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OSCILLATION THEOREMS FOR THIRD ORDER NONLINEAR DELAY DIFFERENCE EQUATIONS

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Abstract. Sufficient conditions are obtained for the third order nonlinear delay difference equation of the form

$$\Delta(a_n(\Delta(b_n(\Delta y_n)^{\alpha}))) + q_n f(y_{\sigma(n)}) = 0$$

to have property (A) or to be oscillatory. These conditions improve and complement many known results reported in the literature. Examples are provided to illustrate the importance of the main results.

Keywords: third order delay difference equation; property (A); comparison theorem

MSC 2010: 39A10

1. Introduction

This paper deals with the nonlinear third order delay difference equation of the form

$$\Delta(a_n(\Delta(b_n(\Delta y_n)^{\alpha}))) + q_n f(y_{\sigma(n)}) = 0, \quad n \geqslant n_0 \tag{1.1}$$

subject to the following conditions:

(H_1) $n_0 \in \mathbb{N} = \{0, 1, 2, \dots\}$, and α is a quotient of odd integers;

(H_2) $\{a_n\}, \{b_n\}$ and $\{q_n\}$ are positive real sequences for all $n \geqslant n_0$;

(H_3) $\{\sigma(n)\}$ is an increasing sequence of integers with $\sigma(n) \leqslant n$, and $\lim\limits_{n\to\infty} \sigma(n) = \infty$;

(H_4) f is a real-valued nondecreasing function with $uf(u) > 0$ for $u \neq 0$, and $f(uv) \geqslant f(u)f(v)$ for $uv > 0$;

(H_5) $\sum\limits_{n=n_0}^{\infty} {a_n}^{-1} = \infty$, $\sum\limits_{n=n_0}^{\infty} b_n^{-1/\alpha} = \infty$.

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By a solution of equation (1.1), we mean a nontrivial real sequence $\{y_n\}$ that is defined for all $n \geqslant n_0 - \sigma(n_0)$ and satisfies equation (1.1) for all $n \geqslant n_0$. A solution $\{y_n\}$ of equation (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and oscillatory otherwise. An equation is called oscillatory if all its solutions are oscillatory. By property (A) of equation (1.1) it is meant that every positive solution $\{y_n\}$ of equation (1.1) is decreasing, that is,

$$\Delta y_n < 0, \quad \Delta(b_n(\Delta y_n)^\alpha) > 0, \quad \Delta(a_n\Delta(b_n(\Delta y_n)^\alpha)) < 0.$$

The investigation of oscillatory properties of third and higher order difference equations received considerable attention in the recent years. This is because such equations often arise in the study of problems in economics, mathematical biology and many other areas of mathematics where discrete models are used, see for example [1], [2], [6].

In the recent papers [3]–[5], [7]–[18], the authors presented criteria for the oscillatory and asymptotic behavior of solutions of third and higher order delay difference equations. Following this trend, in this paper we derive new monotone properties of nonoscillatory solutions of equation (1.1) that permit us to obtain new sufficient conditions for equation (1.1) to have property (A) or to be oscillatory. Our results essentially improve many known results established for delay and ordinary difference equations.

The paper is organized as follows: Section 2 provides some preliminary results that will be used in the main results. In Section 3, we obtain sufficient conditions for equation (1.1) to have property (A), and in Section 4 we present criteria for oscillation of equation (1.1). Finally in Section 5, we provide some examples to illustrate the importance of the main results.

2. Preliminary results

We introduce the following classes of nonoscillatory (let us say positive) solutions of equation (1.1):

$$y_n \in S_0 \Leftrightarrow \Delta y_n < 0, \quad \Delta(b_n(\Delta y_n)^\alpha) > 0, \quad \Delta(a_n\Delta(b_n(\Delta y_n)^\alpha)) < 0$$

or

$$y_n \in S_2 \Leftrightarrow \Delta y_n > 0, \quad \Delta(b_n(\Delta y_n)^\alpha) > 0, \quad \Delta(a_n\Delta(b_n(\Delta y_n)^\alpha)) < 0,$$

eventually for all $n \geqslant N \geqslant n_0$.

We begin with the following lemma.

Lemma 2.1. *Assume that $\{y_n\}$ is an eventually positive solution of equation* (1.1). *Then $\{y_n\}$ satisfies either $y_n \in S_0$ or $y_n \in S_2$ eventually for all $n \geqslant N \geqslant n_0$.*

P r o o f. The proof can be found in [12], and so it can be omitted. □

Next, we derive some important properties of nonoscillatory solutions of equation (1.1) that will be applied in our main results.

Define

$$A_n = \sum_{s=N}^{n-1} \frac{1}{a_s}, \quad B_n = \sum_{s=N}^{n-1} \frac{1}{b_s^{1/\alpha}}, \quad C_n = \sum_{s=N}^{n-1} \frac{1}{b_s^{1/\alpha}} \Big(\sum_{t=N}^{s-1} \frac{1}{a_t} \Big)^{1/\alpha},$$

where $N \geqslant n_0$ is large enough.

Lemma 2.2. *Let $\{y_n\}$ be a positive solution of equation* (1.1) *which belongs to S_2, and*

$$\sum_{n=N}^{\infty} \frac{1}{a_n} \sum_{s=n}^{\infty} q_s f(B_{\sigma(s)}) = \infty. \tag{2.1}$$

Then

(i) $\{y_n B_n^{-1}\}$ *is increasing for all* $n \geqslant N$,
(ii) $\{y_n C_n^{-1}\}$ *is decreasing for all* $n \geqslant N$,
(iii) $\{b_n^{1/\alpha} \Delta y_n A_n^{-1/\alpha}\}$ *is decreasing for all* $n \geqslant N$.

P r o o f. Assume that $\{y_n\}$ is a positive solution of equation (1.1) satisfying $\{y_n\} \in S_2$ for all $n \geqslant N$. Since $a_n \Delta(b_n(\Delta y_n)^\alpha)$ is decreasing, we have

$$b_n(\Delta y_n)^\alpha \geqslant \sum_{s=N}^{n-1} a_s \Delta(b_s(\Delta y_s)^\alpha) \frac{1}{a_s} \geqslant a_n \Delta(b_n(\Delta y_n)^\alpha) A_n. \tag{2.2}$$

This implies

$$\Delta\Big(\frac{b_n(\Delta y_n)^\alpha}{A_n}\Big) = \frac{A_n \Delta(b_n(\Delta y_n)^\alpha) - b_n(\Delta y_n)^\alpha a_n^{-1}}{A_n A_{n+1}} \leqslant 0.$$

Thus $\{b_n^{1/\alpha}(\Delta y_n) A_n^{-1/\alpha}\}$ is decreasing and further, this fact yields

$$y_n \geqslant \sum_{s=N}^{n-1} \frac{A_s^{1/\alpha} b_s^{1/\alpha} \Delta y_s}{A_s^{1/\alpha} b_s^{1/\alpha}} \geqslant \frac{b_n^{1/\alpha} \Delta y_n}{A_n^{1/\alpha}} \sum_{s=N}^{n-1} \frac{1}{b_s^{1/\alpha}} \Big(\sum_{t=N}^{s-1} \frac{1}{a_t} \Big)^{1/\alpha}. \tag{2.3}$$

Hence

$$\Delta\Big(\frac{y_n}{C_n}\Big) = \frac{C_n \Delta y_n - y_n A_n^{1/\alpha} b_n^{-1/\alpha}}{C_n C_{n+1}} \leqslant 0,$$

which implies that $\{y_n C_n^{-1}\}$ is decreasing.

Next, since $\{b_n^{1/\alpha}\Delta y_n\}$ is increasing for all $n \geqslant N$, it is easy to see that for all $n \geqslant N_1 \geqslant N$

$$\begin{aligned} y_n &= y_{N_1} + \sum_{s=N_1}^{n-1} \frac{b_s^{1/\alpha}\Delta y_s}{b_s^{1/\alpha}} \leqslant y_{N_1} + b_n^{1/\alpha}\Delta y_n \sum_{s=N_1}^{n-1} \frac{1}{b_s^{1/\alpha}} \\ &= y_{N_1} - b_n^{1/\alpha}\Delta y_n \sum_{s=N_1}^{N-1} \frac{1}{b_s^{1/\alpha}} + b_n^{1/\alpha}\Delta y_n \sum_{s=N}^{n-1} \frac{1}{b_s^{1/\alpha}}. \end{aligned} \tag{2.4}$$

It follows from condition (2.1) that $b_n^{1/\alpha}\Delta y_n \to \infty$ as $n \to \infty$. If not, then $b_n^{1/\alpha}\Delta y_n \to 2d < \infty$ as $n \to \infty$. Summing equation (1.1) from n to ∞, we get

$$\Delta(b_n(\Delta y_n)^\alpha) \geqslant \frac{1}{a_n}\sum_{s=n}^{\infty} q_s f(y_{\sigma(s)}).$$

On the other hand, $b_n^{1/\alpha}\Delta y_n \to 2d$ as $n \to \infty$, we have $b_n^{1/\alpha}\Delta y_n > d$ for n large enough. This implies $y_n \geqslant dB_n$. Combining the last two inequalities and summing once more, we obtain

$$2d \geqslant f(d)\sum_{s=N_2}^{n-1}\frac{1}{a_s}\sum_{t=s}^{\infty} q_s f(B_{\sigma(s)}).$$

A contradiction with (2.1) and we conclude that $b_n^{1/\alpha}\Delta y_n \to \infty$ as $n \to \infty$. Therefore, for any $n \geqslant N_2 > N_1$, we have from (2.4) that

$$y_n \leqslant b_n^{1/\alpha}\Delta y_n B_n.$$

Now, one can see that

$$\Delta\Big(\frac{y_n}{B_n}\Big) = \frac{B_n \Delta y_n - y_n b_n^{-1/\alpha}}{B_n B_{n+1}} \geqslant 0$$

eventually, and we conclude that $\{y_n B_n^{-1}\}$ is increasing. The proof is now complete. $\square$

3. Criteria for property (A)

In this section, we present several criteria for the class S_2 of equation (1.1) to be empty. In the literature such case is referred to as property (A) of equation (1.1).

Theorem 3.1. *Let condition* (2.1) *hold, and*

$$\lim_{u\to\pm\infty}\frac{u^\alpha}{f(u)}=K_1<\infty. \tag{3.1}$$

If

$$\begin{aligned}\limsup_{n\to\infty}\bigg\{\frac{C^\alpha_{\sigma(n)}}{A_{\sigma(n)}}f\Big(\frac{1}{C_{\sigma(n)}}\Big)&\sum_{s=N}^{\sigma(n)-1}q_sf(C_{\sigma(s)})A_{s+1}\\&+C^\alpha_{\sigma(n)}f\Big(\frac{1}{C_{\sigma(n)}}\Big)\sum_{s=\sigma(n)}^{n-1}q_sf(C_{\sigma(s)})\\&+C^\alpha_{\sigma(n)}f\Big(\frac{1}{B_{\sigma(n)}}\Big)\sum_{s=n}^{\infty}q_sf(B_{\sigma(s)})\bigg\}>K_1,\end{aligned} \tag{3.2}$$

then class S_2 is empty for equation (1.1).

P r o o f. Assume that equation (1.1) possesses an eventually positive solution $\{y_n\}$ belonging to S_2 for all $n\geqslant N$. Summation of equation (1.1) from n to ∞ yields

$$\Delta(b_n(\Delta y_n)^\alpha)\geqslant\frac{1}{a_n}\sum_{s=n}^{\infty}q_sf(y_{\sigma(s)}).$$

Summing the last inequality from N to $n-1$, we obtain

$$\begin{aligned}b_n(\Delta y_n)^\alpha&\geqslant\sum_{s=N}^{n-1}\frac{1}{a_s}\sum_{t=s}^{\infty}q_tf(y_{\sigma(t)})\\&=\sum_{s=N}^{n-1}\frac{1}{a_s}\sum_{t=s}^{n-1}q_tf(y_{\sigma(t)})+\sum_{s=N}^{n-1}\frac{1}{a_s}\sum_{t=n}^{\infty}q_tf(y_{\sigma(t)})\\&=\sum_{s=N}^{n-1}A_{s+1}q_sf(y_{\sigma(t)})+A_n\sum_{s=n}^{\infty}q_sf(y_{\sigma(s)}).\end{aligned} \tag{3.3}$$

Using (2.3), we have

$$\frac{A_ny_n^\alpha}{C_n^\alpha}\geqslant\sum_{s=N}^{n-1}A_{s+1}q_sf(y_{\sigma(s)})+A_n\sum_{s=n}^{\infty}q_sf(y_{\sigma(s)}),$$

or

$$\frac{A_{\sigma(n)}y^{\alpha}_{\sigma(n)}}{C^{\alpha}_{n}} \geqslant \sum_{s=N}^{\sigma(n)-1} A_{s+1}q_s f(y_{\sigma(s)}) + A_{\sigma(n)} \sum_{s=\sigma(n)}^{n-1} q_s f(y_{\sigma(s)}) + A_{\sigma(n)} \sum_{s=n}^{\infty} q_s f(y_{\sigma(s)}).$$

From the monotonicity properties (i)–(iii) of Lemma 2.2 and taking (H_4) into account, we see that

$$\begin{aligned}\frac{A_{\sigma(n)}y^{\alpha}_{\sigma(n)}}{C^{\alpha}_{n}} &\geqslant f\Big(\frac{y_{\sigma(n)}}{C_{\sigma(n)}}\Big) \sum_{s=N}^{\sigma(n)-1} A_{s+1}q_s f(C_{\sigma(s)}) \\ &\quad + A_{\sigma(n)} f\Big(\frac{y_{\sigma(n)}}{C_{\sigma(n)}}\Big) \sum_{s=\sigma(n)}^{n-1} q_s f(C_{\sigma(s)}) \\ &\quad + A_{\sigma(n)} f\Big(\frac{y_{\sigma(n)}}{B_{\sigma(n)}}\Big) \sum_{s=N}^{\infty} q_s f(B_{\sigma(s)}),\end{aligned} \tag{3.4}$$

or

$$\begin{aligned}\frac{y^{\alpha}_{\sigma(n)}}{f(y_{\sigma(n)})} &\geqslant \frac{C^{\alpha}_{\sigma(n)}}{A_{\sigma(n)}} f\Big(\frac{1}{C_{\sigma(n)}}\Big) \sum_{s=N}^{\sigma(n)-1} A_{s+1}q_s f(C_{\sigma(s)}) \\ &\quad + C^{\alpha}_{\sigma(n)} f\Big(\frac{1}{C_{\sigma(n)}}\Big) \sum_{s=\sigma(n)}^{n-1} q_s f(C_{\sigma(s)}) \\ &\quad + C^{\alpha}_{\sigma(n)} f\Big(\frac{1}{B_{\sigma(n)}}\Big) \sum_{s=N}^{\infty} q_s f(B_{\sigma(s)}).\end{aligned}$$

Taking lim sup as $n \to \infty$ on both sides of the last inequality, we are led to contradiction with (3.2). This completes the proof. □

The above theorem is suitable to apply for the half-superlinear and half-linear case of equation (1.1). Indeed, we may formulate the following results:

Corollary 3.2. *Let condition* (2.1) *hold, and*

$$\limsup_{n\to\infty}\Big\{ \frac{1}{A_{\sigma(n)}C^{\beta-\alpha}_{\sigma(n)}} \sum_{s=N}^{\sigma(n)-1} q_s C^{\beta}_{\sigma(s)} A_{s+1} + \frac{1}{C^{\beta-\alpha}_{\sigma(n)}} \sum_{s=\sigma(n)}^{n-1} q_s C^{\beta}_{\sigma(s)} + \frac{C^{\alpha}_{\sigma(n)}}{B^{\beta}_{\sigma(n)}} \sum_{s=n}^{\infty} q_s B^{\beta}_{\sigma(s)} \Big\} > 0.$$

Then the class S_2 is empty for the equation

$$\Delta(a_n(\Delta(b_n(\Delta y_n)^{\alpha}))) + q_n y^{\beta}_{\sigma(n)} = 0, \quad \beta > \alpha. \tag{3.5}$$

Corollary 3.3. *Let condition* (2.1) *hold, and*

$$\limsup_{n\to\infty}\Big\{\frac{1}{A_{\sigma(n)}}\sum_{s=N}^{\sigma(n)-1} q_s C^{\alpha}_{\sigma(s)} A_{s+1} + \sum_{s=\sigma(n)}^{n-1} q_s C^{\alpha}_{\sigma(s)} + \frac{C^{\alpha}_{\sigma(n)}}{B^{\beta}_{\sigma(n)}}\sum_{s=n}^{\infty} q_s B^{\alpha}_{\sigma(s)}\Big\} > 1. \tag{3.6}$$

Then the class S_2 is empty for the equation

$$\Delta(a_n(\Delta(b_n(\Delta y_n)^{\alpha}))) + q_n y^{\alpha}_{\sigma(n)} = 0. \tag{3.7}$$

In the following, we present another criterion for property (A) of equation (1.1) that will be used for half-sublinear case of equation (1.1).

Theorem 3.4. *Let condition* (2.1) *hold, and*

$$\sum_{n=N_1}^{\infty} q_n f(C_{\sigma(n)}) = \infty. \tag{3.8}$$

Assume that

$$\lim_{u\to 0}\frac{u^{\alpha}}{f(u)} = K_2 < \infty. \tag{3.9}$$

If

$$\limsup_{n\to\infty}\Big\{\frac{1}{A_{\sigma(n)}}\sum_{s=N}^{\sigma(n)-1} q_s f(C_{\sigma(s)}) A_{s+1} + \sum_{s=\sigma(n)}^{n-1} q_s f(C_{\sigma(s)}) + f\Big(\frac{C_{\sigma(n)}}{B_{\sigma(n)}}\Big)\sum_{s=n}^{\infty} q_s f(B_{\sigma(s)})\Big\} > K_2,$$

then the class S_2 is empty for equation (1.1).

P r o o f. Assume that equation (1.1) possesses an eventually positive solution $\{y_n\} \in S_2$ for $n \geqslant N$. First, we shall show that (3.8) implies

$$\lim_{n\to\infty} \frac{y_n}{C_n} = 0.$$

Assume the contrary, that is, $\lim\limits_{n\to\infty} y_n/C_n = d > 0$. Then the discrete L'Hospital rule implies that

$$d = \lim_{n\to\infty} \frac{y_n}{C_n} = \Big(\lim_{n\to\infty} a_n\Delta(b_n(\Delta y_n)^\alpha)\Big)^{1/\alpha}.$$

On the other hand, summation of equation (1.1) from N to ∞ yields

$$K = a_s\Delta(b_s(\Delta y_s)^\alpha)|_{s=N} \geqslant \sum_{n=N}^{\infty} q_n f(y_{\sigma(n)}). \tag{3.10}$$

Combining (2.2) and (2.3), we obtain

$$y_n \geqslant C_n(a_n\Delta(b_n(\Delta y_n)^\alpha))^{1/\alpha} \geqslant dC_n.$$

Which, in view of (3.10), gives

$$K \geqslant f(d)\sum_{n=N}^{\infty} q_n f(C_{\sigma(n)}).$$

This contradicts (3.8), and we conclude that $y_n/C_n \to 0$ as $n\to\infty$. Set

$$z_n = \frac{y_{\sigma(n)}}{C_{\sigma(n)}}.$$

Then condition (3.4) together with (H_4) implies

$$\frac{z_n^\alpha}{f(z_n)} \geqslant \frac{1}{A_{\sigma(n)}}\sum_{s=N}^{\sigma(n)-1} q_s f(C_{\sigma(s)})A_{s+1} + \sum_{s=\sigma(n)}^{n-1} q_s f(C_{\sigma(s)}) + f\Big(\frac{C_{\sigma(n)}}{B_{\sigma(n)}}\Big)\sum_{s=n}^{\infty} q_s f(B_{\sigma(s)}).$$

Taking lim sup as $n\to\infty$ on both sides of the last inequality, we are led to contradiction with the assumption of the theorem. This completes the proof. $\square$

For $f(u) = u^\alpha$, Theorem 3.4 reduces to Corollary 3.3, while for the half-sublinear case we obtain the following corollary.

Corollary 3.5. *Let conditions* (2.1) *and* (3.8) *hold. If*

$$\limsup_{n\to\infty}\Big\{\frac{1}{A_{\sigma(n)}}\sum_{s=N}^{\sigma(n)-1} q_s C_{\sigma(s)}^\beta A_{s+1} + \sum_{s=\sigma(n)}^{n-1} q_s C_{\sigma(s)}^\beta + \frac{C_{\sigma(n)}^\alpha}{B_{\sigma(n)}}\sum_{s=n}^{\infty} q_s B_{\sigma(s)}^\beta\Big\} > 0,$$

then the class S_2 is empty for the equation

$$\Delta(a_n(\Delta(b_n(\Delta y_n)^\alpha))) + q_n y_{\sigma(n)}^\beta = 0, \quad \alpha > \beta. \tag{3.11}$$

For the difference equations with $\sigma(n) = n+1$, Corollaries 3.2, 3.3 and 3.5 yield the following results.

Corollary 3.6. *Let $\sigma(n) \equiv n+1$. Assume that condition* (2.1) *holds, and*

$$\limsup_{n\to\infty}\left\{\frac{1}{A_{n+1}C_{n+1}^{\beta-\alpha}}\sum_{s=N}^{n} q_s C_{s+1}^{\beta} A_{s+1} + \frac{C_{n+1}^{\alpha}}{B_{n+1}^{\beta}}\sum_{s=n}^{\infty} q_s B_{s+1}^{\beta}\right\} > 0.$$

Then equation (3.5) *has property* (A).

Corollary 3.7. *Let $\sigma(n) \equiv n+1$. Assume that condition* (2.1) *holds, and*

$$\limsup_{n\to\infty}\left\{\frac{1}{A_{n+1}}\sum_{s=N}^{n} q_s C_{s+1}^{\beta} A_{s+1} + \frac{C_{n+1}^{\alpha}}{B_{n+1}^{\alpha}}\sum_{s=n}^{\infty} q_s B_{s+1}^{\beta}\right\} > 1.$$

Then equation (3.7) *has property* (A).

Corollary 3.8. *Let $\sigma(n) \equiv n+1$. Assume that conditions* (2.1) *and* (3.8) *hold. If*

$$\limsup_{n\to\infty}\left\{\frac{1}{A_{n+1}}\sum_{s=N}^{n} q_s C_{s+1}^{\beta} A_{s+1} + \frac{C_{n+1}^{\beta}}{B_{n+1}^{\beta}}\sum_{s=n}^{\infty} q_s B_{s+1}^{\beta}\right\} > 0,$$

then equation (3.11) *has property* (A).

Remark 3.9. Corollaries 3.6–3.8 essentially improve and extend the results in [15], [16], [18] for the equation $\Delta^3 y_n + q_n y_{n+1} = 0$.

4. Oscillation results

In this section, we present oscillation criteria for equation (1.1). To achieve this, we need to eliminate the class S_0 also.

Theorem 4.1. *Assume that*

$$\lim_{u\to 0}\frac{u}{f^{1/\alpha}(u)} = K_3 < \infty. \tag{4.1}$$

If

$$\limsup_{n\to\infty}\sum_{s=\sigma(n)}^{n-1}\frac{1}{b_s^{1/\alpha}}\left(\sum_{t=s}^{n-1}\frac{1}{a_t}\sum_{j=t}^{n-1} q_j\right)^{1/\alpha} > K_3, \tag{4.2}$$

then the class S_0 is empty for equation (1.1).

P r o o f. Assume that equation (1.1) possesses an eventually positive solution $\{y_n\} \in S_0$ for $n \geqslant N_1 \geqslant N$. First, note that condition (4.2) implies

$$\sum_{n=N_1}^{\infty} \frac{1}{b_s^{1/\alpha}} \left(\sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} q_t \right)^{1/\alpha} = \infty,$$

which guarantees that $y_n \to 0$ as $n \to \infty$.

On the other hand, summation of equation (1.1) from s to $n-1$ gives

$$\Delta(b_s(\Delta y_s)^{\alpha}) \geqslant \frac{1}{a_s} \sum_{t=s}^{n-1} q_t f(y_{\sigma(t)}) \geqslant \frac{1}{a_s} f(y_{\sigma(n)}) \sum_{t=s}^{n-1} q_s.$$

Summing in s, we obtain

$$-\Delta y_s \geqslant \frac{f^{1/\alpha}(y_{\sigma(n)})}{b_s^{1/\alpha}} \left(\sum_{t=s}^{n-1} \frac{1}{a_t} \sum_{j=t}^{n-1} q_j \right)^{1/\alpha}.$$

Summing once more, we obtain

$$y_s \geqslant f^{1/\alpha}(y_{\sigma(n)}) \sum_{t=s}^{n-1} \frac{1}{b_t^{1/\alpha}} \left(\sum_{j=t}^{n-1} \frac{1}{a_j} \sum_{i=j}^{n-1} q_i \right)^{1/\alpha}.$$

Setting $s = \sigma(n)$, we have

$$\frac{y_{\sigma(n)}}{f^{1/\alpha}(y_{\sigma(n)})} \geqslant \sum_{s=\sigma(n)}^{n-1} \frac{1}{b_s^{1/\alpha}} \left(\sum_{t=s}^{n-1} \frac{1}{a_t} \sum_{j=t}^{n-1} q_j \right)^{1/\alpha}.$$

Taking lim sup as $n \to \infty$ on both sides of the last inequality, we are led to contradiction with (4.2). This completes the proof. □

For special cases of equation (1.1), Theorem 4.1 reduces to the following criteria.

Corollary 4.2. *If*

$$\limsup_{n\to\infty} \left\{ \sum_{s=\sigma(n)}^{n-1} \frac{1}{b_s^{1/\alpha}} \left(\sum_{t=s}^{n-1} \frac{1}{a_t} \sum_{j=t}^{n-1} q_j \right)^{1/\alpha} \right\} > 1, \tag{4.3}$$

then the class S_0 *is empty for equation* (3.7).

Corollary 4.3. *If*

$$\limsup_{n\to\infty}\Big\{\sum_{s=\sigma(n)}^{n-1}\frac{1}{b_s^{1/\alpha}}\Big(\sum_{t=s}^{n-1}\frac{1}{a_t}\sum_{j=t}^{n-1}q_j\Big)^{1/\alpha}\Big\}>0, \tag{4.4}$$

then the class S_0 *is empty for equation* (3.11).

Combining the criteria obtained for both classes S_0 and S_2 to be empty, we obtain results for the oscillation of all solutions of equation (1.1).

Theorem 4.4. *Let all conditions of Theorem* 3.1 (*Theorem* 3.4) *and Theorem* 4.1 *hold. Then every solution of equation* (1.1) *is oscillatory.*

Corollary 4.5. *Let all conditions of Corollary* 3.3 *and Corollary* 4.2 *hold. Then every solution of equation* (3.7) *is oscillatory.*

Corollary 4.6. *Let all conditions of Corollary* 3.5 *and Corollary* 4.3 *hold. Then every solution of equation* (3.11) *is oscillatory.*

5. Examples

In this section, we present two examples to illustrate the importance of the main results.

Example 5.1. Consider the third order delay difference equation

$$\Delta(n^{1/3}\Delta(n^{1/4}(\Delta y_n)^{1/3}))+\frac{8}{n^{7/4}}y_{n-3}^{1/3}=0,\quad n\geqslant 1. \tag{5.1}$$

Here $a_n=n^{1/3}$, $b_n=n^{1/4}$, $q_n=8n^{-7/4}$, $\sigma(n)=n-3$ and $\alpha=\beta=\frac{1}{3}$. Simple computation shows that

$$A_n\sim\tfrac{3}{2}n^{2/3},\quad B_n\sim 4n^{1/4}\quad\text{and}\quad C_n\sim\tfrac{3}{2}n^{9/4}.$$

Then it is easy to see that conditions (3.6) and (4.3) are satisfied. Therefore by Corollary 4.5 every solution of equation (5.1) is oscillatory.

Example 5.2. Consider the third order delay difference equation

$$\Delta\Big(\frac{1}{n}\Delta\Big(\frac{1}{n^3}(\Delta y_n)^3\Big)\Big)+2^n y_{n-3}=0,\quad n\geqslant 1. \tag{5.2}$$

Here $a_n = n^{-1}$, $b_n = n^{-3}$, $q_n = 2^n$, $\sigma(n) = n - 3$ and $\alpha = 3$, $\beta = 1$. Simple computation shows that

$$A_n \sim \tfrac{1}{2}n^2, \quad B_n \sim \tfrac{1}{2}n^2 \quad \text{and} \quad C_n \sim \tfrac{3}{8}n^{8/3}.$$

It is easy to see that all conditions of Corollary 3.5 and Corollary 4.3 are satisfied. Therefore by Corollary 4.6 every solution of equation (5.2) is oscillatory.

6. Conclusion

In this paper, we derived new monotonic properties of the nonoscillatory solutions and using these results some new sufficient conditions were presented for the studied equation to have the so called property (A) or to be oscillatory. Our results essentially improve and complement many known results not only for delay difference equations but for ordinary difference equations as well, see [2]–[4], [9]–[18]. Finally, we provided two examples that illustrate the significance of the main results.

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