NORM CONTINUITY OF POINTWISE QUASI-CONTINUOUS MAPPINGS

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Abstract. Let X be a Baire space, Y be a compact Hausdorff space and $\varphi\colon X \to C_p(Y)$ be a quasi-continuous mapping. For a proximal subset H of $Y \times Y$ we will use topological games $\mathcal{G}_1(H)$ and $\mathcal{G}_2(H)$ on $Y \times Y$ between two players to prove that if the first player has a winning strategy in these games, then φ is norm continuous on a dense G_δ subset of X. It follows that if Y is Valdivia compact, each quasi-continuous mapping from a Baire space X to $C_p(Y)$ is norm continuous on a dense G_δ subset of X.

Keywords: function space; weak continuity; generalized continuity; quasi-continuous function; pointwise topology

MSC 2010: 54C35, 54C08, 54C05

1. Introduction

Let X and Z be topological spaces. A function $\varphi\colon X \to Z$ is called quasi-continuous at $x_0 \in X$ if for any neighborhood U of x_0 in X and any neighborhood V of $z_0 = \varphi(x_0)$ in Z there exists a nonempty open subset G of U such that $\varphi(G) \subset V$. The mapping $\varphi\colon X \to Z$ is called quasi-continuous if it is quasi-continuous at any point of X.

Let Y be a compact space and $C(Y)$ be the space of all continuous real-valued functions on Y. We consider two topologies on $C(Y)$, the norm topology, which is the topology generated by the supremum norm $\|f\| = \sup\limits_{y\in Y} |f(y)|$, $f \in C(Y)$, and the pointwise topology, which is the topology inherited from $\mathbb{R}^Y$ with product topology. The space $C(Y)$ equipped with the pointwise topology will be denoted by $C_p(Y)$.

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In 1974, Namioka [15] proved that every continuous mapping $\varphi\colon X \to C_p(Y)$ is norm continuous at the points of a dense G_δ subset of X provided that X is countably Čech-complete. Christensen [5] showed Namioka's theorem is still valid when X is σ-β-unfavorable. It was expected that the result of Namioka remains true when X is an arbitrary Baire space. However, Talagrand [17] provided an example of a pointwise continuous mapping $\varphi\colon X \to C_p(X)$, where X is on an α-favorable space X which is nowhere norm continuous. The result of Talagrand raises the following question:

What are compact spaces Y such that for every Baire space X and continuous (*or quasi-continuous*) *mapping $\varphi\colon X \to C_p(Y)$ must be norm continuous at each point of some dense G_δ subset of X?*

Several partial answers to the above question have been obtained by some authors (see e.g. [3], [6]–[14]). In particular, Bouziad [2] introduced two person games $\mathcal{G}_1(H)$ and $\mathcal{G}_2(H)$ on product $Y \times Y$, where H is a proximal subset of $Y \times Y$, to show that if the first player has winning strategies in both plays, then Y is a co-Namioka compact space.

In this paper, we will show that if in a compact space Y the second player in games $\mathcal{G}_1(H)$ and $\mathcal{G}_2(H)$ has no winning strategies, then every quasi-continuous mapping $\varphi\colon X \to C_p(Y)$ is norm continuous on a dense G_δ subset of X.

2. Results

We start this section by introducing the following topological games. The first one is known as *"Banach-Mazur game"* (or *"Choquet game"*, see [4] or [16]).

The Banach-Mazur game $\mathcal{BM}(X)$: Two players β and α select alternately nonempty open subsets of X as follows. Player β starts the game by selecting a nonempty open subset U_1 of X. In return, α replies by selecting some nonempty open subset V_1 of U_1. At the n-th stage of the game, $n \geqslant 1$, player β chooses a nonempty open subset $U_n \subset V_{n-1}$ and α answers by choosing a nonempty open subset V_n of U_n. Proceeding in this fashion, the players generate a sequence $(U_n, V_n)_{n=1}^{\infty}$ which is called a *play*. Player α wins the play $(U_n, V_n)_{n\geqslant 1}^{\infty}$ if $\bigcap\limits_{n\geqslant 1} U_n = \bigcap\limits_{n\geqslant 1} V_n \neq \emptyset$; otherwise player β wins this play. A *partial play* is a finite sequence of sets consisting of the first few moves of a play. A *strategy* for player α is a rule by means of which the player makes his/her choices. An *s-play* is a play in which α selects his/her moves according to the strategy s. The strategy s for the player α is said to be a *winning strategy* if every s-play is won by α. A space X is called *α-favorable* if there exists a winning strategy for α in $\mathcal{BM}(X)$.

It is easy to verify that every α-favorable space X is a Baire space. There are examples of Baire spaces which are not α-favorable (see for example [10]). It is known that X is a Baire space if and only if player β does not have a winning strategy in the game $\mathcal{BM}(X)$.

Let Y be a compact Hausdorff space and Δ denote the diagonal of $Y \times Y$. Following [2], a subset H of $Y \times Y$ is called proximal if it intersects every neighborhood of Δ. For a proximal set $H \subset \Delta$ we consider the following two player topological games.

$\mathcal{G}_1(H)$: At the n-th stage, $\mathfrak{a}$ selects a pair (W_n, D_n), where W_n is an open neighborhood of Δ and $D_n \cap H$ is a dense subset of H. Then $\mathfrak{b}$ answers by taking a point $(y_n, y'_n) \in W_n \cap H \cap D_n$. This play is won by $\mathfrak{a}$ if for every neighborhood W of Δ there is some $n \in \mathbb{N}$ such that $(y_n, y'_n) \in W$. Otherwise, $\mathfrak{b}$ wins the play. The space Y is called $\mathcal{G}_1(H)$-$\mathfrak{b}$-*favorable* if $\mathfrak{b}$ has a winning strategy in $\mathcal{G}_1(H)$. Otherwise, Y is called $\mathcal{G}_1(H)$-$\mathfrak{b}$-*unfavorable*.

$\mathcal{G}_2(H)$: At the n-th stage, $\mathfrak{a}$ selects a pair (W_n, D_n), where W_n is an open neighborhood of Δ and D_n is a dense subset of W_n. Then the answer of $\mathfrak{b}$ will be a point $(y_n, y'_n) \in W_n \cap D_n$. The play is won by $\mathfrak{a}$ if for every neighborhood W of Δ containing H there is some $n \in \mathbb{N}$ such that $(y_n, y'_n) \in W$. Otherwise, $\mathfrak{b}$ wins the game. The space Y is called $\mathcal{G}_2(H)$-$\mathfrak{b}$-*favorable* if $\mathfrak{b}$ has a winning strategy in $\mathcal{G}_2(H)$. Otherwise, Y is called $\mathcal{G}_2(H)$-$\mathfrak{b}$-*unfavorable*.

Hereafter, we will assume that Y is a compact space and H is a proximal subset of Y. In order to prove the main result of this paper, we need the following lemmas.

Lemma 1. *Let $A \subset C(Y)$ be such that for some $\varepsilon > 0$ there is a neighborhood W of Δ such that $|f(y) - f(y')| < \frac{1}{4}\varepsilon$ for each $f \in A$ and $(y, y') \in W$. Then for every $f \in A$ there is a relatively open, with respect to pointwise topology on A, set $B \subseteq A$ such that $f \in B$ and $\|\cdot\| - \operatorname{diam}(B) < \varepsilon$.*

Proof. For each $y \in Y$ let $W_y = \{z\colon (y, z) \in W\}$. Then each W_y is open and $|f(y) - f(z)| < \frac{1}{4}\varepsilon$ for each $f \in A$ and $z \in W_y$. Since Y is compact, there are points $y_1, \ldots, y_n \in Y$ such that $Y = \bigcup_{i=1}^{n} W_{y_i}$. Choose an element $f_0 \in A$ and define

$$B = \Big\{ f \in A\colon\ |f(y_i) - f_0(y_i)| < \frac{\varepsilon}{8},\ 1 \leqslant i \leqslant n \Big\}.$$

Then for each $f, g \in B$ and $y \in Y$ there is some $1 \leqslant i \leqslant n$ such that $y \in W_{y_i}$. Therefore we have

$$\begin{aligned} |f(y) - g(y)| &\leqslant |f(y) - f(y_i)| + |f(y_i) - f_0(y_i)| + |f_0(y_i) - g(y_i)| + |g(y_i) - g(y)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}. \end{aligned}$$

It follows that $\|f - g\| < \varepsilon$. □

Lemma 2. *Let X be a topological space and $\varphi\colon X \to C_p(Y)$ be a quasi-continuous mapping. If X is α-favorable and $\mathfrak{b}$ has no winning strategy in $\mathcal{G}_1(H)$ or X is Baire and $\mathfrak{a}$ has a winning strategy in $\mathcal{G}_1(H)$, then for each $\varepsilon > 0$ and a nonempty open subset U of X there are an open neighborhood E of Δ and a nonempty open subset $O \subset U$ such that for each $f \in \varphi(O)$ and $(y, y') \in E \cap H$ we have $|f(y) - f(y')| < \varepsilon$.*

P r o o f. If the result of the lemma were not true, then there are some $\varepsilon > 0$ and an open subset U of X such that for each open subset $O \subset U$ and open neighborhood E of Δ, $|f(y) - f(y')| \geqslant \varepsilon$ for some $f \in \varphi(O)$ and $(y, y') \in E \cap H$. Let $U_1 = U$ be the first move of player β in $\mathcal{BM}(X)$ and $V_1 \subset U_1$ be the answer of α to this movement. Suppose that (W_1, D_1) is the first move of $\mathfrak{a}$ in $\mathcal{G}_1(H)$. By our assumption, there is some $f_1 \in \varphi(V_1)$ and $(y_1, y_1') \in W_1 \cap D_1 \cap H$ such that $|f_1(y_1) - f_1(y_1')| > \frac{1}{2}\varepsilon$. Let (y_1, y_1') be the answer of $\mathfrak{b}$ to (W_1, D_1). In step n, when $V_1, \ldots, V_n$ and $(W_1, D_1), \ldots, (W_n, D_n)$ are specified by α and $\mathfrak{a}$, respectively, we select some $f_n \in \varphi(V_n)$ and $(y_n, y_n') \in W_n \cap D_n \cap H$ such that $|f_n(y_n) - f_n(y_n')| > \frac{1}{2}\varepsilon$. Let $\delta_n = |f_n(y_n) - f_n(y_n')| - \frac{1}{2}\varepsilon$ and define

$$B_n = \Big\{f\colon\ |f(y_n) - f_n(y_n)| < \frac{\delta_n}{2} \text{ and } |f(y_n') - f_n(y_n')| < \frac{\delta_n}{2}\Big\}.$$

If $f \in B_n$, we have

$$\begin{aligned} (2.1)\quad |f(y_n) - f(y_n')| &\geqslant |f_n(y_n) - f_n(y_n')| - \{|f(y_n) - f_n(y_n)| + |f(y_n') - f_n(y_n')|\} \\ &> |f_n(y_n) - f_n(y_n')| - \delta_n = \frac{\varepsilon}{2}. \end{aligned}$$

Thanks to the quasi-continuity of φ, there is some nonempty subset U_{n+1} of V_n such that $\varphi(U_{n+1}) \subset B_n$. Let U_{n+1} be the answer of β to the partial play $(U_1, V_1, \ldots, U_n, V_n)$ and (y_n, y_n') be the response of $\mathfrak{b}$ to $((W_1, D_1), \ldots, (W_n, D_n))$. In this way by induction on n, a strategy for β in $\mathcal{BM}(X)$ and a strategy for $\mathfrak{b}$ in $\mathcal{G}_1(H)$ is defined. Under either every assumption of the lemma, there are related games $\{(W_n, D_n), (y_n, y_n')\}$ and $\{(U_n, V_n)\}$ which are won by $\mathfrak{a}$ and α, respectively. Let $z \in \bigcap_{n\geqslant 1} U_n$ and $f = \varphi(z)$. Define

$$W = \Big\{(y, y')\colon\ |f(y) - f(y')| < \frac{\varepsilon}{3}\Big\}.$$

Then W is a neighborhood of Δ, hence there is some $n \in \mathbb{N}$ such that $(y_n, y_n') \in W$. However, $f \in \varphi(U_{n+1}) \subset B_n$, hence by (2.1), $|f(y_n) - f(y_n')| > \frac{1}{2}\varepsilon$. This contradiction proves the lemma. □

Lemma 3. *Let X and φ satisfy the assumptions of Lemma 2 and let Y be $\mathfrak{b}$-unfavorable for play $\mathcal{G}_2(H)$. Then for every nonempty open subset U of X and every $\varepsilon > 0$ there is a nonempty open subset O of U and an open neighborhood W of Δ such that $|f(y) - f(y')| < \varepsilon$ for each $f \in \varphi(O)$ and $(y, y') \in W$.*

P r o o f. Suppose that the lemma is not true. Then there is some $\varepsilon > 0$ and a nonempty open subset U of X such that for every nonempty open subset O' of U and every open neighborhood E of Δ there are $f \in \varphi(O')$ and $(y, y') \in E$ such that $|f(y) - f(y')| \geqslant \varepsilon$. By Lemma 2, there is a nonempty open subset O' of U and an open neighborhood E of Δ such that $|f(y) - f(y')| < \frac{1}{2}\varepsilon$ for each $(y, y') \in E \cap H$ and $f \in \varphi(O')$. Let $U_1 = O'$ be the first choice of β in $\mathcal{BM}(X)$ and $V_1 \subset U_1$ be the response of α to U_1. Let E' be an open neighborhood of Δ such that $\overline{E'} \subset E$. Let (W_1, D_1) be the first choice of $\mathfrak{a}$ in the play $\mathcal{G}_2(H)$. Then there is some $f \in \varphi(V_1)$ such that $|f(y_1) - f(y_1')| > \frac{1}{2}\varepsilon$ for some $(y_1, y_1') \in W_1 \cap E'$. Since $D_1 \cap E'$ is dense in $W_1 \cap E'$, we can assume that $(y_1, y_1') \in W_1 \cap E' \cap D_1$. Let (y_1, y_1') be the answer of $\mathfrak{b}$ to (W_1, D_1).

Let the partial plays $(U_1, \ldots, U_n, V_n)$ in $\mathcal{BM}(X)$ and $((W_1, D_1), \ldots, (W_n, D_n))$ in $\mathcal{G}_2(H)$ for some $n \in \mathbb{N}$ be specified. Then by our assumption, there is some $f_n \in \varphi(V_n)$ and $(y_n, y_n') \in W_n \cap E' \cap D_n$ such that $|f_n(y_n) - f_n(y_n')| > \frac{1}{2}\varepsilon$. Let (y_n, y_n') be the answer of $\mathfrak{b}$ to $(W_1, D_1), \ldots, (W_n, D_n)$. Define $\delta_n = |f_n(y_n) - f_n(y_n')| - \frac{1}{2}\varepsilon$ and

$$B_n = \Big\{f\colon\ |f(y_n) - f_n(y_n)| < \frac{\delta_n}{2} \text{ and } |f(y_n') - f_n(y_n')| < \frac{\delta_n}{2}\Big\}.$$

Then B_n is a pointwise open subset of $C(Y)$ which contains $f_n \in \varphi(V_n)$. Thanks to quasi-continuity of φ, there is an open subset $U_{n+1} \subset V_n$ such that $\varphi(U_{n+1}) \subset B_n$. Let U_{n+1} be the next move of player β. By (2.1), $|f(y_n) - f(y_n')| > \frac{1}{2}\varepsilon$ for each $f \in \varphi(U_{n+1})$. In this way, by induction on n a strategy for β in $\mathcal{BM}(X)$ and a strategy for $\mathfrak{b}$ in $\mathcal{G}_2(H)$ are determined. Since $\mathfrak{b}$ does not have a winning strategy, there is a play $\{(W_n, D_n), (y_n, y_n')\}_{n\geqslant 1}$ which is won by $\mathfrak{a}$. Let $\{(U_n, V_n)\}_{n\geqslant 1}$ be its corresponding $\mathcal{BM}(X)$ game. Then $\bigcap\limits_{n\geqslant 1} U_n \neq \emptyset$. Let $f = \varphi(z) \in \varphi\Big(\bigcap\limits_{n\geqslant 1} U_n\Big)$ and define

$$W = \Big\{(y, y')\colon\ |f(y) - f(y')| < \frac{\varepsilon}{3}\Big\} \cup (Y \times Y \setminus \overline{E'}).$$

Then W is a neighborhood of Δ which contains H. Therefore, there is some n such that $(y_n, y_n') \in W$. Since $(y_n, y_n') \in E'$, it follows that $|f(y_n) - f(y_n')| < \frac{1}{3}\varepsilon$. However, $f \in \varphi(U_n) \subset B_n$. This contradiction proves the lemma. $\square$

Now, we are ready to state the main result of this section.

Theorem 4. *Let X be a topological space and $\varphi\colon X \to C_p(Y)$ be a quasi-continuous mapping. Suppose that X is α-favorable and $\mathfrak{b}$ has no winning strategy in $\mathcal{G}_1(H)$ or X is Baire and $\mathfrak{a}$ has a winning strategy in $\mathcal{G}_1(H)$. If Y is $\mathfrak{b}$-unfavorable for play $\mathcal{G}_2(H)$, there is a dense G_δ subset D of X such that φ is norm continuous on D.*

Proof. Let $\varphi\colon X \to C_p(Y)$ be a quasi-continuous mapping. Define

$$G_n = \bigcup \Big\{O\colon\ O \text{ is open in } X \text{ and norm-diam}(\varphi(O)) < \frac{1}{n}\Big\}.$$

Then each G_n is open in X. Let U be an arbitrary nonempty open subset of X. By Lemma 3, there is a nonempty open subset O of U and an open neighborhood W of Δ such that $|f(y) - f(y')| < \frac{1}{5}n^{-1}$ for each $f \in \varphi(O)$ and $(y, y') \in W$. In view of Lemma 1, there is a pointwise open set $B \subset C_p(Y)$ such that $B \cap \varphi(O) \neq \emptyset$ and norm-diam$(B \cap \varphi(O)) < n^{-1}$. Since φ is quasi-continuous, the set $\varphi^{-1}(B) \cap O$ is semi-open and nonempty, and consequently, it contains a nonempty open set V. Thus $V \subset G_n \cap U$, hence G_n is dense in X. Clearly φ is norm continuous on $D = \bigcap\limits_{n\geqslant 1} G_n$. □

Let Γ be a set and

$$\sigma(\Gamma) = \{x \in [0,1]^\Gamma\colon\ \{\gamma \in \Gamma\colon\ x(\gamma) \neq 0 \text{ is countable}\}\}.$$

A compact space Y is called *Corson compact* if it can be embedded in some $\sigma(\Gamma)$. The space Y is called *Valdivia compact* if it can be embedded in some subset K of $[0,1]^\Gamma$ such that $K \cap \sigma(\Gamma)$ is dense in K. It follows from the definition that every Corson compact space is Valdivia compact but the converse is not true in general (see [8]). Debs [6] proved that if X is a Baire space and Y is a Corson compact, then every continuous mapping $\varphi\colon X \to C_p(Y)$ is norm continuous at any point of a dense G_δ subset of X. Bouziad [2] improved this result by showing that Y can be any $\mathfrak{a}$-favorable space for the games $\mathcal{G}_1(H)$ and $\mathcal{G}_2(H)$, where H is a proximal subset of $Y \times Y$. So the above result holds when Y is Valdivia compact (see [1]).

Kendeov et al. [11], Corollaries 5 and 8, have shown that this result remains true if X is α-favorable, Y is Valdivia compact and φ is quasi-continuous. Theorem 4 enables us to give a simultaneous generalization of these results.

Corollary 5. *Let X be a Baire space and Y be a Valdivia compact space. Then every quasi-continuous mapping $\varphi\colon X \to C_p(Y)$ is norm continuous at any point of a dense G_δ subset of X.*

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