the *Lagrange variational problem*.

Definition 2.3. Let $\varphi \in \Phi(\mathbf{M})$ and $\mathbf{n}\colon (a \leqslant t \leqslant b) \to \mathbf{M}$ be a solution of Ω. Every expression

$$\int_a^b \mathbf{n}^* \widetilde{\varphi}, \quad \widetilde{\varphi} = \varphi + \widetilde{\omega},\ \widetilde{\omega} \in \Omega \tag{2.3}$$

will be referred to as a variational integral with the constraint Ω.

The formal abbreviation $\int \varphi$ for the integral (2.3) is useful in practice and not confusing. In reality, we in fact deal with the family of all forms $\widetilde{\varphi} = \varphi + \widetilde{\omega}$, $\widetilde{\omega} \in \Omega$ and the integration over the interval $a \leqslant t \leqslant b$ is introduced for a mere historical reason. Still more explicitly, it should be noted that the subsequent extremals are genuinely local concepts which correspond to Definition 2.4. On the contrary, the historical approach rests on the global stationarity including some special boundary conditions for the variations. While the definition (2.4) can be literally carried over to all multidimensional Lagrange problems, already the introduction of appropriate boundary conditions causes terrible difficulties in the case of several independent variables.

Definition 2.4. A solution $\mathbf{n}\colon (a \leqslant t \leqslant b) \to \mathbf{M}$ of Ω is called an *extremal of the integral* (2.3) if

$$\mathbf{n}^* A \rfloor \, \mathrm{d}\widetilde{\varphi} = 0, \quad \text{all } A,\ \widetilde{\varphi} = \varphi + \widetilde{\omega},\ \widetilde{\omega} = \widetilde{\omega}[\mathbf{n}] \in \Omega \tag{2.4}$$

for all variations A of $\mathbf{n}$ and an appropriate choice of the form $\widetilde{\omega}$.

The admissible variations A are in full accordance with the classical theory and the concept of variational integrals and extremals is inspired by [13]. The ambiguity of the form $\widetilde{\omega}$ corresponds to the common concept of the Lagrange multipliers, however, it will be completely deleted here. Assuming (2.4) with certain $\widetilde{\omega} = \widetilde{\omega}[\mathbf{n}]$, the global stationarity

$$\int_a^b \mathbf{n}^* \mathcal{L}_A \varphi = 0, \quad A \text{ satisfying } \mathbf{n}^* \widetilde{\varphi}(A)|_{t=a}^{t=b} = 0,\ \widetilde{\varphi} = \varphi + \widetilde{\omega}[\mathbf{n}]$$

with the original form $\varphi = \widetilde{\varphi} - \widetilde{\omega}[\mathbf{n}]$ is obvious.

We are passing to the crucial point of this article.

Definition 2.5. For a special choice $\breve{\omega} \in \Omega$, the form $\breve{\varphi} = \varphi + \breve{\omega}$ is called a *Poincaré-Cartan* ($\mathcal{PC}$) *form* related to the integral (2.3) if

$$A \rfloor \mathrm{d}\breve{\varphi} \cong Z \rfloor \, \mathrm{d}\breve{\varphi} \ (\operatorname{mod} \Omega), \tag{2.5}$$

where $Z \in \mathcal{T}(\mathbf{M})$ is an arbitrary vector field and $A = A[Z]$ an appropriate variation of the solution $\mathbf{n}$. Moreover, we postulate the existence of a formula

$$\varphi^j(A) = \sum f^j_{kr} D^r \varphi^k(Z) \quad (\text{finite sum, } j = 1, 2, \ldots), \tag{2.6}$$

where $\varphi^1, \varphi^2, \ldots$ is any (or, arbitrary) basis of module $\Phi(\mathbf{M})$ and coefficients $f^j_{kr} \in \mathcal{F}(\mathbf{M})$ do not depend on the choice of the integral (2.3), the vector field Z and the solution $\mathbf{n}$.

Condition (2.5) is clearly equivalent to the more illustrative identity

$$\mathbf{n}^* A \rfloor \mathrm{d}\breve{\varphi} = \mathbf{n}^* Z \rfloor \mathrm{d}\breve{\varphi} \quad (\text{all solutions } \mathbf{n}). \tag{2.7}$$

The construction of the vector field A will soon appear. It is of local nature: if Z vanishes near a certain point, then $A = A[Z]$ vanishes at the same place, too.

The above definition provides the simplest possible "absolute approach" to the theory of $\mathcal{PC}$ forms without any use of accidental structures. In order to simplify the exposition, the definition is not the most general one and can be applied with success only to the controllable diffieties Ω introduced below. In full generality, some additional imposition on the vector field Z is necessary, but we abstain from more detail at this place.

Theorem 2.1. *A solution $\mathbf{n}$ of Ω is extremal of integral* (2.3) *if and only if*

$$\mathbf{n}^* Z \rfloor \mathrm{d}\breve{\varphi} = 0, \quad Z \in \mathcal{T}(\mathbf{M}) \tag{2.8}$$

for some (or, for every) $\mathcal{PC}$ form $\breve{\varphi}$.

Proof. Condition (2.8) trivially implies (2.4). Let us conversely assume (2.4), hence

$$0 = \mathbf{n}^* A \rfloor \mathrm{d}\widetilde{\varphi} = \mathbf{n}^* A \rfloor \mathrm{d}(\breve{\varphi} + \omega) = \mathbf{n}^* Z \rfloor \mathrm{d}\breve{\varphi} + \mathbf{n}^* A \rfloor \mathrm{d}\omega, \quad A = A[Z]$$

for all vector fields Z and the form $\omega = \widetilde{\varphi} - \breve{\varphi} \in \Omega$. However,

$$0 = \mathbf{n}^* \mathcal{L}_A \omega = \mathbf{n}^* A \rfloor \mathrm{d}\omega + \mathrm{d}\mathbf{n}^* \omega(A),$$

whence

$$0 = \int_a^b \mathbf{n}^* Z \rfloor \mathrm{d}\breve{\varphi} - \mathbf{n}^* \omega(A)|_{t=a}^{t=b}.$$

The boundary summands disappear if the vector field Z (and therefore A) vanishes near the endpoints $\mathbf{n}(a), \mathbf{n}(b) \in \mathbf{M}$. This implies (2.8) in the interior points $a < t < b$, hence everywhere. □

The last fundamental concept concerns the infinitesimal symmetries.

Definition 2.6. A vector field $V \in \mathcal{T}(\mathbf{M})$ is called a *variation of diffiety* $\Omega \subset \Phi(\mathbf{M})$ if $\mathcal{L}_V\Omega \subset \Omega$ and a *variation of integral* (2.3) if moreover $\mathcal{L}_V\varphi \in \Omega$. Variations which generate a (local) Lie group are called *infinitesimal symmetries.*

Important warning: our variations are called *generalized* (or *higher-order*, or *Lie-Bäcklund*) *infinitesimal symmetries* in contemporary literature though they do not necessarily generate any group. This highly misleading terminology is made more precise here. Explicit formula for all variations will be soon stated.

The variations V of diffiety Ω may be regarded as the "universal" admissible variations A satisfying $\mathbf{n}^*\mathcal{L}_A\Omega = 0$ for every solution $\mathbf{n}$ of Ω. In the crucial definition (2.5), the variation $A = A[Z]$ does not depend on the choice of $\mathbf{n}$ and therefore may be regarded as a variation $V = V[Z]$ of diffiety Ω as well. So we have the alternative definition

$$V \rfloor \mathrm{d}\breve{\varphi} \cong Z \rfloor \mathrm{d}\breve{\varphi} \pmod{\Omega} \tag{2.9}$$

of $\mathcal{PC}$ form $\breve{\varphi}$ where $V = V[Z]$ is a variation of Ω.

Theorem 2.2. *A variation $V \in \mathcal{T}(\mathbf{M})$ of diffiety $\Omega \subset \Phi(\mathbf{M})$ is an infinitesimal symmetry if and only if V preserves a certain good filtration Ω_* in the sense that $\mathcal{L}_V\Omega_l \subset \Omega_l$ for all l.*

We do not need urgently this result (see [22], [9]) at this place and delay the proof. It should be only noted that the sufficience of the condition is easy since the inclusion declares that the Lie derivative $\mathcal{L}_V$ acts on finite-dimensional spaces where the classical theory can be applied and such V does generate a group. Alas, there are as a rule many good filtrations to be examined for the explicit determination of all symmetries. No finite algorithm as yet exists.

Theorem 2.3. *Every infinitesimal symmetry $V \in \mathcal{T}(\mathbf{M})$ of variational integral* (2.3) *preserves a certain $\mathcal{PC}$ form $\breve{\varphi}$ in the sense that $\mathcal{L}_V\breve{\varphi} = 0$.*

We again delay the proof.

Our approach and the terminology differ from the common use and the following brief examples are stated for better clarity. We however start with some technical tools.

Definition 3.1. A function $x \in \mathcal{F}(\mathbf{M})$ is called an *independent variable* for a diffiety $\Omega \subset \Phi(\mathbf{M})$ if the differential $\mathrm{d}x$ together with Ω generate the module $\Phi(\mathbf{M})$.

Alternatively, if $\omega^1, \omega^2, \dots$ is a basis of Ω, then $\mathrm{d}x, \omega^1, \omega^2, \dots$ is a basis of $\Phi(\mathbf{M})$. So we may introduce the *total derivative* $D = D_x$ (abbreviation) for the independent variable x. It is defined by

$$\mathrm{d}x(D) = Dx = 1, \quad \omega^1(D) = \omega^2(D) = \ldots = 0, \quad \text{hence} \quad D = \frac{\partial}{\partial x}\left(+\sum 0 \cdot \frac{\partial}{\partial \omega^j}\right).$$

Obviously $D \in \mathcal{H} = \mathcal{H}(\Omega)$ and this vector field D can be taken for a basis of $\mathcal{H}$. It follows that the condition

$$\mathcal{L}_D \Omega_l \subset \Omega_{l+1} \quad (\text{all } l), \quad \Omega_l + \mathcal{L}_D \Omega_l = \Omega_{l+1} \quad (l \text{ large enough}) \tag{3.1}$$

is equivalent to (2.1) and (2.2). Moreover, we introduce the *contact forms*

$$\omega_f = \mathrm{d}f - Df\,\mathrm{d}x \in \Omega, \quad f \in \mathcal{F}(\mathbf{M})$$

satisfying the obvious identities

$$\mathrm{d}\omega \cong \mathrm{d}x \wedge \mathcal{L}_D \omega \ (\mathrm{mod}\ \Omega), \quad \omega_{Df} = \mathcal{L}_D \omega_f, \quad \mathrm{d}f = Df\,\mathrm{d}x + \sum \frac{\partial f}{\partial h^j} \omega_{h^j}, \tag{3.2}$$

where $\omega \in \Omega$, $f \in \mathcal{F}(\mathbf{M})$.

Lemma 3.1. *A vector field $V \in \mathcal{T}(\mathbf{M})$ is a variation of diffiety $\Omega \subset \Phi(\mathbf{M})$ if and only if*

$$\mathcal{L}_D \omega(V) = D\omega(V), \quad \omega \in \Omega. \tag{3.3}$$

Proof. First formula (3.2) implies

$$0 \cong \mathcal{L}_V \omega = V \rfloor \mathrm{d}\omega + \mathrm{d}\omega(V) \cong -\mathcal{L}_D \omega(V)\,\mathrm{d}x + \mathrm{d}\omega(V) \ (\mathrm{mod}\ \Omega),$$

where $\mathrm{d}\omega(V) \cong D\omega(V)\,\mathrm{d}x$ by using the last formula (3.2). □

Lemma 3.2. *A vector field $A \in \mathcal{T}(\mathbf{M})$ is a variation of a solution $\mathbf{n}$ of diffiety Ω if and only if*

$$\mathbf{n}^*\mathcal{L}_D\omega(A) = \mathbf{n}^* D\omega(A) = \frac{\mathrm{d}}{\mathrm{d}\mathbf{n}^* x}\mathbf{n}^*\omega(A), \quad \omega \in \Omega. \tag{3.4}$$

The proof may be omitted. Both conditions (3.3) and (3.4) are sufficient if the form ω runs only over some generators $\omega^1, \omega^2, \dots$ of the module Ω.

We are passing to examples proper.

3.1. The jet diffieties. Informally, they correspond to the trivial constraints where the "true" differential equations are absent and we deal with "all curves". It should be noted, however, that this property is not of absolute nature and is destroyed after "higher order" change of the jet coordinates (3.5), see [22], [21].

In rigorous terms, let us introduce the space $\mathbf{M}(m)$ with coordinates

$$x, w^j_r, \quad j = 1, \dots, m, \ r = 0, 1, \dots \tag{3.5}$$

and the $\mathcal{F}(\mathbf{M}(m))$-submodule $\Omega(m) \subset \Phi(\mathbf{M}(m))$ of differential forms

$$\omega = \sum a^j_r \omega^j_r \quad (\text{finite sum, } \omega^j_r = \mathrm{d}w^j_r - w^j_{r+1}\,\mathrm{d}x). \tag{3.6}$$

This is in reality the well-known infinite-order jet space. On the other hand, we also have a diffiety where the term $\Omega(m)_l$ of the order-preserving good filtration $\Omega(m)_*$ involves all forms (3.6) with $r \leqslant l$. In geometrical terms, the solutions of $\Omega(m)$ are the infinite prolongations

$$x = t, \quad w^j_r(t) = \frac{\mathrm{d}^r w^j(t)}{\mathrm{d}t^r}, \quad a \leqslant t \leqslant b$$

of the curves $w^1 = w^1(t), \dots, w^m = w^m(t)$ lying in the space $\mathbb{R}^{m+1}$.

The total derivative

$$D = \frac{\partial}{\partial x} + \sum w^j_{r+1}\frac{\partial}{\partial w^j_r} \in \mathcal{H}(\Omega(m))$$

with respect to the independent variable x satisfies $\mathcal{L}_D\omega^j_r = \omega^j_{r+1}$. Lemma 3.2 can be therefore comfortably applied for the choice $\omega = \omega^j_r$ and we obtain all variations

$$V = v\frac{\partial}{\partial x} + \sum D^r v^j \frac{\partial}{\partial \omega^j_r}, \quad v = Vx,\ v^j = \omega^j_0(V) = Vw^j_r - w^j_{r+1}Vx, \tag{3.7}$$

where $v, v^1, \dots, v^m$ are arbitrary functions. The familiar prolongation formula

$$V = v\frac{\partial}{\partial x} + \sum v_r^j \frac{\partial}{\partial w_r^j}, \quad v_{r+1}^j = Dv_r^j - w_{r+1}^j Dv$$

easily follows since

$$v_r^j = \mathrm{d}w_r^j(V) = \omega_r^j(V) + w_{r+1}^j Vx = D^r v^j + w_{r+1}^j v,$$

which implies the above recurrence for the coefficients v_r^j. The result (3.7) is, however, simpler and better for use.

Turning to the variational integrals, let us denote

$$\varphi = f\,\mathrm{d}x, \quad f = f(x, \cdot\cdot, w_r^j, \cdot\cdot) \in \mathcal{F}(\mathbf{M}(m)).$$

Then the "increasing" recurrence

$$\breve{\varphi} = \varphi + \breve{\omega}, \quad \breve{\omega} = \sum f_{r+1}^j \omega_r^j, \quad f_r^j = \frac{\partial f}{\partial w_r^j} - Df_{r+1}^j \tag{3.8}$$

provides the common $\mathcal{PC}$ form. This follows from the directly verifiable formula

$$\mathrm{d}\breve{\varphi} \cong \sum f_0^j \omega_0^j \wedge \mathrm{d}x \ (\mathrm{mod}\ \Omega(m) \wedge \Omega(m)), \quad f_0^j = \sum (-1)^r D^r \frac{\partial f}{\partial w_r^j}. \tag{3.9}$$

Condition (2.9) is satisfied: for a given vector field Z, we may choose even the quite explicit variation $V = V[Z]$ given by (3.7), where $v = Zx$ and $v^j = \omega_0^j(Z)$. Employing this $\mathcal{PC}$ form, the condition (2.8) for the extremals reads

$$0 = \mathbf{n}^* Z \rfloor \mathrm{d}\breve{\varphi} = \mathbf{n}^* \sum f_0^j \omega_0^j(Z)\,\mathrm{d}x = \mathbf{n}^* \sum f_0^j v^j\,\mathrm{d}t \quad (\text{all } Z),$$

whence the common Euler-Lagrange system $\mathbf{n}^* f_0^j = 0$, $j = 0, \dots, m$ immediately follows.

3.2. The Monge equation. Let us deal with the equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = F\Big(x, y, z, \frac{\mathrm{d}y}{\mathrm{d}x}\Big).$$

The corresponding diffiety $\Omega \subset \Phi(\mathbf{M})$ in the space $\mathbf{M}$ with coordinates x, y_r, z, $r = 0, 1, \dots$ involves all differential forms

$$\omega = \sum a_r \eta_r + a\zeta, \quad \eta_r = \mathrm{d}y_r - y_{r+1}\,\mathrm{d}x, \ \zeta = \mathrm{d}z - F(x, y_0, z, y_1)\,\mathrm{d}x. \tag{3.10}$$

The total derivative
$$D = \frac{\partial}{\partial x} + \sum y_{r+1}\frac{\partial}{\partial y_r} + F\frac{\partial}{\partial z}$$
satisfies
$$\mathcal{L}_D\eta_r = \eta_{r+1}, \quad \mathcal{L}_D\zeta = F_{y_0}\eta_0 + F_{y_1}\eta_1 + F_z\zeta$$
and there is the obvious order-preserving good filtration Ω_*, where the term Ω_l consists of forms (3.10) with $r \leqslant l$.

Lemma 3.2 cannot be directly applied, however, let us introduce the form $\pi_0 = \zeta - F_{y_1}\eta_0 \in \Omega_0$. Then
$$\begin{aligned}
&\pi_1 = \mathcal{L}_D\pi_0 = a\eta_0 + F_z\zeta = (a + F_zF_{y_1})\eta_0 + F_z\pi_0, \quad a = F_{y_0} - DF_{y_1},\\
&\pi_2 = \mathcal{L}_D\pi_1 = (a + F_zF_{y_1})\eta_1 + \dots,\\
&\pi_3 = \mathcal{L}_D\pi_2 = (a + F_zF_{y_1})\eta_2 + \dots,\\
&\quad \dots
\end{aligned}$$
Assuming $a + F_zF_{y_1} \neq 0$, the forms $\pi_0, \pi_1, \dots$ can be taken for an alternative basis of diffiety Ω and then the variations
$$V = v\frac{\partial}{\partial x} + \sum D^r p\frac{\partial}{\partial \pi_r}, \quad v = Vx,\ p = \pi_0(V) \tag{3.11}$$
with arbitrary functions $v, p \in \mathcal{F}(\mathbf{M})$ are determined.

Let us turn to the variational integrals. For instance, let
$$\varphi = f(x, y_0, z)\,\mathrm{d}x, \quad \mathrm{d}\varphi = (f_{y_0}\eta_0 + f_z\zeta) \wedge \mathrm{d}x.$$
In terms of the alternative basis, one can obtain the formula
$$\mathrm{d}\varphi = (b\pi_0 + c\pi_1) \wedge \mathrm{d}x$$
with certain clumsy coefficients not stated here and then the $\mathcal{PC}$ form $\breve{\varphi} = \varphi - c\pi_0$ easily follows since
$$\mathrm{d}(c\pi_0) \cong Dc\,\mathrm{d}x \wedge \pi_0 + c\,\mathrm{d}\pi_0, \quad \mathrm{d}\pi_0 \cong \mathrm{d}x \wedge \pi_1 \pmod{\Omega \wedge \Omega}$$
by using (3.2) and therefore clearly
$$\mathrm{d}\breve{\varphi} = \mathrm{d}(\varphi - c\pi_0) \cong (b - Dc)\pi_0 \wedge \mathrm{d}x \pmod{\Omega \wedge \Omega}.$$
The condition (2.5) can be verified by the same arguments as above. We also have the Euler-Lagrange equation $\mathbf{n}^*(b - Dc) = 0$ for the extremals $\mathbf{n}$.

Let us return to the exceptional case $a + F_z F_{y_1} = 0$. Then

$$\pi_1 = \mathcal{L}_D \pi_0 = F_z \pi_0$$

and we refer to a general theory (see [7], [22]), which implies that π_0 is a multiple of a differential $\mathrm{d}f$. We have a noncontrollable diffiety (see below) but the variations V can be determined as well (see [22], [10]).

3.3. The second-order constraints. Let us deal with the Hilbert-Cartan equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{1}{2}\Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2.$$

The corresponding diffiety $\Omega \subset \Phi(\mathbf{M})$ in the space $\mathbf{M}$ with coordinates x, y_r, z_0, z_1, $r = 0, 1, \ldots$ involves all differential forms

$$\omega = \sum a_r \eta_r + b_0 \zeta_0 + b_1 \zeta_1 \quad \text{(finite sum)},$$

where

$$\eta_r = \mathrm{d}y_r - y_{r+1}\,\mathrm{d}x, \quad \zeta_0 = \mathrm{d}z_0 - z_1\,\mathrm{d}x, \quad \zeta_1 = \mathrm{d}z_1 - \frac{1}{2}(y_1)^2\,\mathrm{d}x.$$

The total derivative

$$D = \frac{\partial}{\partial x} + \sum y_{r+1}\frac{\partial}{\partial y_r} + z_1 \frac{\partial}{\partial z_0} + \frac{1}{2}(y_1)^2 \frac{\partial}{\partial z_1}$$

satisfies $\mathcal{L}_D \eta_r = \eta_{r+1}$, $\mathcal{L}_D \zeta_0 = \zeta_1$, $\mathcal{L}_D \zeta_1 = y_1 \eta_1$. There is the obvious order-preserving good filtration quite analogous as above.

In order to apply Lemma 3.2, let us introduce the forms

$$\begin{aligned}
\pi_0 &= (y_1)^2 \eta_0 - y_2 \zeta_0 - y_1 \zeta_1,\\
\pi_1 &= \mathcal{L}_D \pi_0 = 2y_1 y_2 \eta_1 - y_0 \zeta_0 - 2y_2 \zeta_1,\\
\pi_2 &= \mathcal{L}_D \pi_1 = 2\{(y_2)^2 + y_1 y_0\}\eta_0 - y_4 \zeta_0 - 3y_2 \zeta_1,\\
\pi_3 &= \mathcal{L}_D \pi_2 = 2\{\ldots\}\eta_1 + \ldots,\\
&\ \vdots\\
\pi_{r+2} &= 2\{\ldots\}\eta_r + \ldots,\\
&\ \vdots
\end{aligned}$$

which can be taken for the alternative basis of diffiety Ω. Then formally the same formula (3.11) for the variations holds true without any change. The $\mathcal{PC}$ form $\breve{\varphi}$ related to a variational integral $\int \varphi$ does not bring any difficulties, see [22].

4. The standard basis

We return to the general theory where the true sense of the above differential forms π occurring in the examples will be clarified.

For every submodule $\Theta \subset \Omega$ of a diffiety Ω, let $\operatorname{Ker}\Theta \subset \Theta$ be the submodule of all forms $\vartheta \in \Theta$ such that $\mathcal{L}_{\mathcal{H}}\vartheta \subset \Theta$ or, equivalently, with the property $\mathcal{L}_D\vartheta \in \Theta$ where $D \in \mathcal{H}$ is a total derivative.

Definition 4.1. A filtration $\overline{\Omega}_*\colon\ \overline{\Omega}_0 \subset \overline{\Omega}_1 \subset \ldots \subset \Omega = \bigcup \overline{\Omega}_l$ of diffiety Ω is called a *standard* one if $\overline{\Omega}_*$ is good and moreover

$$\operatorname{Ker}\overline{\Omega}_{l+1} = \overline{\Omega}_l, \quad l > 0, \quad \operatorname{Ker}\overline{\Omega}_0 = \operatorname{Ker}^2\overline{\Omega}_0 \neq \overline{\Omega}_0. \tag{4.1}$$

Theorem 4.1. *Let Ω_* be a good filtration of diffiety Ω. There exists a unique standard filtration $\overline{\Omega}_*$ of Ω such that*

$$\overline{\Omega}_{l+c} = \Omega_l \quad (l \textit{ large enough}) \tag{4.2}$$

with appropriate $c \geqslant 0$.

P r o o f ([7], [22], [9]). The naturally induced mapping

$$\mathcal{L}_D\colon\ \Omega_l/\Omega_{l-1} \to \Omega_{l+1}/\Omega_l \quad (\text{formally } \Omega_{-1} = 0) \tag{4.3}$$

is $\mathcal{F}$-linear and surjective for l large enough. It is therefore even bijective for large l and then $\operatorname{Ker}\Omega_l = \Omega_{l-1}$, $l \geqslant L$. We obtain the increasing sequence

$$\ldots \supset \operatorname{Ker}\Omega_L = \Omega_{L-1} \supset \operatorname{Ker}\Omega_{L-1} \supset \operatorname{Ker}^2\Omega_{L-1} \supset \ldots,$$

which terminates with equalities

$$\ldots \supset \operatorname{Ker}^K\Omega_{L-1} \supset \operatorname{Ker}^{K+1}\Omega_{L-1} = \operatorname{Ker}^{K+2}\Omega_{L-1} = \ldots$$

So we may put

$$\overline{\Omega}_0 = \operatorname{Ker}^K\Omega_{L-1},\ \overline{\Omega}_1 = \operatorname{Ker}^{K-1}\Omega_{L-1}, \ldots,\ \overline{\Omega}_{K-1} = \operatorname{Ker}\Omega_{L-1},\ \overline{\Omega}_K = \Omega_{L-1}, \ldots$$

where $\overline{\Omega}_{l+c} = \Omega_l$, $l \geqslant L$, $c = K - L + 1$. $\square$

For the reader's amusement, the following figures may illustrate the intuitive sense of the above construction.

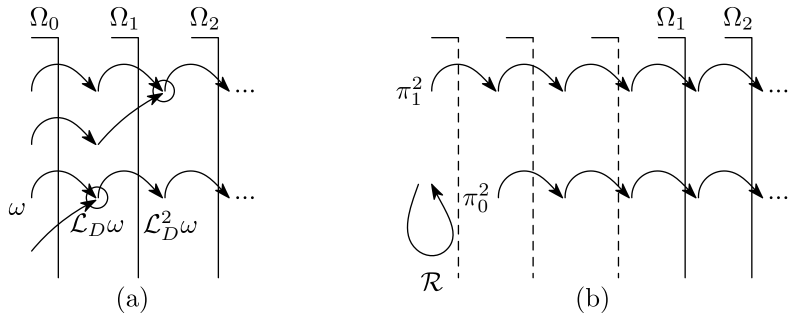

Figure 1. The original filtration (a), the standard filtration (b)

Theorem 4.2. *The submodule*

$$\mathcal{R} = \bigcap \operatorname{Ker}^k \Omega_l = \operatorname{Ker} \overline{\Omega}_0 \subset \overline{\Omega}_0 \quad (\textit{any fixed } l)$$

is generated by all differentials $\mathrm{d}f \in \Omega$.

Proof. Assuming $\mathrm{d}f \in \Omega$ we have

$$Df = \mathrm{d}f(D) = 0, \quad \mathcal{L}_D\,\mathrm{d}f = \mathrm{d}Df = 0$$

and it follows that $\mathrm{d}f$ lies in all Ker-modules hence in $\mathcal{R}$. The converse is more involved (see [7], [10]) and we omit the proof for the reasons to follow. $\square$

Definition 4.2. Diffiety Ω is called *controllable* if $\mathcal{R} = 0$ is the trivial module, that is, the diffiety Ω does not contain any differentials $\mathrm{d}f \neq 0$.

In a certain sense, the noncontrollable diffieties can be regarded as diffieties depending on a finite number of parameters $f = c \in \mathbb{R}$ ($\mathrm{d}f \in \Omega$). Though this property does not cause many difficulties, the results become somewhat clumsy. We therefore deal only with the controllable diffieties from now on. In this case, the standard filtration

$$(4.4) \qquad \overline{\Omega}_*\colon\ \overline{\Omega}_0 = \operatorname{Ker}\overline{\Omega}_1 \subset \overline{\Omega}_1 = \operatorname{Ker}\overline{\Omega}_2 \subset \ldots \subset \overline{\Omega}_{l+c} = \Omega_l \subset \ldots, \quad \operatorname{Ker}\overline{\Omega}_0 = 0$$

with strict inclusions simplifies a little and, moreover, we obtain the *standard basis* of diffiety Ω as follows. Let

π_0^j $(0 < j \leqslant j_0)$ be a basis of $\overline{\Omega}_0$ and together with
$\pi_1^j = \mathcal{L}_D\pi_0^j$ $(0 < j \leqslant j_0)$, π_0^j $(j_0 < j \leqslant j_1)$ a basis of $\overline{\Omega}_1$ and together with

$\pi_2^j = \mathcal{L}_D^2 \pi_0^j$ $(0 < j \leqslant j_0)$, $\pi_1^j = \mathcal{L}_D \pi_0^j$ $(j_0 < j \leqslant j_1)$, π_0^j $(j_1 < j \leqslant j_2)$ a basis of $\overline{\Omega}_2$ and so on with $\overline{\Omega}_3$, $\overline{\Omega}_4$, ...

This is a finite algorithm. The stationarity

$$0 < j_0 < j_1 < \ldots < j_c = j_{c+1} = \ldots = \dim \Omega_{l+1}/\Omega_l = \mu(\Omega) \quad (l \text{ large enough})$$

easily follows from (4.3) for a certain integer c. In reality, $\mu = \mu(\Omega)$ does not depend on the choice of the filtration Ω_* (see [7], [22], [10]). We obtain a hierarchy of forms organized in the infinite table

$$\begin{array}{cccccccc} \pi_0^j, & \pi_1^j = \mathcal{L}_D \pi_0^j, & \pi_2^j = \mathcal{L}_D^2 \pi_0^j, & \ldots, & \ldots, & \ldots, & (0 < j \leqslant j_0) \\ & \pi_0^j, & \pi_1^j = \mathcal{L}_D \pi_0^j, & \ldots, & \ldots, & \ldots, & (j_0 < j \leqslant j_1) \\ & & \pi_0^j, & \ldots, & \ldots, & \ldots, & (j_1 < j \leqslant j_2) \\ & & & \ldots, & \ldots, & \ldots, & \\ & & & \pi_0^j, & \pi_1^j = \mathcal{L}_D \pi_0^j, & \ldots, & (j_{c-1} < j \leqslant j_c) \end{array} \tag{4.5}$$

with a finite number $\mu(\Omega)$ of lines where the forms π_r^j, $r \leqslant l$ lying in the 1-st up to l-th column constitute a basis of the module Ω_{l-1}, $l = 1, 2, \ldots$ The table will be frequently referred to.

Definition 4.3. The forms π_0^j, $1 \leqslant j \leqslant \mu(\Omega)$ are called *initial* to the standard filtration $\overline{\Omega}_*$.

We recall that the total number $j_c = \mu(\Omega)$ of initial forms does not depend on the choice of the filtration.

Lemma 4.1. *All variations V of diffiety Ω are given by the formula*

$$V = v\frac{\partial}{\partial x} + \sum D^r p^j \frac{\partial}{\partial \pi_r^j}, \quad v = Vx,\ p^j = \pi_0^j(V) \tag{4.6}$$

and all variations A of a solution $\mathbf{n}$ of Ω are

$$A = a\frac{\partial}{\partial x} + \sum a_r^j \frac{\partial}{\partial \pi_r^j}, \quad \mathbf{n}^* a_r^j = \mathbf{n}^* D^r a^j, \tag{4.7}$$

where v, p^j, a and a^j may be arbitrary functions.

P r o o f. Apply Lemma 3.1 or Lemma 3.2. □

Theorem 4.3. *Let (2.3) be a given variational integral. For every choice of initial forms π_0^j of a given standard filtration Ω_*, there exists a unique $\mathcal{PC}$ form $\breve{\varphi} = \varphi + \breve{\omega}$ such that*

$$\mathrm{d}\breve{\varphi} \cong 0 \ (\mathrm{mod}\ \textit{all}\ \pi_0^j\ \textit{and}\ \Omega \wedge \Omega). \tag{4.8}$$

P r o o f ([7], [9], [10]). The obvious congruences

$$\mathrm{d}\pi^j_r \cong \mathrm{d}x \wedge \mathcal{L}_D \pi^j_r = \mathrm{d}x \wedge \pi^j_{r+1} \ (\mathrm{mod}\ \Omega \wedge \Omega),$$
$$\mathrm{d}(g\pi^j_{r-1}) \cong \mathrm{d}x \wedge g\pi^j_r \ (\mathrm{mod}\ \pi^j_{r-1},\ \Omega \wedge \Omega)$$

permit us to delete successively the higher-order summands in the congruence

$$\mathrm{d}\varphi \cong \sum g^j_r \pi^j_r \wedge \mathrm{d}x \ (\mathrm{mod}\ \Omega \wedge \Omega)$$

up to the final "zeroth-order" formula

$$\mathrm{d}(\varphi + \breve{\omega}) = \sum e^j \pi^j_0 \wedge \mathrm{d}x \ (\mathrm{mod}\ \Omega \wedge \Omega). \tag{4.9}$$

In more detail: A summand $g^j_R \pi^j_R \wedge \mathrm{d}x$ with large $R \geqslant 1$ disappears if the form φ is replaced with $\varphi + g^j_R \pi^j_{R-1}$ since then

$$\mathrm{d}(\varphi + g^j_R \pi^j_{R-1}) \cong \ldots + g^j_R \pi^j_R \wedge \mathrm{d}x + Dg^j_R \cdot \mathrm{d}x \wedge \pi^j_{R-1} + g^j_r \, \mathrm{d}x \wedge \pi^j_R,$$

where the R-th order summands cancel. This procedure is repeatedly applied to the final result (4.9). The correction $\breve{\omega}$ of the form φ to the form $\breve{\varphi} = \varphi + \breve{\omega}$ is unique.

Assuming (4.9), the condition (2.9) reads

$$V \rfloor \mathrm{d}\breve{\varphi} \cong \sum e^j \pi^j_0(V) \, \mathrm{d}x = Z \rfloor \mathrm{d}\breve{\varphi} \cong \sum e^j \pi^j_0(Z) \, \mathrm{d}x.$$

It is trivially satisfied if $\pi^j_0(V) = \pi^j_0(Z)$ for all $j = 1, \ldots, m$. We may choose the variation V=V[Z] given by formula (4.6), where

$$p^j = \pi^j_0(V) = \pi^j_0(Z), \quad \text{hence} \quad \pi^j_r(V) = D^r p^j = D^r \pi^j_0(Z)$$

and $v = \mathrm{d}x(V)$ is arbitrary. Since $\mathrm{d}x, \pi^j_r$, $j = 1, \ldots, m$; $r = 0, 1, \ldots$ is a basis of $\Phi(\mathbf{M})$, the clumsy assumption (2.6) holds true for this variation $A = V[Z]$. So we indeed have a $\mathcal{PC}$ form $\breve{\varphi}$ in the sense of Definition 2.5. □

The following result is a consequence of (2.8).

Theorem 4.4. *Assuming* (4.9), *a solution* $\mathbf{n}$ *of diffiety* Ω *is extremal of integral* (2.3) *if and only if* $\mathbf{n}^* e^j = 0$, $j = 1, \ldots, \mu(\Omega)$.

The uniqueness of $\mathcal{PC}$ form $\breve{\varphi}$ in Theorem 4.3 should be taken with caution. First of all, the standard filtration of a diffiety Ω is unique if and only if $\mu(\Omega) = 1$, see [7], [22]. (On this occasion, we cannot pass in silence the remarkable history concerning the beautiful but hopelessly forgotten Monge problem, see [15], [5], related to the origins of the primary Cartan's version of the "absolute theory" of differential equations. The Monge problem is resolved just in the case $\mu(\Omega) = 1$.) Assuming $\mu(\Omega) > 1$, there are too many standard filtrations and a certain favourable case appears only if $\mu(\Omega) = j_0$. Then the module generated by all initial forms π_0^j, $1 \leqslant j \leqslant j_0 = \mu(\Omega)$ is unique and the congruence (4.8) does determine a unique $\mathcal{PC}$ form already for a given filtration. In general, the module generated by the initial forms π_0^j, $j_0 < j \leqslant \mu(\Omega)$ and the $\mathcal{PC}$ form $\breve{\varphi}$ are not uniquely determined by a given filtration Ω_*, which causes some difficulties.

5. The main result

The symmetries were as yet only marginally occurring and we turn to more detailed investigation. We shall not directly deal with the corresponding Lie group but only with the infinitesimal generator instead, the variation V. Though all the interrelations between the variational integral $\int \varphi$ with the constraint Ω and the variation V are not self-evident at the first glance, the final result can be understood already at this early place.

Theorem 5.1. *Let $\Omega \subset \Phi(\mathbf{M})$ be a controllable diffiety and $V \in \mathcal{T}(\mathbf{M})$ an infinitesimal symmetry of a variational integral $\int \varphi$ with the constraint Ω. If $\breve{\varphi}$ is any fixed $\mathcal{PC}$ form, every extremal is lying in a certain subspace $\mathbf{M}[c] \subset \mathbf{M}$ determined by the equation $\breve{\varphi}(V) = c$, where $c \in \mathbb{R}$. If $\breve{\varphi}$ is an invariant $\mathcal{PC}$ form, there exists the orbit subspace $\mathbf{M}[c]/V \subset \mathbf{M}/V$ of the total orbit space $\mathbf{M}/\mathbf{V}$, the orbit diffiety $\Omega[c]/V \subset \Phi(\mathbf{M}[c]/V)$ naturally induced by Ω and the Routh variational integral*

$$\int \breve{\varphi}[c], \quad \breve{\varphi}[c] = \breve{\varphi} - c\,\mathrm{d}w,\ w \in \mathcal{F}(\mathbf{M}),\ Vw = 1 \tag{5.1}$$

defined on the space $\mathbf{M}[c]/V$. Altogether we have the variational integral (5.1) *with the constraint $\Omega[c]/V \subset \Phi(\mathbf{M}[c]/V)$. Projections on the orbit space $\mathbf{M}[c]/V$ of the original extremals which are lying in $\mathbf{M}[c]$ are extremals of the Routh integral. Moreover, if $\breve{\varphi}[c] \in \Phi(\mathbf{M}[c]/V)$ is a $\mathcal{PC}$ form of the Routh integral, the original extremals lying in $\mathbf{M}[c]$ are surjectively projected onto the family of all extremals of the integral* (5.1).

P r o o f. The following proof consists of many short steps.

5.1. The underlying space. Let a vector field $V \in \mathcal{T}(\mathbf{M})$ generate a (local) one-parameter Lie group of transformations on $\mathbf{M}$. Assuming $V \neq 0$ at every point of $\mathbf{M}$, we obtain the (local) orbit space denoted $\mathbf{M}/V$. Let $\mathbf{v}\colon\ \mathbf{M} \to \mathbf{M}/V$ be the natural projection. The functions $g \in \mathcal{F}(\mathbf{M}/V)$ bijectively correspond to the *invariants* (in classical terminology: *first integrals*) $\overline{g} \in \mathcal{F}(\mathbf{M})$ satisfying $V\overline{g} = 0$, namely $\overline{g} = \mathbf{v}^* g$. We abbreviate (formally identify)

$$g = \mathbf{v}^* g = \overline{g}, \quad \text{hence} \quad \mathcal{F}(\mathbf{M}/V) \subset \mathcal{F}(\mathbf{M}).$$

Analogously, the forms $\psi \in \Phi(\mathbf{M}/V)$ bijectively correspond to the *integral invariants* $\overline{\psi} = \mathbf{v}^*\psi \in \Phi(\mathbf{M})$ satisfying either of the equivalent conditions (see [6], [7])

$$\mathcal{L}_{fV}\overline{\psi} = 0 \quad (f \in \mathcal{F}(\mathbf{M})) \quad \text{or} \quad V \rfloor \overline{\psi} = \overline{\psi}(V) = 0. \tag{5.2}$$

We again abbreviate

$$\psi = \mathbf{v}^*\psi = \overline{\psi}, \quad \text{hence} \quad \Phi(\mathbf{M}/V) \subset \Phi(\mathbf{M}).$$

The integral invariants ψ should not be confused with invariant forms $\varphi \in \Phi(\mathbf{M})$ satisfying the weaker condition $\mathcal{L}_V\varphi = 0$.

5.2. Adapted coordinates. We suppose the existence of alternative coordinates, especially the first integrals g^k, with a somewhat strange notation at this place

$$w, g^k\colon\ \mathbf{M} \to \mathbb{R}; \quad Vw = 1,\ Vg^k = 0,\ k = 0, 1, \dots$$

This is a reliable fact but rigorous proofs are nontrivial, see [7], [23]. Then the functions

$$g^k = \mathbf{v}^* g^k\colon\ \mathbf{M}/V \to \mathbb{R}, \quad k = 0, 1, \dots$$

may be taken for coordinates on the orbit space. Invariant forms are

$$\varphi = W\,\mathrm{d}w + \sum G^k\,\mathrm{d}g^k, \quad W, G^k \in \mathcal{F}(\mathbf{M}/V)$$

and integral invariants appear if $W = 0$.

5.3. The entrance of diffieties. Let V be a symmetry of diffiety $\Omega \subset \Phi(\mathbf{M})$. We choose $x = g^0$ for the independent variable. If $D = D_x$ is the corresponding total derivative, the contact forms

$$\eta (= \omega_w) = \mathrm{d}w - Dw\,\mathrm{d}x, \quad \eta^j (= \omega_{g^j}) = \mathrm{d}g^j - Dg^j\,\mathrm{d}x, \quad j = 1, 2, \dots$$

may be taken for a basis of Ω. We may also introduce the contact forms

$$\eta_r = \omega_{D^r w} = \mathcal{L}_D^r \eta, \quad \eta_r^j = \omega_{D^r g^j} = \mathcal{L}_D^r \omega^j, \quad r = 0, 1, \ldots;\ j = 1, 2, \ldots,$$

though they are of little importance for the general theory.

5.4. The orbit diffiety. The inclusion $\mathcal{L}_V \Omega \subset \Omega$ implies

$$0 = V(\omega(D)) = (\mathcal{L}_V \omega)(D) + \omega([V, D]) = \omega([V, D]), \quad \omega \in \Omega,$$

whence $[V, D] \in \mathcal{H}$ is a multiple of D. However, $[V, D]x = VDx - DVx = 0$ and therefore $[V, D] = 0$. It follows that the vector field $D \in \mathcal{T}(\mathbf{M})$ is V-projectable, that is,

$$D(\mathbf{v}^* g) = (\mathbf{v}_* D) g, \quad g \in \mathcal{F}(\mathbf{M}/V)$$

for a certain vector field $\mathbf{v}_*\mathbf{D} \in \mathcal{T}(\mathbf{M}/\mathbf{V})$. We again abbreviate

$$D = \mathbf{v}_* D, \quad D \in \mathcal{T}(\mathbf{M}/V).$$

One can see that η is an invariant form and all forms

$$\eta_r = \mathbf{v}^* \eta_r,\ r = 1, 2, \ldots, \quad \eta_r^j = \mathbf{v}^* \eta_r^j,\ r = 0, 1, \ldots;\ j = 1, 2, \ldots$$

are integral invariants. Let us introduce the module $\Omega/V = \Omega \cap \Phi(\mathbf{M}/V)$ of all forms $\omega \in \Omega$ which are integral invariants. The contact forms η^j, $j = 1, 2, \ldots$ may be taken for the basis of this $\mathcal{F}(\mathbf{M}/V)$-submodule of the module $\Phi(\mathbf{M}/V)$. We recall that x, g^j, $j = 1, 2, \ldots$ are coordinates on the orbit space $\mathbf{M}/V$, whence $\Omega/V \subset \Phi(\mathbf{M}/V)$ is $\mathcal{F}(\mathbf{M}/V)$-submodule of codimension one.

5.5. Adapted filtrations. In order to see that $\Omega/V \subset \Phi(\mathbf{M}/V)$ is a diffiety on the orbit space, some arrangements are necessary. Let Ω_* be a good filtration where the second condition of (2.1) holds true if $l \geqslant L$. Let $\widetilde{\Omega}_0 \subset \Omega$ be any submodule generated by a finite number of forms η_r and η_r^j ensuring moreover the inclusion $\Omega_l \subset \widetilde{\Omega}_0$. (In fact the forms with $r = 0$ are sufficient here but not quite useful in practice.) Then the good filtration $\widetilde{\Omega}_*$ defined by the recurrence $\widetilde{\Omega}_{l+1} = \widetilde{\Omega}_l + \mathcal{L}_D \widetilde{\Omega}_l$ is clearly invariant. (The proof of Theorem 2.2 is done.)

We turn to a better result. If a submodule $\Theta \subset \Omega$ is invariant, hence $\mathcal{L}_V \Theta \subset \Theta$, then $\operatorname{Ker} \Theta \subset \Omega$ is also invariant. If the proof of Theorem 4.1 is applied to invariant filtration Ω_*, we obtain a standard and, moreover, invariant filtration $\overline{\Omega}_*$. (Theorem 2.2 is somewhat improved.)

Analogous reasonings can be applied to the submodule $\Omega/V \subset \Phi(\mathbf{M}/V)$. Roughly, the form $\eta = \eta_0$ is omitted and we obtain the good filtration $(\Omega/V)_*$. Altogether, the $\mathcal{F}(\mathbf{M}/V)$-submodule $\Omega/V \subset \Phi(\mathbf{M}/V)$ of all integral invariants $\omega \in \Omega$ is a diffiety on the orbit space $\mathbf{M}/V$ with (formally) the same total derivative D.

5.6. On the solutions. If $\mathbf{n}\colon (a \leqslant t \leqslant b) \to \mathbf{M}$ is a solution of Ω, then the corresponding projection $\mathbf{vn}\colon (a \leqslant t \leqslant b) \to \mathbf{M}/V$ is a solution of diffiety Ω/V. Informally, the coordinate $w \in \mathcal{F}(\mathbf{M})$ is omitted. Conversely, a solution of diffiety Ω/V is expressed in terms of coordinates x, g^j, $j = 1, 2, \ldots$ In order to obtain the solution $\mathbf{n}$ of Ω, the remaining coordinate w should be (not uniquely) determined from the easy Pfaffian equation $\mathbf{n}^*\eta = 0$.

5.7. The entrance of variational integral. Let $\varphi \in \Phi(\mathbf{M})$ and $V \in \mathcal{T}(\mathbf{M})$ be a variation of the integral $\int \varphi$. If $\breve{\varphi}$ is a $\mathcal{PC}$ form and $\mathbf{n}$ is an extremal, the identities

$$\mathcal{L}_V \Omega \subset \Omega, \quad \mathcal{L}_V \varphi \in \Omega, \quad \mathbf{n}^*\Omega = 0, \quad \mathbf{n}^* Z \rfloor \mathrm{d}\breve{\varphi} = 0, \quad \breve{\varphi} = \varphi + \breve{\omega}$$

imply

$$0 = \mathbf{n}^* \mathcal{L}_V \breve{\varphi} = \mathbf{n}^* V \rfloor \mathrm{d}\breve{\varphi} + \mathbf{n}^* \, \mathrm{d}\breve{\varphi}(V) = \mathrm{d}\mathbf{n}^* \breve{\varphi}(V),$$

whence

$$\mathbf{n}^* \breve{\varphi}(V) = c, \quad c = c[\mathbf{n}] \in \mathbb{R}.$$

This is the *Noether theorem* for the Lagrange variational problem.

5.8. The Noether subspace. If a solution $\mathbf{n}$ of Ω satisfies $\mathbf{n}^* f = 0$ for a certain $f \in \mathcal{F}(\mathbf{M})$, then also

$$0 = \mathrm{d}\mathbf{n}^* f = \mathbf{n}^* \, \mathrm{d}f = \mathbf{n}^* Df \, \mathrm{d}x, \quad \mathbf{n}^* \, \mathrm{d}x = \mathrm{d}\mathbf{n}^* x \neq 0$$

and therefore $\mathbf{n}^* Df = 0$. It follows that

$$\mathbf{n}^* D^r (\breve{\varphi}(V) - c) = 0, \quad r = 0, 1, \ldots \tag{5.3}$$

In geometrical terms, every extremal lies in a certain subspace $\mathbf{i}[c]\colon \mathbf{M}[c] \subset \mathbf{M}$ defined by equations (5.3). Assuming $\breve{\varphi}(V) = c \in \mathbb{R}$, we have $\mathbf{M}[c] = \mathbf{M}$ and the theory becomes much easier than in the general case $\breve{\varphi}(V) \neq \text{const}$. We therefore suppose $\mathrm{d}\breve{\varphi}(V) \neq 0$ from now on unless otherwise stated. Equations (5.3) can be also expressed in the global form

$$\mathbf{n}^* \mathcal{H}^r (\breve{\varphi}(V) - c) = 0, \quad \mathcal{H} = \mathcal{H}(\Omega); \quad r = 0, 1, \ldots$$

without the use of the accidental technical tool, the derivative $D = D_x$.

5.9. The Routh integral. Let $\breve{\varphi}$ be an invariant $\mathcal{PC}$ form, hence

$$0 = \mathcal{L}_V \breve{\varphi} = V \rfloor \mathrm{d}\breve{\varphi} + \mathrm{d}\breve{\varphi}(V).$$

It follows that $V\breve{\varphi}(V) = 0$ and therefore

$$VD^r(\breve{\varphi}(V) - c) = 0, \quad r = 0, 1, \ldots$$

In geometrical terms, the vector field $V \in \mathcal{T}(\mathbf{M})$ is tangent to the Noether subspace and may be regarded as a vector field on $\mathbf{M}[c]$ as well. We may introduce the orbit subspace $\mathbf{M}[c]/V \subset \mathbf{M}/V$. On the other hand, the elementary equation

$$\mathcal{L}_{fV}\breve{\varphi} = f\mathcal{L}_V\breve{\varphi} + \breve{\varphi}(V)\,\mathrm{d}f = \breve{\varphi}(V)\,\mathrm{d}f, \quad f \in \mathcal{F}(\mathbf{M})$$

implies the identity

$$\mathcal{L}_{fV}(\breve{\varphi} - c\,\mathrm{d}w) = (\breve{\varphi}(V) - c)\,\mathrm{d}f, \quad w \in \mathcal{F}(\mathbf{M}),\ Vw = 1$$

with the restriction

$$\mathcal{L}_{fV}\breve{\varphi}[c] = 0, \quad \breve{\varphi}[c] = \mathbf{i}[c]^*(\breve{\varphi} - c\,\mathrm{d}w) \in \Phi(\mathbf{M}[c])$$

to the subspace $\mathbf{i}[c]\colon \mathbf{M}[c] \subset \mathbf{M}$. Due to (5.2), the latter identity declares that the form $\breve{\varphi}[c] \in \Phi(\mathbf{M}[c])$ is integral invariant. It may be identified with the corresponding form on the orbit space $\mathbf{M}[c]/V$. In more detail

$$\mathbf{i}[c]^*(\breve{\varphi} - c\,\mathrm{d}w) = \breve{\varphi}[c] = \mathbf{v}^*\breve{\varphi}[c], \quad \mathbf{v}\colon \mathbf{M}[c] \to \mathbf{M}[c]/V. \tag{5.4}$$

We have the desired *Routh variational integral* $\int \breve{\varphi}[c]$ on the space $\mathbf{M}[c]/V$ with the constraint diffiety of integral invariants Ω/V restricted to the subspace $\mathbf{M}[c]/V$. In quite explicit terms, the submodule

$$\mathbf{i}[c]^*\Omega/V \subset \Phi(\mathbf{M}[c]/V)$$

is the constraint diffiety for the Routh integral (5.4).

5.10. The overall survey. Altogether we recall

$$\begin{array}{ll} \text{coordinates } w, x, g^j \ \text{ on } \mathbf{M},\ \text{ the basis } \eta,\, \eta^j \ \text{ of } \Omega, & \\ \text{coordinates } x, g^j \ \text{ on } \mathbf{M/V},\ \ \text{the basis } \eta^j \text{ of } \Omega/V, & \end{array} \quad j = 1, 2, \ldots$$

Assuming $\breve{\varphi}(V) \neq \text{const}$, we may choose $\breve{\varphi}(V) = g^1$. Let us denote $g^{2r+1} = D^r g^1$, $r = 0, 1, \ldots$ Then

$$\begin{matrix} w, x, g^{2r} \text{ are coordinates on } \mathbf{M}[c] \text{ and } \eta, \eta^{2r} \text{ is a basis of } \Omega[c], \\ x, g^{2r} \text{ are coordinates on } \mathbf{M}[c]/V \text{ and } \eta^{2r} \text{ is a basis of } \Omega[c]/V \end{matrix} \quad r = 1, 2, \ldots$$

In fact we do not need such artificial arrangement in the proof. In all particular examples to follow, much better coordinates can be found.

There are commutative diagrams

$$\begin{tikzcd} \mathbf{i}[c] \colon \mathbf{M} \arrow[d, "\mathbf{v}"'] & \mathbf{M}[c] \arrow[l] \arrow[d] \\ \mathbf{i}[c] \colon \mathbf{M}/V & \mathbf{M}[c]/V \arrow[l] \end{tikzcd} \quad \begin{tikzcd} \mathbf{i}[c]^* \colon \Phi(\mathbf{M}) \arrow[r] \arrow[d, "\mathbf{v}^*"'] & \Phi(\mathbf{M}[c]) \arrow[d] \\ \mathbf{i}[c]^* \colon \Phi(\mathbf{M}/V) \arrow[r] & \Phi(\mathbf{M}[c]/V) \end{tikzcd} \quad \begin{tikzcd} \mathbf{i}[c] \colon \Omega \arrow[r] \arrow[d, "\mathbf{v}"'] & \Omega[c] \arrow[d] \\ \mathbf{i}[c] \colon \Omega/V \arrow[r] & \Omega[c]/V \end{tikzcd}$$

Since $\Omega \subset \Phi(\mathbf{M})$ and $\Omega/V \subset \Phi(\mathbf{M}/V)$ are diffieties, the restrictions

$$\mathbf{i}[c]^*\Omega = \Omega[c] \subset \Phi(\mathbf{M}[c]), \quad \mathbf{i}[c]^*\Omega/V \subset \Phi(\mathbf{M}[c]/V)$$

are also diffieties since the restrictions of good filtrations are (obviously) good. We recall the abbreviated notation of variational integrals

$\int \varphi = \int_a^b \mathbf{n}^*(\varphi + \widetilde{\omega})$ (solution $\mathbf{n}$ of Ω, arbitrary $\widetilde{\omega} \in \Omega$),

$\int \breve{\varphi}[c] = \int_a^b \mathbf{n}^*(\breve{\varphi}[c] + \widetilde{\omega})$ (solution $\mathbf{n}$ of $\Omega[c]/V$, arbitrary $\widetilde{\omega} \in \Omega[c]/V$).

Let us at least accentuate the strengh of $\mathcal{PC}$ forms. The above condition

$$\mathbf{n}^* Z \rfloor \, \mathrm{d}\breve{\varphi} = 0 \quad (\text{arbitrary } Z \in \mathcal{T}(\mathbf{M}))$$

for the extremal $\mathbf{n}$ is equivalent to any of the seemingly weaker conditions

$$\mathbf{n}^* A \rfloor \, \mathrm{d}(\breve{\varphi} + \omega) = 0, \quad \mathbf{n}^* V \rfloor \, \mathrm{d}(\breve{\varphi} + \omega) = 0, \quad \mathbf{n}^* Z \rfloor \, \mathrm{d}(\breve{\varphi} + \omega) = 0 \tag{5.5}$$

with arbitrary variations A or V, appropriate $\omega \in \Omega$, and arbitrary vector field Z.

5.11. The finale. In order to conclude the proof, let us introduce the classical Pfaff-Darboux normal form

$$\mathrm{d}\breve{\varphi} = \sum \mathrm{d}a_k \wedge \mathrm{d}b_k, \quad a_k, b_k \in \mathcal{F}(\mathbf{M}); \ k = 0, 1, \ldots, K \tag{5.6}$$

of the exterior differential $\mathrm{d}\breve{\varphi}$. Condition (2.8) for the extremals $\mathbf{n}$ in the space $\mathbf{M}$ reads

$$\mathbf{n}^* \, \mathrm{d}a_k = \mathbf{n}^* \, \mathrm{d}b_k = 0, \quad k = 0, \ldots, K, \ \mathbf{n}^*\Omega = 0. \tag{5.7}$$

On the other hand, if $\breve{\varphi}$ is an invariant $\mathcal{PC}$ form, then

$$0 = \mathcal{L}_V \breve{\varphi} = V \rfloor \mathrm{d}\breve{\varphi} + \mathrm{d}\breve{\varphi}(V), \quad \text{therefore} \quad V \rfloor \mathrm{d}\breve{\varphi} = -\mathrm{d}\breve{\varphi}(V),$$

where

$$V \rfloor \mathrm{d}\breve{\varphi} = \sum V a_k \,\mathrm{d}b_k - \sum V b_k \,\mathrm{d}a_k.$$

We may suppose $a_0 = \breve{\varphi}(V)$ without loss of generality, whence

$$Va_0 = \ldots = Va_K = 0, \quad Vb_0 = 1, \quad Vb_1 = \ldots = Vb_K = 0.$$

Denoting $w = b_0$, we have

$$\mathrm{d}\breve{\varphi} = \mathrm{d}\breve{\varphi}(V) \wedge \mathrm{d}w + \sum \mathrm{d}a_j \wedge \mathrm{d}b_j, \quad j = 1, \ldots, K,$$

whence

$$\breve{\varphi} = \breve{\varphi}(V)\,\mathrm{d}w + \sum a_j \,\mathrm{d}b_j + \mathrm{d}F, \quad F \in \mathcal{F}(\mathbf{M}).$$

Trivially, $\mathbf{i}[c]^* \breve{\varphi}(V) = c$, which implies that the form

$$\breve{\varphi}[c] = \mathbf{i}[c]^*(\breve{\varphi} - c\,\mathrm{d}w) = \mathbf{i}[c]^*\Big(\sum a_j \,\mathrm{d}b_j + \mathrm{d}F\Big)$$

makes sense on the orbit space $\mathbf{M}[c]/V$. It follows that

$$\mathrm{d}\breve{\varphi}[c] = \mathbf{i}[c]^* \sum \mathrm{d}a_j \wedge \mathrm{d}b_j$$

and we obtain certain extremals $\mathbf{n}$ in the space $\mathbf{M}[c]/V$ by applying Definition (2.4). In more detail, we have the condition

$$\mathbf{n}^* A \rfloor \mathrm{d}\breve{\varphi}[c] = 0 \quad (\text{all } A),$$

which is satisfied if

$$\mathbf{n}^* \,\mathrm{d}a_j = \mathbf{n}^* \,\mathrm{d}b_j = 0, \quad j = 1, \ldots, K, \quad \mathbf{n}^* \Omega[c]/V = 0. \tag{5.8}$$

Conditions (5.8) are weaker than (5.7). However, if $\breve{\varphi}[c]$ is a $\mathcal{PC}$ form, there exist no other extremals than those satisfying (2.8) and the proof of Theorem 5.1 is done.

5.12. Easier finale. The Pfaff-Darboux normal form provides an interesting complementary insight into the role of the Routh integral and was mentioned just for this reason here. In fact, the direct arguments are enough and quite easy. Indeed, if $\mathbf{n}$ is a solution of Ω lying in $\mathbf{M}[c]$, then the projection $\mathbf{vn}$ on the orbit space $\mathbf{M}[c]/V$ is trivially a solution of diffiety $\Omega[c]/V$ involving only the integral invariants of Ω. If, moreover, $\mathbf{n}$ is an extremal, condition (2.8) restricted to $\mathbf{M}[c]$ implies (2.4) with the form $\breve{\varphi}[c]$ instead of $\breve{\varphi}$ and with a general vector field Z projectable on the orbit space $\mathbf{M}[c]/V$ instead of a variation A. So the projections $\mathbf{vn}$ are even the "very strong" extremals of the Routh integral, moreover, if $\breve{\varphi}[c]$ is a $\mathcal{PC}$ form, then other extremals satisfying (5.5) do not exist. □

If the variation V and the orbit diffiety are explicitly described, the Routh integral (5.1) with the reduced constraint $\Omega[c]/V$ on the space $\mathbf{M}[c]/V$ can be easily written down, however, the last sentence of Theorem 5.1 latently contains the true difficulty.

Definition 5.1. We speak of a *normal case of the reduction problem* if every extremal of the integral (5.1) is a projection of an appropriate extremal of the primary integral $\int \varphi$.

We will return to the criteria of normality later on. Some statements with particular examples are needful. At this place, we delete one gap also occurring in the proof, namely the postulated existence of an invariant $\mathcal{PC}$ form.

Theorem 5.2. *The invariant $\mathcal{PC}$ form $\breve{\varphi}$ with $\mathcal{L}_V \breve{\varphi} = 0$ exists for every variation V of integral $\int \varphi$ with controllable constraint Ω.*

Proof. Let Ω_* be a standard filtration and let us (temporarily) denote by Θ the module generated by all initial forms $\pi_0^1, \ldots, \pi_0^\mu$. If the module Θ is invariant in the sense that $\mathcal{L}_V \Theta \subset \Theta$, then Theorem 4.3 ensures that the corresponding $\mathcal{PC}$ form $\breve{\varphi}$ is invariant, too. So we refer to the following lemma. □

Lemma 5.1. *Let Ω_* be a standard and invariant filtration. Then the module Θ can be made invariant if the initial forms are appropriately modified.*

Proof. We state only the rough scheme of reasonings. Assume $j_0 = j_\mu$, so we have only initial forms $\pi_0^1, \ldots, \pi_0^{j_0} \in \Omega_0$. Then $\Theta = \Omega_0$ and since we suppose $\mathcal{L}_V \Omega_0 \subset \Omega_0$, the assertion holds true.

Assume $j_1 = j_\mu$, so we have the initial forms $\pi_0^1, \ldots, \pi_1^{j_1} \in \Omega_1$. Then

$$\mathcal{L}_V \pi_0^j = \sum a_{j'}^j \pi_0^{j'}, \quad j, j' = 1, \ldots, j_0$$

(since Ω_0 is preserved) and moreover

$$\mathcal{L}_V \pi_0^k = \sum b_{k'}^k \pi_0^{k'} + \sum c_j^k \pi_0^j + \sum d_j^k \pi_1^j, \quad k, k' = j_0 + 1, \ldots, j_1$$

(since Ω_1 is preserved). The forms π_1^j are not initial and must be deleted in order to obtain the sought result. Let us introduce the correction

$$\overline{\pi}_0^k = \pi_0^k + \sum u_j^k \pi_1^j, \quad k = j_0 + 1, \ldots, j_1$$

of initial forms. Then

$$\mathcal{L}_V \overline{\pi}_0^k = \sum \left(d_j^k + V u_j^k + \sum u_{j'}^k a_j^{j'} \right) \pi_1^j + \ldots$$

and we obtain the system of equations

$$d_j^k + Vu_j^k + \sum u_{j'}^k a_j^{j'} = u_j^k, \quad j = 1, \ldots, j_0;\ k = j_0 + 1, \ldots, j_1$$

for the unknown functions u_j^k. They ensure the invariance of the corrected module Θ with the initial forms $\pi_0^1, \ldots, \pi_0^{j_0}$ (as before), $\overline{\pi}_0^{j_0+1}, \ldots, \overline{\pi}_0^{j_1}$ (corrected).

Analogous adaptations can be made if $j_2 = j_\mu$, and so on. □

Let us conclude with the case which was as yet passed over.

Theorem 5.3. *Let $\Omega \subset \Phi(\mathbf{M})$ be a controllable diffiety and $V \in \mathcal{T}(\mathbf{M})$ an infinitesimal symmetry of a variational integral $\int \varphi$ with the constraint Ω. If $\breve{\varphi}$ is an invariant $\mathcal{PC}$ form and $\breve{\varphi}(V) = c \in \mathbb{R}$, then the Routh variational integral* (5.1) *is defined on the orbit space $\mathbf{M}/V$. Altogether we have the variational integral $\int \breve{\varphi}[c]$ with constraint $\Omega/V \subset \Phi(\mathbf{M}/V)$ and the original extremals in $\mathbf{M}$ are naturally projected on the family of all extremals of the integral* (5.1).

Proof. This is formally Theorem 5.1 if we put $\mathbf{i}[c] = \mathrm{id}$, that is, $\mathbf{M}[c] = \mathbf{M}$ and $\mathbf{i}[c]^* = \mathrm{id}$, $\Omega[c] = \Omega$. The proof may be omitted. □

Warning: We tacitly suppose the controllability. If Ω/V or $\Omega[c]/V$ are noncontrollable diffieties, the above results fail. A certain caution is necessary also for the choice of the independent variable $x = g^0$ since it is employed both on $\mathbf{M}$ and on the space $\mathbf{M}[c] \subset \mathbf{M}$.

6. Introductory applications

The notation of the general theory is insufficient and cannot be as a rule mechanically preserved in particular reduction problems. Two kinds of coordinates then appear, the primary coordinates on the space $\mathbf{M}$ and $\mathbf{M}[c]$ together with coordinates adapted to the orbit spaces $\mathbf{M}/V$ and $\mathbf{M}[c]/V$. The reduction problem is expressed in terms of the primary coordinates, while the adapted coordinates determine the geometrical sense of the final achievement (maybe) in a somewhat latent form.

Let us turn to simple instructive examples.

6.1. The point symmetry of a first-order integral in the jet space. We recall the coordinates on the jet space $\mathbf{M}(m)$, the contact forms of the diffiety $\Omega(m)$ and the total derivative

$$x, w_r^j,\ \omega_r^j = \mathrm{d}w_r^j - w_{r+1}^j\,\mathrm{d}x,\ D = \frac{\partial}{\partial x} + \sum w_{r+1}^j \frac{\partial}{\partial w_r^j}, \quad j = 1, \ldots, m;\ r = 0, 1, \ldots$$

for our convenience. The order-preserving filtration $\Omega(m)_*$ is a standard one, the forms

$$\omega_0^j\ (=\pi_0^j,\ j=1,\ldots,\ j_0=\mu(\Omega(m))=m)$$

are initial and generate the first term $\Omega(m)_0$ of the filtration. We recall variation (3.7) of diffiety $\Omega(m)$. Let this variation V be the infinitesimal point symmetry in the common sense:

$$Vx=v(x,\cdot\cdot,w_0^j,\cdot\cdot),\quad Vw_0^j=v_0^j+w_1^jv=v^j(x,\cdot\cdot,w_0^j,\cdot\cdot),\quad j=1,\ldots,m.$$

Then V generates a group of point transformations of $\mathbf{M}(m)$. In full detail, there exist functionally independent functions

$$w=w(x,\cdot\cdot,w_0^j,\cdot\cdot),\ g=g(x,\cdot\cdot,w_0^j,\cdot\cdot),\ g^k=g^k(x,\cdot\cdot,w_0^j,\cdot\cdot),\quad k=1,\ldots,m-1$$

satisfying

$$Vw=1,\quad Vg=Vg^k=0,\quad k=1,\ldots,m-1.$$

The reason for this notation is as follows. If g is taken for alternative independent variable, we obtain the useful alternative total derivative

$$\mathcal{D}_g\,(=\mathcal{D}\text{ abbreviation})\ =\frac{1}{Dg}D\quad\text{with }[V,\mathcal{D}]=0$$

and the remaining invariant functions (first integrals)

$$w_r=\mathcal{D}^rw,\quad g_r^k=\mathcal{D}^rg^k,\quad k=1,\ldots,m-1;\ r=1,2,\ldots$$

of the order r. Then the functions

$$\text{(6.1)}\qquad w_0\,(=w),\ g,\ g_0^k\,(=g^k),\ w_r,\ g_r^k,\quad k=1,\ldots,m-1;\ r=1,2,\ldots$$

may be taken for alternative coordinates on $\mathbf{M}(m)$ and if w_0 is omitted, we have the coordinates on $\mathbf{M}(m)/V$. Consequently all forms

$$\text{(6.2)}\qquad \eta_r(=\omega_{w_r})=\mathrm{d}w_r-w_{r+1}\,\mathrm{d}g,\quad \eta_r^k(=\omega_{g_r^k})=\mathrm{d}g_r^k-g_{r+1}^k\,\mathrm{d}g$$

constitute the alternative basis of diffiety $\Omega(m)$ and if η_0 is omitted, we have a basis of diffiety $\Omega(m)/V$. We eventually introduce the variational integral $\int\varphi$ and the corresponding $\mathcal{PC}$ form, where

$$\varphi=f(x,\cdot\cdot,w_0^j,w_1^j,\cdot\cdot)\,\mathrm{d}x,\quad \breve{\varphi}=f\,\mathrm{d}x+\sum\frac{\partial f}{\partial w_1^j}\omega_0^j.$$

The point symmetry V preserves the filtration $\Omega(m)_*$ and especially the first term $\Omega(m)_0$. Moreover, due to congruence (4.8) and the equality $j_0 = \mu(\Omega)$, the first term uniquely determines the $\mathcal{PC}$ form, which implies that $\breve{\varphi}$ is an invariant form. Theorem 5.1 can be applied. Assuming

$$\mathrm{d}\breve{\varphi}(V) \neq 0, \quad \breve{\varphi}(V) = fv + \sum \frac{\partial f}{\partial w_1^j} v_0^j,$$

we may introduce the Noether subspace $\mathbf{i}[c]\colon \mathbf{M}[c] \subset \mathbf{M}$ defined by

$$D^r(\breve{\varphi}(V) - c) = 0 \quad (\text{equivalently } \mathcal{D}^r(\breve{\varphi}(V) - c) = 0) \tag{6.3}$$

and the Routh integral $\int \breve{\varphi}[c]$. The differential form

$$\breve{\varphi}[c] = \mathbf{i}[c]^*(\breve{\varphi} - c\,\mathrm{d}w) \tag{6.4}$$

is an integral invariant defined even on the orbit space $\mathbf{M}[c]/V$ and we obtain the sought reduction with the constraint diffiety $\Omega[c]/V = \mathbf{i}[c]^*\Omega/V$.

Altogether we have deleted the coordinate w and the contact form η, moreover, the conservation law $\breve{\varphi}(V) = c$ holds true in the reduced space.

The same result was derived in [1], [2] together with very explicit particular examples using a direct construction, however, the true sense of the result is better clarified in the general theory.

6.2. Continuation: the normality condition. Abbreviating $N = \breve{\varphi}(V)$, we wish to determine some coordinates on the Noether subspace $\mathbf{M}[c] \subset \mathbf{M}(m)$ and on the orbit subspace $\mathbf{M}[c]/V \subset \mathbf{M}(m)/V$ by applying the common implicit functions theorem. In terms of the alternative coordinates clearly

$$\mathcal{D}^r N = \ldots + \frac{\partial N}{\partial w_1} w_{r+1} + \sum \frac{\partial N}{\partial g_1^k} g_{r+1}^k, \quad r = 1, 2, \ldots$$

as the top order summands are concerned. Therefore

$$\mathrm{d}\mathcal{D}^r N = \ldots + \frac{\partial N}{\partial w_1}\,\mathrm{d}w_{r+1} + \sum \frac{\partial N}{\partial g_1^k}\,\mathrm{d}g_{r+1}^k, \quad r = 1, 2, \ldots$$

Assuming

$$\frac{\partial N}{\partial w_1} \left(= \frac{\partial \breve{\varphi}(V)}{\partial w_1}\right) \neq 0, \tag{6.5}$$

it follows that coordinates $w_1, w_2, \dots$ in (6.1) can be replaced with functions N, $\mathcal{D}N, \dots$, that is, the family of functions

$$w_0\,(=w),\ g,\ \mathcal{D}^r N,\ g_r^k, \quad k=1,\dots,m-1;\ r=0,1,\dots \tag{6.6}$$

may be also taken for coordinates on $\mathbf{M}(m)$. Omitting all functions $\mathcal{D}^r N$, we have coordinates on $\mathbf{M}[c]$. Even more is true. We may refer to the identity

$$VN = V\breve{\varphi}(V) = 0, \quad \text{where } V = \frac{\partial}{\partial w} = \frac{\partial}{\partial w_0}$$

in terms of the alternative coordinates. Then the same arguments as above imply that (6.6) are coordinates on the orbit space $\mathbf{M}[c]/V$ if the first term w_0 is omitted.

Let us turn to the $\mathcal{PC}$ form $\breve{\varphi}$. Formula (3.9) reads

$$\mathrm{d}\breve{\varphi} \cong \sum f_0^j \omega_0^j \wedge \mathrm{d}x \ \ (\operatorname{mod} \Omega(m) \wedge \Omega(m))$$

and therefore a certain congruence

$$\mathrm{d}\breve{\varphi} \cong \left(e\eta_0 + \sum e^k \eta_0^k\right) \wedge \mathrm{d}g \ \ (\operatorname{mod} \Omega(m) \wedge \Omega(m))$$

holds true in terms of alternative coordinates (6.1). (Briefly saying: $\breve{\varphi}$ is a $\mathcal{PC}$ form after applying any pointwise transformation and the forms $\eta_0, \eta_0^1, \dots, \eta_0^{m-1}$ may be taken for initial as well.) Since the form

$$\mathrm{d}\breve{\varphi}[c] = \mathbf{i}[c]^*(\mathrm{d}\breve{\varphi} - c\,\mathrm{d}w) = \mathbf{i}[c]^*\,\mathrm{d}\breve{\varphi}$$

on the orbit space is independent of the coordinate $w = w_0$, it follows that

$$\mathrm{d}\breve{\varphi}[c] \cong \mathbf{i}[c]^*\left(\sum e^k \eta_0^k \wedge \mathrm{d}g\right) \ \ (\operatorname{mod} \Omega[c]/V \wedge \Omega[c]/V). \tag{6.7}$$

If e^k, η_0^k, g are alternatively regarded as functions and forms on the orbit space, the mapping $\mathbf{i}[c]^*$ can be formally omitted and condition (6.7) declares that $\breve{\varphi}[c]$ is a $\mathcal{PC}$ form on the orbit space. We conclude that condition (6.5) ensures the normal case of the reduction problem in terms of coordinates on the orbit space.

Though the above reasoning clarifies the substance of normality, the result (6.5) is of little use in practice. A normality condition expressed in terms of the primary jet coordinates would be better. This is indeed possible by applying a tricky argument (see [1], [2]) which is somewhat simplified here as follows.

First of all, one can easily derive the formulae

$$\mathrm{d}D^r N = \ldots + \sum \frac{\partial N}{\partial w_1^j}\,\mathrm{d}w_{r+1}^j,\ r=0,1,\dots, \qquad \frac{\partial N}{\partial w_1^j} = \sum \frac{\partial^2 f}{\partial w_1^j \partial w_1^{j'}} v^{j'} \tag{6.8}$$

by direct calculation. Analogous formulae

$$\mathrm{d}g^k_r = \ldots + \sum \frac{\partial g^k_r}{\partial w^j_r}\,\mathrm{d}w^j_r, \quad k = 1, \ldots, m-1;\ r = 0, 1, \ldots, \tag{6.9}$$

where

$$\frac{\partial g^k_r}{\partial w^j_r} = \frac{1}{(Dg)^{r+1}}\Big(-\frac{\partial g}{\partial w^j_0} Dg^k_0 + \frac{\partial g^k_0}{\partial w^j_0} Dg\Big), \quad r = 1, 2, \ldots \tag{6.10}$$

need more effort, see below.

Assuming (6.8), (6.9) and (6.10), the sufficient normality condition

$$\sum \frac{\partial^2 f}{\partial w^j_1 \partial w^{j'}_1} v^j v^{j'} \neq 0, \quad v^j = \omega^j_0(V) = v^j_0 - w^j_1 v \tag{6.11}$$

in terms of primary coordinates can be proved as follows. By using the identities

$$0 = Vg = vDg + \sum v^j \frac{\partial g}{\partial w^j_0}, \quad 0 = Vg^k_0 = vDg^k_0 + \sum v^j \frac{\partial g^k_0}{\partial w^j_0}$$

we obtain the nontrivial dependences

$$\sum \frac{\partial g^k_r}{\partial w^j_r} v^j = \frac{1}{(Dg)^{r+1}}(vDg \cdot Dg^k_0 - vDg^k_0 \cdot Dg) = 0, \quad r = 1, 2, \ldots$$

Inequality (6.11) together with (6.8) therefore declares that

$$\det \begin{pmatrix} \dfrac{\partial D^{r-1}N}{\partial w^1_1} & \cdots & \dfrac{\partial D^{r-1}N}{\partial w^m_1} \\ \dfrac{\partial g^1_r}{\partial w^1_1} & \cdots & \dfrac{\partial g^1_r}{\partial w^m_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g^{m-1}_r}{\partial w^1_1} & \cdots & \dfrac{\partial g^{m-1}_r}{\partial w^m_1} \end{pmatrix} \neq 0, \quad r = 1, 2, \ldots$$

and we may apply the same arguments as above: functions (6.6) can be taken for coordinates on $\mathbf{M}(m)$ with the same consequences for the $\mathcal{PC}$ form $\breve{\varphi}[c]$.

Let us finally outline the proof of formula (6.10). If $r = 1$, then

$$g^k_1 = \mathcal{D}g^k_0 = \frac{1}{Dg} Dg^k_0, \quad \frac{\partial g^k_1}{\partial w^j_1} = -\frac{1}{(Dg)^2}\frac{\partial Dg}{\partial w^j_1} Dg^k_0 + \frac{1}{Dg}\frac{\partial Dg^k_0}{\partial w^j_1}$$

and inserting

$$\frac{\partial Dg}{\partial w_1^j} = \frac{\partial g}{\partial w_0^j}, \qquad \frac{\partial Dg_0^k}{\partial w_1^j} = \frac{\partial g_0^k}{\partial w_0^j},$$

we obtain (6.10). Analogous equalities

$$g_2^k = \frac{1}{Dg} Dg_1^k, \qquad \frac{\partial g_2^k}{\partial w_2^j} = \frac{1}{Dg}\frac{\partial Dg_1^k}{\partial w_2^j} = \frac{1}{Dg}\frac{\partial g_1^k}{\partial w_2^j}$$

imply (6.10) with $r = 2$, and so on with $r > 2$.

6.3. Still a continuation: on the Jacobi-Maupertuis principle. Let us in particular choose

$$V = \frac{\partial}{\partial x}, \quad w = x, \quad g = w_0^m, \quad g^k = w_0^k, \quad k = 1, \ldots, m-1.$$

Informally saying, then the primary jet coordinates differ from the alternative coordinates adapted to the orbit spaces only within the choice of the independent variable. For instance,

$$\text{the form}\, \mathrm{d}w_r^k - w_{r+1}^k \,\mathrm{d}x \text{ turns into } \mathrm{d}w_r^k - \frac{w_{r+1}^k}{w_1^m}\,\mathrm{d}g = \mathrm{d}w_r^k - \frac{w_{r+1}^k}{w_1^m}\,\mathrm{d}w_0^m.$$

Let us mention the variational integral $\int \varphi$ where

$$\varphi = T(\cdot\cdot, w_0^j, w_1^j, \cdot\cdot) - V(\cdot\cdot, w_0^j, \cdot\cdot), \quad 2T = \sum \frac{\partial T}{\partial w_1^j} w_1^j. \tag{6.12}$$

Then, trivially,

$$\breve{\varphi} = (T - V)\,\mathrm{d}x + \sum \frac{\partial T}{\partial w_1^j}\omega_0^j = -(T + V)\,\mathrm{d}x + \sum \frac{\partial T}{\partial w_1^j}\,\mathrm{d}w_0^j,$$
$$\breve{\varphi}(V) = -(T + V), \quad \breve{\varphi}[c] = \breve{\varphi} - c\,\mathrm{d}x = \sum \frac{\partial T}{\partial w_1^j}\,\mathrm{d}w_0^j$$

and the normality condition (6.11) can be simplified as $T \neq 0$ by using the second formula in (6.12)

If $\mathrm{d}w_0^j = w_1^j\,\mathrm{d}x$ is inserted into the $\mathcal{PC}$ form $\breve{\varphi}[c]$, three results

$$2T\,\mathrm{d}x, \quad -2(V + c)\,\mathrm{d}x, \quad \pm 2\sqrt{\mp(V + c)T}\,\mathrm{d}x$$

appear if the conservation law $-(T+V)=c$ is taken into account. Due to the differential $\mathrm{d}x$, these forms are not defined on the orbit spaces, however, the independent variable in the last form can be arbitrarily changed. So we obtain, e.g., the form

$$\widetilde{\varphi}[c] = \pm 2\sqrt{\mp(V+c)T\Big(\cdot\cdot,\frac{\mathrm{d}w_0^j}{\mathrm{d}w_0^m},\cdot\cdot\Big)}\,\mathrm{d}w_0^m,$$

which is already defined on the orbit spaces. The above Routh reduction $\breve{\varphi}[c]$ is a $\mathcal{PC}$ form related to the variational integral $\int\widetilde{\varphi}[c]$ on the underlying orbit space $\mathbf{M}[c]/V$ with the constant energy $T+V$.

6.4. A constrained variational integral. It is not easy to state a short and nontrivial example. Let us deal with the constrained variational integral

$$\int f(y^1,\ldots,y^m,z)\,\mathrm{d}x,\quad \frac{\mathrm{d}z}{\mathrm{d}x} = F\Big(\frac{\mathrm{d}y^1}{\mathrm{d}x},\ldots,\frac{\mathrm{d}y^m}{\mathrm{d}x}\Big).$$

First of all, we introduce the coordinates and contact forms

$$x,\ y_r^j,\quad \eta_r^j = \mathrm{d}y_r^j - y_{r+1}^j\,\mathrm{d}x,\quad \zeta = \mathrm{d}z - F(y_1^1,\ldots,y_1^m)\,\mathrm{d}x,$$

where $j=1,\ldots,m$ and $r=0,1,\ldots$ Then

$$\mathcal{L}_D\eta_r^j = \eta_{r+1}^j,\quad \mathcal{L}_D\zeta = \sum\frac{\partial F}{\partial w_1^j}\eta_1^j,\quad D = \frac{\partial}{\partial x} + \sum y_{r+1}^j\frac{\partial}{\partial y_r^j} + F\frac{\partial}{\partial z}.$$

We have the diffiety Ω, where the submodules $\Omega_l\subset\Omega$ $(l=0,1,\ldots)$ of forms

$$\omega = \sum a_r^j\eta_r^j + a\zeta\quad (\text{sum with } r\leqslant l)$$

provide a good but not the standard filtration. However, clearly

$$\mathcal{L}_D\pi = -\sum D\frac{\partial F}{\partial y_1^j}\eta_0^j\in\Omega_0,\quad \pi = \zeta - \sum\frac{\partial F}{\partial y_0^j}\eta_0^j\in\Omega_0$$

and it follows that π generates the module $\operatorname{Ker}\Omega_0$. So we may introduce the initial forms

$$\pi_0^1 = \eta_0^1,\ldots,\quad \pi_0^{m-1} = \eta_0^{m-1},\quad \pi_0^m = \pi$$

satisfying the equations

$$\pi_r^k = \mathcal{L}_D^r\pi_0^k = \eta_r^k,\quad k=1,\ldots,m-1;\ r=0,1,\ldots,$$

$$\pi_0^m = \pi = \zeta - \ldots,\quad \pi_{r+1}^m = \mathcal{L}_D^{r+1}\pi = \ldots - A\eta_r^m,\quad A = D\frac{\partial F}{\partial y_1^m}.$$

We have tacitly supposed $A \neq 0$ here. The forms η_r^m not appearing in the first line can be expressed in terms of forms π_r^j of the second line.

On the other hand, we have the variational integral $\int \varphi$ where $\varphi = f(\cdot\cdot, y_0^j, \cdot\cdot, z)\,\mathrm{d}x$. It follows that

$$\mathrm{d}\varphi = \Big(\sum \frac{\partial f}{\partial y_0^j}\eta_0^j + \frac{\partial f}{\partial z}\zeta\Big) \wedge \mathrm{d}x = \Big(\sum B^j \pi_0^j + B\pi_1^m\Big) \wedge \mathrm{d}x$$

in terms of the standard basis with the coefficients

$$B^k = \frac{\partial f}{\partial y_0^k} + \frac{\partial f}{\partial z}\frac{\partial F}{\partial y_1^k} + \frac{1}{A}D\frac{\partial F}{\partial y_1^k}, \quad B^m = \frac{\partial f}{\partial z}, \quad B = -\frac{1}{A}\Big(\frac{\partial f}{\partial y_0^m} + \frac{\partial f}{\partial z}\frac{\partial F}{\partial y_1^m}\Big)$$

and we obtain the $\mathcal{PC}$ form $\breve{\varphi} = \varphi - B\pi_0^m$ since

$$\mathrm{d}\breve{\varphi} = \mathrm{d}\varphi - \mathrm{d}B \wedge \pi_0^m - B\,\mathrm{d}\pi_0^m \cong \Big(\sum B^j\pi_0^j - DB\pi_0^m\Big) \wedge \mathrm{d}x \ (\mathrm{mod}\ \Omega \wedge \Omega).$$

For the choice $V = \partial/\partial x$, clearly

$$\breve{\varphi}(V) = f - B\Big(-F + \sum \frac{\partial F}{\partial y_1^j}y_1^j\Big), \quad \breve{\varphi}[c] = \breve{\varphi} - c\,\mathrm{d}x = (f-c)\,\mathrm{d}x - B\pi_0.$$

The curiosity $\breve{\varphi}[c] = -B\pi_0$ independent of the integral $\int \varphi$ appears if one supposes the identity

$$F = \sum \frac{\partial F}{\partial y_1^j}y_1^j.$$

We shall not discuss the choice of the variables in the orbit spaces here, they are analogous as in the Jacobi-Maupertuis principle and let us also pass the normality condition with silence.

6.5. Two symmetries of a second-order integral. We conclude with an "incorrect" example. The above mechanisms will be mechanically simulated for the case of two symmetries.

Let us recall the jet space $\mathbf{M}(m+1)$ where the coordinates and contact forms are denoted

$$x,\ w_r^j,\ z_r, \quad \omega_r^j = \mathrm{d}w_r^j - w_{r+1}^j\,\mathrm{d}x, \quad \xi_r = \mathrm{d}z_r - z_{r+1}\,\mathrm{d}x \quad j = 1,\dots,m;\ r = 0,1,\dots,$$

for better clarity of formulae to follow. We shall deal with two examples of the second-order variational integral $\int \varphi$.

First, suppose $\varphi = f(x, \cdot\cdot, w_0^j, w_1^j, w_2^j, \cdot\cdot, z_2)\,\mathrm{d}x$. Then

$$\breve{\varphi} = f\,\mathrm{d}x + \sum\Big(\frac{\partial f}{\partial w_1^j} - D\frac{\partial f}{\partial w_2^j}\Big)\omega_0^j + \sum \frac{\partial f}{\partial w_2^j}\omega_1^j - D\frac{\partial f}{\partial z_2}\xi_0 + \frac{\partial f}{\partial z_2}\xi_1$$

is the $\mathcal{PC}$ form and

$$V = \frac{\partial}{\partial z_0}, \quad W = x\frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1}$$

are two symmetries. The Noether subspace $\mathbf{M}[a,b] \subset \mathbf{M}(m+1)$ depends on two parameters $a, b \in \mathbb{R}$ and is defined by the equations

$$D^r(\breve{\varphi}(V) - a) = 0, \quad D^r(\breve{\varphi}(W) - b) = 0, \quad r = 0, 1, \dots, \tag{6.13}$$

where

$$\breve{\varphi}(V) = -\frac{\partial f}{\partial z_2}, \quad \breve{\varphi}(W) = -xD\frac{\partial f}{\partial z_2} + \frac{\partial f}{\partial z_2}.$$

The Routh reduction

$$\breve{\varphi}[a,b] = \breve{\varphi} - a\,\mathrm{d}(z_0 - xz_1) - b\,\mathrm{d}z_1$$

satisfies the conditions

$$\breve{\varphi}[a,b](V) = 0, \quad \breve{\varphi}[a,b](W) = 0, \tag{6.14}$$

which ensure that $\breve{\varphi}[a,b]$ is defined on the orbit space $\mathbf{M}[a,b]/(V,W)$ of two symmetries V, W. Clearly $[V,W] = 0$ and we have the abelian symmetry group.

Second, let us analogously choose

$$\varphi = f(x, \cdot\cdot, w_0^j, w_1^j, w_2^j, \cdot\cdot)\frac{z_2}{z_1}\,\mathrm{d}x, \quad V = \frac{\partial}{\partial z_0}, \quad W = \sum z_r\frac{\partial}{\partial z_r}.$$

This is already a nonabelian case since $[V,W] = V$. Then

$$\breve{\varphi} = \ldots - \left\{f\frac{z_2}{(z_1)^2} + D\frac{f}{z_1}\right\}\xi_0 + \frac{f}{z_1}\xi_1$$

and the Noether subspace $\mathbf{M}[a,b] \subset \mathbf{M}(m+1)$ is defined by the equations (6.13), where

$$\breve{\varphi}(V) = -\{\ldots\}, \quad \breve{\varphi}(W) = z_0\{\ldots\} + f.$$

The Routh reduction should be of the form $\breve{\varphi}[a,b] = \breve{\varphi} - \mathrm{d}F$, where the function $F = F(z_0, z_1)$ satisfies the equations

$$a = \breve{\varphi}(V) = VF = \frac{\partial F}{\partial z_0}, \quad b = \breve{\varphi}(W) = WF = z_0\frac{\partial F}{\partial z_0} + z_1\frac{\partial F}{\partial z_1}$$

analogous to (6.14). Alas, the equations are incompatible and such a reduction does not exist. On the other hand, the two-form $\mathrm{d}\breve{\varphi}$ makes a good sense on the orbit space $\mathbf{M}[a,b]/(V,W)$ and it follows that the symplectical structures can be reduced without any difficulty.

7. Perspectives

We ask some questions and raise some problems to be investigated.

7.1. Noncontrollable diffieties. The *residual submodule* $\mathcal{R} = \mathcal{R}(\Omega) \subset \Omega$ of diffiety Ω appearing in Theorem 4.2 is unique. It follows that $\mathcal{L}_V\mathcal{R} \subset \mathcal{R}$ for every variation V of Ω, see [7], [21]. In order to introduce the $\mathcal{PC}$ form, additional imposition $\mathcal{L}_Z\mathcal{R} \subset \mathcal{R}$ for the vector field Z in definition (2.5) is necessary.

7.2. On the $\mathcal{PC}$ form. We do not know if there exist "reasonable" forms $\breve{\varphi}$ satisfying (2.5) but not (2.6). Alternatively: can the clumsy condition (2.6) be omitted? Still in other terms, does (2.5) ensure the local nature of $A = A[Z]$ and therefore the Theorem 2.1?

7.3. Several independent variables. If n is the number of independent variables, the diffieties $\Omega \subset \Phi(\mathbf{M})$ are of codimension n and we have the multiple integral $\int \varphi$ with given differential n-form φ and arbitrary n-forms $\widetilde{\omega}$, $\breve{\omega} \cong 0 \pmod{\Omega}$. The definitions of the variations, extremals and $\mathcal{PC}$ forms can be preserved. However, the standard filtrations and the controllability rest on the involutivity concept, see [8]. It should be noted that even a formal definition of the $\mathcal{PC}$ form for the Lagrange variational problem of multiple variational integrals cannot be regarded as a trivial task, see [14].

7.4. Several symmetries. The orbit spaces and the Noether theorem do not cause difficulty, however, invariant $\mathcal{PC}$ forms, the Noether subspaces and the Routh variational integrals are ambiguous.

7.5. The Lie-Cartan pseudogroups. Somewhat paradoxically, the more symmetries we have, the worse are the results. For instance, the Noether theorem is easy for the Lie group but the general case of the Lie-Cartan pseudogroups where the symmetries depend on an "infinite number of parameters" was not clarified yet.

7.6. On the determination of symmetries. It should be noted that even the structure of all symmetries of the family of curves in $\mathbb{R}^3$ is actually unknown. There are too many nonclassical symmetries, in particular the "higher order" invertible contact transformations generated by "multiple waves".

7.7. On the Noether theorem. It may be generalized to include the *divergence symmetries*, that is, the symmetries of the Euler-Lagrange system where the $\mathcal{PC}$ form is preserved modulo a differential of a function. Does there exist a reasonable symmetry reduction?

8. Concluding comments

At least, we highly appreciate the professionality and prompt cooperation of the anonymous referee with the belief that the following notes may delete some ambiguities concerning the distinction between our approach and the common theories which rest on the advanced jet formalisms.

First, the article concerns the Routh reduction in the primary sense given in [12]. So we start with the variational integral $\int \varphi$ and the result again is a variational integral $\int \breve{\varphi}[c]$ on the orbit space. As a result, the nonabelian case does not give any reasonable Routh reduction, see [2]. In articles like [19], [4], [11], [20], the Routh procedure mainly focuses on the reduction of the Lagrange system and the symplectical structures.

Second, we deal with variational integrals subject to arbitrary differential constraint Ω, the quite general Lagrange variational problem. On the contrary, the common theories are devoted to minutely precise analysis of the first-order variational integrals appearing in applied mechanics. This is very valuable but rather narrow theory.

Third, our approach is coordinate-free (intrinsical) in the widest possible sense. For instance, the independent and dependent variables with various projections and connections appear only in particular examples as technical tools. As a result, the exposition is extremely short and does not need any subtle geometrical concepts of accidental nature proper only to the special problems under consideration.

At last, let us briefly point out some details: Definitions 2.4 and 2.5, the distinction between variations and infinitesimal symmetries, the explicit formula (4.6) for all variations, the Euler-Lagrange system without any uncertain multipliers for the extremals and the three-lines proof of the Noether theorem for the general Lagrange problem, formula (3.3) avoiding the common "linearization procedure" (see [16]), and last but not least, the survey of open problems which are still waiting for solution.

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