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ON MINIMAL SPECTRUM OF MULTIPLICATION LATTICE MODULES

Sachin Ballal, Vilas Kharat, Pune

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Abstract. We study the minimal prime elements of multiplication lattice module M over a C-lattice L. Moreover, we topologize the spectrum $\pi(M)$ of minimal prime elements of M and study several properties of it. The compactness of $\pi(M)$ is characterized in several ways. Also, we investigate the interplay between the topological properties of $\pi(M)$ and algebraic properties of M.

Keywords: prime element; mimimal prime element; Zariski topology

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1. Introduction

The notion of minimal prime elements of a lattice module is a generalization of minimal prime elements of a multiplicative lattice. The prime and minimal prime elements of multiplicative lattice were introduced and studied by Thakare, Manjarekar and Maeda [12], Thakare and Manjarekar [11], and the minimal prime ideals of 0-distributive lattices by Pawar and Thakare [9]. Keimel [7] unified the study of minimal prime ideals for various structures, e.g. commutative rings, distributive lattices, lattice ordered groups, f-rings. In this paper, we have carried out investigations leading to the study of generalizations of notions in commutative rings and multiplicative lattices along the lines of Dilworth (see [6]).

A complete lattice L with the least element 0 and the greatest element 1 is said to be a *multiplicative lattice* if a binary operation "$\cdot$" called multiplication on L satisfying the following conditions is defined:

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(1) $a \cdot b = b \cdot a$ for all $a, b \in L$,

(2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in L$,

(3) $a \cdot \bigvee_{\alpha} b_{\alpha} = \bigvee_{\alpha} (a \cdot b_{\alpha})$ for all $a, b_{\alpha} \in L$,

(4) $a \cdot 1 = a$ for all $a \in L$.

Henceforth, $a \cdot b$ will be simply denoted by ab.

An element $p \neq 1$ of a multiplicative lattice L is said to be *prime* if $ab \leqslant p$ implies either $a \leqslant p$ or $b \leqslant p$. A prime element $p \in L$ is said to be a *minimal prime* over an element $a \in L$ if $a \leqslant p$ and whenever there is a prime element $q \in L$ with $a < q \leqslant p$, then $q = p$. In L, a minimal prime element over 0 will be called a *minimal prime element of* L. For $a \in L$, its radical is denoted by $\sqrt{a}$ and defined as $\sqrt{a} = \bigvee\{x \in L\colon\ x^n \leqslant a$ for some $n \in \mathbb{Z}^+\}$. An element $a \in L$ is called *semiprime* or *radical* if $\sqrt{a} = a$.

An element $a \in L$ is said to be *compact* if $a \leqslant \bigvee X$, $X \subseteq L$ implies that there exists a finite number of elements $x_1, x_2, \ldots, x_n \in X$ such that $a \leqslant \bigvee\limits_{i=1}^{n} x_i$. We denote the set of all compact elements of a multiplicative lattice L by L_*. In a multiplicative lattice L, an element $a \in L$ is said to be *nilpotent* if $a^n = 0$ for some $n \in \mathbb{Z}^+$ and is said to be *reduced* if the only nilpotent element of L is 0.

An element $e \in L$ is said to be *meet principal* or *join principal* if it satisfies the identity $a \wedge be = ((a : e) \wedge b)e$ or $(ae \vee b) : e = (b : e) \vee a$, respectively, for $a, b \in L$. Also, e is said to be *principal* if it is both join and meet principal. A multiplicative lattice L is said to be *principally generated* (PG) if every element of L is a join of principal elements of L. A multiplicative lattice L is said to be *compactly generated* (CG) if every element of L is the join of compact elements of L. According to Alarcon et al. [1], if L is a compactly generated multiplicative lattice with 1 compact, then maximal elements exist in L and every maximal element is a prime element. Further, in a compactly generated multiplicative lattice, if every finite product of compact elements is a compact element, then prime elements and minimal primes over $a \in L$ exist (see [1]).

By a *C-lattice* we mean a multiplicative lattice L with the greatest element 1, which is compact as well as multiplicative identity, that is, generated under joins by a multiplicatively closed subset C of compact elements of L.

A complete lattice M with the smallest element 0_M and the greatest element 1_M is said to be a *lattice module* over the multiplicative lattice L or *L-module* if there is a multiplication between elements of M and L, denoted by aN for $a \in L$ and $N \in M$, which satisfies the following properties:

(1) $(ab)N = a(bN)$;

(2) $\bigvee_{\alpha} a_{\alpha} \bigvee_{\beta} N_{\beta} = \bigvee_{\alpha,\beta} (a_{\alpha} N_{\beta})$;

(3) $1_L N = N$;

(4) $0_L N = 0_M$ for $a, b, a_\alpha \in L$ and for $N, N_\beta \in M$.

Let M be a lattice module over a multiplicative lattice L. For $N \in M$ and $b \in L$, denote $(N : b) = \bigvee\{X \in M\colon aX \leqslant N\}$. If $a, b \in L$, we write $(a : b) = \bigvee\{x \in L\colon bx \leqslant a\}$. If $A, B \in M$, then $(A : B) = \bigvee\{x \in L\colon xB \leqslant A\}$.

An element $A \in M$ is called *weak meet principal* if $(B : A)A = B \wedge A$ for all $B \in M$; A is called *weak join principal* if $bA : A = b \vee (0 : A)$ for all $b \in L$; and A is *weak principal* if A is both weak meet principal and weak join principal. Lattice module M over a multiplicative lattice L is called a *multiplication lattice module* if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$.

An element $N \neq 1_M$ in M is said to be *prime* if $aX \leqslant N$ implies $X \leqslant N$ or $a1_M \leqslant N$, i.e. $a \leqslant (N : 1_M)$ for every $a \in L$ and $X \in M$. An element $N \neq 1_M$ of M is called a *maximal element* if for every element B of M such that $N \leqslant B$, either $N = B$ or $B = 1_M$. Let M be an L-module. An element N in M is called *compact* if $N \leqslant \bigvee\limits_{\alpha \in I} A_\alpha$ (I is an indexed set) implies $N \leqslant A_{\alpha_1} \vee A_{\alpha_2} \vee \ldots \vee A_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I.

In this paper, a lattice module M will be a multiplication lattice module, which is compactly generated with the largest element 1_M being compact and L will be a C-lattice.

For general background and terminology of multiplicative lattice and multiplication lattice module, the reader may consult [1], [2], [4]–[6], [12], [11].

2. The Zariski topology

In [3], the Zariski topology over the prime spectrum $\operatorname{Spec}(M)$ of a lattice module M over a C-lattice L has been studied by Ballal and Kharat. In [10], Phadatare et al. introduced and studied the concept of quasi-prime elements as a generalization of prime elements and also the Zariski topology on the quasi-prime spectrum of a lattice module M over a C-lattice L.

In this paper most of the results in [12] and [11] are generalized.

Definition 2.1. Let M be a lattice module over a multiplicative lattice L. An element $P \in M$ is called a *minimal prime over an element* $N \in M$ if $N \leqslant P$ and there is no other prime element Q of M such that $N \leqslant Q < P$.

Lemma 2.2. *Let M be a multiplication lattice module over a C-lattice L and $(0_M : 1_M)$ be a radical element. Then for $x \in L$, $(0_M : x) = (0_M : x^n)$ for every integer $n \geqslant 1$.*

P r o o f. Note that $(0_M : x) = \bigvee\{N \in M\colon xN \leqslant 0_M\}$ and as $x^n \leqslant x$, we have $(0_M : x) \leqslant (0_M : x^n)$ for every integer $n \geqslant 1$. Let $N_1 = (0_M : x^n)$. Since M is a multiplication lattice module, $N_1 = a1_M$ for some $a \in L$. So $x^n a^n 1_M \leqslant x^n a 1_M = 0_M$. Hence $xa \leqslant \sqrt{(0_M : 1_M)} = (0_M : 1_M)$. So $xa1_M \leqslant 0_M$, i.e. $N_1 \leqslant (0_M : x)$ and consequently $(0_M : x) = (0_M : x^n)$ for each integer $n \geqslant 1$. $\square$

Theorem 2.3 ([8]). *Let M be a multiplication lattice module over a C-lattice L and $a \in L$ be proper. A prime element $P \in M$ with $a1_M \leqslant P$ is minimal if and only if for $x \in L_*$ with $x1_M \leqslant P$ there is an element $y \in L_*$ such that $y1_M \nleqslant P$ and $x^n y1_M \leqslant a1_M = N$ for some positive integer n.*

The following result characterizes a prime element to be a minimal prime.

Theorem 2.4. *Let M be a multiplication lattice module over a C-lattice L and $(0_M : 1_M)$ be a radical element. A prime element $P \in M$ is a minimal prime if and only if for $x \in L_*$, P contains precisely one of $x1_M$ and $(0_M : x)$.*

P r o o f. Suppose that the condition is true for prime element $P \in M$. Let $x \in L_*$ be such that $x1_M \leqslant P$ and $(0_M : x) \nleqslant P$. Then there exists $y \in L_*$ such that $y1_M \leqslant (0_M : x)$ but $y1_M \nleqslant P$. Thus, $xy1_M \leqslant 0_M$ and hence $x^n y1_M \leqslant 0_M$ for every integer $n \geqslant 1$. This shows that for each $x \in L_*$ with $x1_M \leqslant P$ there exists an element $y \in L_*$ such that $y1_M \nleqslant P$ and $x^n y1_M \leqslant 0_M$. By Theorem 2.3, it follows that P is minimal.

Conversely, suppose that a prime element $P \in M$ is minimal and also that $x1_M \leqslant P$ for $x \in L_*$. Then by Theorem 2.3, there exists $y \in L_*$ such that $y1_M \nleqslant P$ and $x^n y1_M = 0_M$ for some positive integer n. Consequently, $y1_M \leqslant (0_M : x^n)$. By Lemma 2.2, we have $(0_M : x^n) = (0_M : x)$ and hence $y1_M \leqslant (0_M : x)$. This implies that $(0_M : x) \nleqslant P$.

Now, if $x1_M \nleqslant P$ and $(0_M : x) \nleqslant P$, then there exists $y \in L_*$ such that $y1_M \leqslant (0_M : x)$ but $y1_M \nleqslant P$. Hence, we have $xy1_M \leqslant 0_M$ and so $xy1_M \leqslant P$. But $x1_M \nleqslant P$ and $y1_M \nleqslant P$ together contradicts the fact that P is a prime. This shows that P contains precisely one of $x1_M$ and $(0_M : x)$. $\square$

Let $\sigma(M)$ be the set of prime elements of a lattice module M. For an element $N \in M$ we set $V(N) = \{P \in \sigma(M)\colon N \leqslant P\}$. Taking the sets $\{V(N)\colon N \in M\}$ as a base for closed sets, $\sigma(M)$ becomes a topological space and this topology is called the *Zariski topology* (see [3]).

The restriction of the Zariski topology to the set of minimal prime elements $\pi(M)$ makes it a topological space and it is called the minimal prime spectrum of M.

The following results about a minimal prime spectrum are immediate.

Corollary 2.5. *Let M be a multiplication lattice module over a reduced C-lattice L. For $a \in L$, $V(0_M : a) = \pi(M) - V(a1_M)$. In particular, $V(a1_M)$ and $V(0_M : a)$ are disjoint open and closed sets.*

Corollary 2.6. *Let M be a multiplication lattice module over a reduced C-lattice L with 1_M being compact. Then $\pi(M)$ is a Hausdorff space with a base of open and closed sets.*

Definition 2.7 ([11]). A subset S of a multiplicative lattice L is said to be multiplicatively closed if $x, y \in S$ implies $xy \in S$, and is said to be sub-multiplicatively closed if $x, y \in X$ implies $a \leqslant xy$ for some $a \in S$.

In order to characterize prime elements of lattice modules in terms of multiplicatively closed subset of L, we need the following lemma.

Lemma 2.8 ([4]). *Let M be a multiplication lattice module over a* PG *C-lattice L and $N \in M$ with $N < 1_M$. Then the following conditions are equivalent.*

(1) *N is a prime element in M.*
(2) *$(N : 1_M)$ is a prime element in L.*
(3) *There exists a prime element p in L with $(0_M : 1_M) \leqslant p$ such that $N = p1_M$.*

For $N \in M$ we define $C(N) = \{x \in L\colon\ x \nleqslant (N : 1_M)\}$.

Lemma 2.9. *Let M be a multiplication lattice module over a* PG *C-lattice L. An element $P \in M$ is a prime if and only if $C(P)$ is a multiplicatively closed subset of L.*

Proof. Suppose that $P \in M$ is a prime and $x, y \in C(P)$. Then $x \nleqslant (P : 1_M)$ and $y \nleqslant (P : 1_M)$. Since $P \in M$ is a prime, by Lemma 2.8 we have that $(P : 1_M) \in L$ is a prime. As $x \nleqslant (P : 1_M)$, $y \nleqslant (P : 1_M)$ and $(P : 1_M)$ is a prime, $xy \nleqslant (P : 1_M)$, i.e. $xy \in C(P)$ and hence $C(P)$ is multiplicatively closed.

Conversely, suppose that $C(P)$ is a multiplicatively closed subset of L and $xy1_M \leqslant P$ for $x, y \in L$. Then $xy \leqslant (P : 1_M)$ and so $xy \notin C(P)$. If $x \nleqslant (P : 1_M)$ and $y \nleqslant (P : 1_M)$, then $x \in C(P)$, $y \in C(P)$ and this contradicts the fact that $C(P)$ is multiplicatively closed. Therefore $x \leqslant (P : 1_M)$ or $y \leqslant (P : 1_M)$, i.e. $x1_M \leqslant P$ or $y1_M \leqslant p$. Consequently, P is a prime. □

Lemma 2.10 ([11]). *Let a be an element of a C-lattice L and S be a multiplicatively closed subset of L satisfying the property $s \nleqslant a$ for all $s \in S$. Then there is a multiplicatively closed subset S' of L containing S which is maximal with respect to the property $s' \nleqslant a$ for all $s' \in S'$.*

Lemma 2.11 ([11]). *(Separation lemma) Let S be a sub-multiplicatively closed subset of a C-lattice L. Suppose that $a \in L$ and $t \nleq a$ for every $t \in S$. Then there exists a prime element $p \in L$ such that $a \leqslant p$ and it is maximal with respect to $t \nleq p$ for each $t \in S$.*

An element a in a complete lattice L is said to be *completely join prime* if $a \leqslant \bigvee S$, $S \subseteq L$ implies $a \leqslant s$ for some $s \in S$.

Lemma 2.12. *Let M be a multiplication lattice module over a* PG *C-lattice L and suppose every element of L is a completely join prime. A prime element $P \in M$ with $a1_M \leqslant P$ is minimal if and only if $C(P)$ is a maximal multiplicatively closed subset of L with $x \nleq a$ for all $x \in C(P)$ and $a \in L$.*

Proof. Suppose that $C(P)$ is a maximal multiplicatively closed subset of L with $x \nleq a$ for all $x \in C(P)$. By Lemma 2.11 there is a prime element $(Q : 1_M) \geqslant a$ that is maximal with respect to the property that $x \nleq (Q : 1_M)$ for all $x \in C(P)$. Hence, by Lemma 2.9, $C(Q)$ is a multiplicatively closed subset of L. As $a \leqslant (Q : 1_M)$, we have $x \nleq a$ for any $x \in C(Q)$. But $C(P)$ is a maximal multiplicatively closed subset of L with the property that $x \nleq a$ for all $x \in C(P)$, hence we must have $C(Q) \subseteq C(P)$. Now, if $y \in C(P)$, then $y \nleq (Q : 1_M)$ and hence $y \in C(Q)$. Consequently, we have $C(P) = C(Q)$. Now, let $z \leqslant (P : 1_M)$, i.e. $z \in C(P)$. Then $z \notin C(Q)$ and it implies that $z \leqslant (Q : 1_M)$ and it further implies $(P : 1_M) \leqslant (Q : 1_M)$. Similarly, we have $(Q : 1_M) \leqslant (P : 1_M)$ and hence $(P : 1_M) = (Q : 1_M)$. It follows that $P = Q$.

Now we show that P is a minimal prime. Suppose that $P' \in M$ is a prime with $a \leqslant (P' : 1_M) < (P : 1_M)$. Then by Lemma 2.9, $C(P')$ is a multiplicatively closed subset of L with $x \nleq a$ for all $x \in C(P')$ and $C(P) \subseteq C(P')$. This contradicts the maximality of $C(P)$. Hence, P is a minimal prime element of M with $a1_M \leqslant P$.

Conversely, suppose that $P \in M$ is a minimal prime with $a1_M \leqslant P$. Then by Lemma 2.9, $C(P)$ is a multiplicatively closed subset of L with $x \nleq a$ for all $x \in C(P)$. By Lemma 2.10, there is a maximal multiplicatively closed subset S which contains $C(P)$ and $x \nleq a$ for all $x \in S$. We show that $S = C(P')$, where $P' = p1_M$ and $p = \bigvee(L - S)$. Let $y \in C(P') = \{z \in L\colon\ z \nleq \bigvee(L - S)\}$. This gives $y \nleq \bigvee(L - S)$, i.e. $y \in S$ and $C(P') \subseteq S$. On the other hand, if $s \in S$, then $s \notin L - S$ and $s \nleq \bigvee(L - S)$. As each element of L is a completely join prime, we have $s \in C(P')$ and therefore $C(P) = C(P')$.

By the first part, as S is a maximal multiplicatively closed subset of L with respect to $x \nleq a$ for all $x \in S$, we conclude that P' is a minimal prime with $a1_M \leqslant P'$. Clearly, $C(P) \subseteq S = C(P')$ gives that $P' \leqslant P$ and since P is minimal, we must have $P = P'$. Hence, $C(P) = S = C(P')$ is the required maximal multiplicatively closed subset of L with $x \nleq a$ for all $x \in M$ and $a \in L$. $\square$

For $N \in M$ define $\sqrt[M]{N} = \bigvee\{x \in L\colon\ x^n 1_M \leqslant N\} 1_M$.

Theorem 2.13. *Let L be a* PG *C-lattice in which every element is completely join prime and let M be a multiplication lattice module over L. For $N \in M$, the radical $\sqrt[M]{N} = \bigwedge\{P\colon\ P$ is a minimal prime element of M with $N \leqslant P\}$.*

P r o o f. Observe that for a prime element $P \in M$ with $N \leqslant P$ we have $\sqrt[M]{N} \leqslant P$. Therefore $\sqrt[M]{N} \leqslant \bigwedge\{P\colon\ P$ is a minimal prime element of M with $N \leqslant P\}$.

Now, let $x \in L_*$ be such that $x1_M \nleqslant \sqrt[M]{N}$ and let $S = \{x^i\colon\ x^i \nleqslant (N : 1_M)$ and i is an integer$\}$. Observe that S is a multiplicatively closed subset of L. By Lemma 2.10, there is a maximal multiplicatively closed set S' such that $y \nleqslant (N : 1_M)$ for $y \in S'$. Let $p' = \bigvee(L - S')$. Then $S' = C(p'1_M) = C(P')$. By Lemma 2.12, P' is a minimal prime element of M with $N \leqslant P'$. Clearly, $x \in C(P')$ and as such $x \nleqslant (P : 1_M)$. This gives that $\bigwedge\{P\colon\ P$ is a minimal prime element of M with $N \leqslant P\} \leqslant \sqrt[M]{N}$. Consequently, $\sqrt{N} = \bigwedge\{P\colon\ P$ is a minimal prime element of M with $N \leqslant P\}$. $\square$

Corollary 2.14. *Let M be a lattice module over a reduced* PG *C-lattice L and $N \in M$. Then for a prime element $P \in M$ with $N \leqslant P$ there exists a minimal prime element $Q \in M$ such that $N \leqslant Q \leqslant P$.*

P r o o f. Suppose $P \in M$ is a prime element with $N \leqslant P$. Then by Lemma 2.9, $C(P)$ is a multiplicatively closed subset of L with $x \nleqslant (N : 1_M)$ for all $x \in C(P)$. By Lemma 2.10, there is a maximal multiplicatively closed set S such that $y \nleqslant (N : 1_M)$ for all $y \in S$. Also, $C(Q) = S$, where $Q = p1_M = \bigvee(L - S)1_M$ is a minimal prime element of M with $N \leqslant Q$ and $C(P) \subseteq C(Q) = S$ implies that $Q \leqslant P$. $\square$

Lemma 2.15 ([12]). *Let L be a C-lattice. Then each nonzero element of L is contained in a maximal multiplicatively closed subset of L not containing zero.*

For $N \in M$ we set $U(N) = \{P \in \pi(M)\colon\ N \nleqslant P\}$.

Theorem 2.16. *Let L be a* PG *C-lattice in which every element is completely join prime and let M be a multiplication lattice module over L. Then $(0_M : a) = \bigwedge U(a1_M) = \{P \in \pi(M)\colon\ a1_M \nleqslant P\}$, $a \in L$.*

P r o o f. Suppose $P \in M$ is a minimal prime. Then by Theorem 2.4 we have $(0_M : a) \leqslant P$ when $a1_M \nleqslant P$ and therefore $(0_M : a) \leqslant \bigwedge\{P \in \pi(M)\colon a1_M \nleqslant P\} = Q$. If $(0_M : a) < Q$, then there exists $x \in L_*$ such that $x1_M \leqslant Q$ and $x1_M \nleqslant (0_M : a)$. Clearly, $ax1_M \nleqslant 0_M$ and so $ax \neq 0$. By Lemma 2.15, ax is contained in some maximal multiplicatively closed subset S of L not containing 0. As proved in Lemma 2.12, $S = C(P)$, where $P = p1_M$ and $p = \bigvee(L - S)$ is a minimal prime element of L. Now $ax \in S$ implies $ax \nleqslant (P : 1_M)$ and hence $ax1_M \nleqslant P$.

Since P is a minimal prime and $a1_M \nleqslant P$, we have $x1_M \nleqslant P$. Therefore $x1_M \nleqslant Q$, a contradiction and consequently, $(0_M : a) = \bigwedge\{P \in \pi(M)\colon\ a1_M \nleqslant P\}$. $\square$

Theorem 2.17. *Let L be a* PG *C-lattice in which every element is a completely join prime and let M be a multiplication lattice module over L. Then $a1_M = (0_M : (0_M : a1_M))$ if and only if $a1_M = \bigwedge\{P \in \pi(M) : a1_M \leqslant P\}$, $a \in L$.*

Proof. Suppose $a1_M = (0_M : (0_M : a1_M))$, $a \in L$. By Theorem 2.4 we have $\bigwedge\{P \in \pi(M)\colon\ (0_M : a) \nleqslant P\} = \bigwedge\{P \in \pi(M)\colon\ a1_M \leqslant P\}$. But $(0_M : (0_M : a1_M)) = \bigwedge\{P \in \pi(M)\colon\ (0_M : a) \nleqslant P\}$ gives that $a1_M = \bigwedge\{P \in \pi(M)\colon\ a1_M \leqslant P\}$.

Conversely, suppose that $a1_M = \bigwedge\{P \in \pi(M)\colon\ a1_M \leqslant P\}$. By Theorem 2.16 we have $(0_M : (0_M : a1_M)) = \bigwedge\{P \in \pi(M)\colon\ (0_M : a) \nleqslant P\}$. Now, by Theorem 2.4 we have $\bigwedge\{P \in \pi(M)\colon\ (0_M : a) \nleqslant P\} = \bigwedge\{P \in \pi(M)\colon\ a1_M \leqslant P\}$ and by assumption, $a1_M = (0_M : (0_M : a1_M))$. $\square$

Theorem 2.18. *Let M be a multiplication lattice module over a* PG *C-lattice L. Then $(0_M : a) = \bigwedge\{V(0_M : a)\}$, $a \in L$.*

Proof. Note that $(0_M : a) \leqslant \bigwedge\{V(0_M : a)\}$, $a \in L$ follows immediately. Now, let $x \in L_*$ be such that $x1_M \nleqslant (0_M : a)$. Then $ax1_M \nleqslant 0_M$ and so $ax \neq 0$. Therefore ax is contained in some maximal multiplicatively closed subset S of L. Then $S = V(P) = V(p1_M)$, where $p = \bigvee(L - S)$ and p is a minimal prime element of L. Now $ax \in C(P)$ implies $ax \nleqslant (P : 1_M)$ and hence $ax1_M \nleqslant P$. Since P is a minimal prime, we have $x1_M \nleqslant P$ and $a1_M \nleqslant P$. By Theorem 2.4 we have $(0_M : a) \leqslant P$ and hence $P \in V(0_M : a)$. As $x1_M \nleqslant P$, we have $x1_M \nleqslant \bigwedge(V(0_M : a))$. Thus, $x1_M \nleqslant (0_M : a)$ implies $x1_M \nleqslant \bigwedge(V(0_M : a))$, i.e. $\bigwedge(V(0_M : a)) \leqslant (0_M : a)$. $\square$

We now show that the minimal prime spectrum $\pi(M)$ is a completely regular Hausdorff space, i.e. a Tychonoff space.

Theorem 2.19. *Let M be a multiplication lattice module over a* PG *C-lattice L. Then the topology on $\pi(M)$ for which the collection $\{U(a1_M)\colon\ a \in L\}$ is a base for open sets is Tychonoff.*

Proof. Suppose that $P_1, P_2 \in \pi(M)$ with $P_1 \neq P_2$. Clearly $P_1 \nleqslant P_2$ and $P_2 \nleqslant P_1$. Let $x \in L_*$ with $x1_M \leqslant P_1$ be such that $x1_M \nleqslant P_2$. By Theorem 2.3, there is $y \in L_*$ with $y1_M \nleqslant P_1$ and $x^n y1_M = 0_M$ for some integer n. If $y1_M \nleqslant P_2$, then this together with $x1_M \nleqslant P_2$ gives $x^n y1_M \nleqslant P_2$, which is a contradiction to the fact that $0_M \leqslant P_2$. Therefore $y1_M \leqslant P_2$. Clearly, $P_1 \in U(y1_M)$, $P_2 \in U(x1_M)$ and

$U(x1_M) \cap U(y1_M) = \{P \in \pi(M)\colon\ x1_M \nleqslant P,\ y1_M \nleqslant P\} = U(xy1_M) = U(x^n y1_M) = U(0_M) = \varphi$. Consequently, $\pi(M)$ is a Hausdorff space and hence singletons are closed.

Now, let $Q \in \pi(M)$ and F be a closed subset of $\pi(M)$ such that $Q \notin F$. Then $Q \in \pi(M) - F$ and $\pi(M) - F$ is open in $\pi(M)$. Then there is an open set $U(s1_M)$ for some $s \in L$ such that $Q \in U(s1_M) \subseteq \pi(M) - F$. Define a function f on $\pi(M)$ as $f(Q) = 0_M$ if $Q \in U(s1_M)$ and $f(Q) = 1_M$ otherwise. Then $f(Q) = 0_M$ and $f(F) = 1$. Note that f is continuous and hence $\pi(M)$ is completely regular. Consequently, $\pi(M)$ is a completely regular Hausdorff space, i.e. a Tychonoff space. □

Corollary 2.20. *$\pi(M)$ is totally disconnected and zero dimensional space.*

Theorem 2.21. *Let M be a multiplication lattice module over a* PG *C-lattice L. Let $x, y \in L$. Then $U(x1_M) \subseteq U(y1_M)$ if and only if $0_M : (0_M : x1_M) \leqslant 0_M : (0_M : y1_M)$. In addition, $U(x1_M) = U(y1_M)$ if and only if $(0_M : x) \leqslant (0_M : y)$.*

P r o o f. Suppose that $U(x1_M) \subseteq U(y1_M)$ for $x, y \in L$. By Theorem 2.16 we have $(0_M : y) \leqslant (0_M : x)$ and hence $(0_M : y) \nleqslant P$ which implies $(0_M : x) \nleqslant P$ and so $\{P \in \pi(M)\colon\ (0_M : y) \nleqslant P\} \subseteq \{P \in \pi(M)\colon\ (0_M : x) \nleqslant P\}$. By Theorem 2.4 we have $0_M : (0_M : x1_M) \leqslant 0_M : (0_M : y1_M)$.

Conversely, suppose that $0_M : (0_M : x1_M) \leqslant 0_M : (0_M : y1_M)$. Therefore $\{P \in \pi(M)\colon\ (0_M : y) \nleqslant P\} \subseteq \{P \in \pi(M)\colon\ (0_M : x) \nleqslant P\}$ and so $\{P \in \pi(M)\colon\ y1_M \leqslant P\} \subseteq \{P \in \pi(M)\colon\ x1_M \leqslant P\}$ by Theorem 2.4. This gives $\{P \in \pi(M)\colon\ x1_M \nleqslant P\} \subseteq \{P \in \pi(M)\colon\ y1_M \nleqslant P\}$ and therefore $U(x1_M) \subseteq U(y1_M)$.

For the second part, suppose that $U(x1_M) = U(y1_M)$. Then $U(x1_M) \subseteq U(y1_M)$ implies $0_M : (0_M : x1_M) \leqslant 0_M : (0_M : y1_M)$ and $U(y1_M) \subseteq U(x1_M)$ implies $0_M : (0_M : y1_M) \leqslant 0_M : (0_M : x1_M)$. Hence, $0_M : (0_M : y1_M) = 0_M : (0_M : x1_M)$ and $0_M : (0_M : (0_M : y1_M)) \leqslant 0_M : (0_M : (0_M : x1_M))$. Consequently, $(0_M : x) = (0_M : y)$.

Conversely, suppose that $(0_M : x) = (0_M : y)$. Then $0_M : (0_M : x1_M) = 0_M : (0_M : y1_M)$, i.e. $0_M : (0_M : x1_M) \leqslant 0_M : (0_M : y1_M)$ and $0_M : (0_M : y1_M) \leqslant 0_M : (0_M : x1_M)$ and the result follows by the first part. □

Theorem 2.22. *Let M be a multiplication lattice module over a* PG *C-lattice L. Let I be an indexing set and $S = \{x_r\colon\ r \in I\}$ be a set of points in L such that the collection of sets $\{U(x_r1_M)\colon\ r \in I\}$ has the finite intersection property. Then the intersection of all $\{U(x_r1_M)\colon\ r \in I\}$ is nonempty.*

P r o o f. We have $\bigcap\limits_{i=1}^{n} U(x_i) = U(y1_M)$, where $y = x_1x_2 \ldots x_n$. Note that the multiplication of finite number of nonzero x_r, $r \in I$ is nonzero. The collection of all nonzero x_r, $r \in I$ together with finite multiplication of $x_r \in S$ is multiplicatively closed subset of L not containing 0. By Lemma 2.10, there is a maximal multiplicatively closed subset S' of L containing S and not containing 0. We have $S' = C(P) = C(p1_M)$, where $p = \bigvee(L - S')$ and p is a minimal prime element of L. Clearly, $P \in U(x_r 1_M)$ for all $x_r \in S'$. As $S \subseteq S'$, we have $P \in U(x_r 1_M)$ for all $x_r \in S$. Thus, $P \in \bigcap\limits_{r \in I} U(x_r 1_M)$, which implies that $\bigcap\limits_{r \in I} U(x_r 1_M) \neq \varphi$. □

If the family $\{V(x1_M) \colon x \in L\}$ is considered as an open basis for $\pi(M)$, the resulting topology is called the dual topology and denoted by $\tau^{\rm d}$. We denote the topology for which $\{U(x1_M) \colon x \in L\}$ is an open basis by τ.

Theorem 2.23. *Let M be a multiplication lattice module over a* PG *C-lattice L. The topology τ on $\pi(M)$ for which $\{U(x1_M) \colon x \in L\}$ is a basis for open sets is finer than the topology $\tau^{\rm d}$ on $\pi(M)$ for which $\{V(x1_M) \colon x \in L\}$ is a basis for open sets and moreover $\tau = \tau^{\rm d}$.*

Proof. We know that $\{V(x1_M) \colon x \in L\}$ is a basis for open sets for the topology on $\pi(M)$ denoted by $\tau^{\rm d}$. Clearly, $V(x1_M) = \pi(M) - U(x1_M)$ for all $x \in L$. Note that for $x \in L$, $U(x1_M)$ is closed in $\pi(M)$. Hence, $V(x1_M)$ is open in the topology τ for $\pi(M)$, i.e. τ is finer than $\tau^{\rm d}$.

Now, for any $x \in L$ we have $U(x1_M) = V(0_M : x)$. Thus, every basic open set in τ is open in $\tau^{\rm d}$ and so we conclude that $\tau = \tau^{\rm d}$. □

Theorem 2.24. *Let M be a multiplication lattice module over a* PG *C-lattice L. The following statements are equivalent in M.*

(1) *$\pi(M)$ is compact.*
(2) *The poset $\{U(x1_M) \colon x \in L\}$, under set inclusion, is a Boolean lattice.*
(3) *For $x \in L$ there exist $N_1 = y_1 1_M, N_2 = y_2 1_M, \ldots, N_n = y_n 1_M \in M$ with $y_i 1_M = N_i \leqslant (0_M : x)$ for $i = 1, 2, \ldots, n$ and $(0_M : x) \wedge \bigwedge\limits_{i=1}^{n} (0_M : y_i) = 0_M$.*
(4) *For $x \in L$ there exist $N_1 = y_1 1_M, N_2 = y_2 1_M, \ldots, N_n = y_n 1_M \in M$ such that $0_M : (0_M : x1_M) = \bigwedge\limits_{i=1}^{n} (0_M : y_i)$.*
(5) *$\tau = \tau^{\rm d}$.*
(6) *$\{U(x1_M) \colon x \in L\}$ is a subbasis for open sets of $\pi(M)$ with respect to the topology τ.*
(7) *$\{V(x1_M) \colon x \in L\}$ is a subbasis for open sets of $\pi(M)$ with respect to the topology $\tau^{\rm d}$.*

Proof. (1) ⇒ (2): Clearly the set $\{U(x1_M) \colon x \in L\}$ is partially ordered under set inclusion.

Now, we first show that

(i) $U(x1_M) \cup U(y1_M) = U(x1_M \vee y1_M)$;

(ii) $U(x1_M) \cap U(y1_M) = U(xy1_M)$.

Let $P \in U(x1_M) \cup U(y1_M)$, then $P \in U(x1_M)$ or $P \in U(y1_M)$ and so $x1_M \nleqslant P$ or $y1_M \nleqslant P$. Therefore $x1_M \vee y1_M \nleqslant P$ and this implies $P \in U(x1_M \vee y1_M)$. Now, let $Q \in U(x1_M \vee y1_M)$, then $x1_M \vee y1_M \nleqslant Q$ and this implies that $x1_M \nleqslant Q$ or $y1_M \nleqslant Q$. Therefore $Q \in U(x1_M) \cup U(y1_M)$. Consequently, $U(x1_M) \cup U(y1_M) = U(x1_M \vee y1_M)$. Similarly, $U(x1_M) \cap U(y1_M) = U(xy1_M)$.

From this we conclude that $(\{U(x1_M)\colon x \in L\}, \cup, \cap)$ is a lattice.

Now, $U(0.1_M) = U(0_M) = \varphi$ and $U(1.1_M) = U(1_M) = \pi(M)$. This shows that $(\{U(x1_M)\colon x \in L\}, \cup, \cap)$ is a bounded lattice. Again, observe that $U(x1_M) \cup (U(y1_M) \cap U(z1_M)) = (U(x1_M) \cup U(y1_M)) \cap (U(x1_M) \cup U(z1_M))$ and $U(x1_M) \cap (U(y1_M) \cup U(z1_M)) = (U(x1_M) \cap U(y1_M)) \cup (U(x1_M) \cap U(z1_M))$. This shows that $(\{U(x1_M)\colon x \in L\}, \cup, \cap)$ is a distributive lattice.

Finally, we show that $(\{U(x1_M)\colon x \in L\}, \cup, \cap)$ is complemented. Note that for $x \in L$ we have $V(x1_M) \cap V(0_M : x) = \varphi$. Then $V(x1_M) \cap \{V(N)\colon N \leqslant (0_M : x)\} = \varphi$. Since $\pi(M)$ is compact, there exist $N_1, N_2, \ldots, N_n \leqslant (0_M : x)$ such that $V(x1_M) \cap \{V(N_i)\colon N_i \leqslant (0_M : x),\ i = 1, 2, \ldots, n\} = \varphi$. By taking complements in $\pi(M)$, we get $\pi(M) = U(x1_M) \cup U(N_1) \cup \ldots \cup U(N_n)$. Since each $N_i \leqslant (0_M : x)$ for $i = 1, 2, \ldots, n$, we have $U(x1_M) \cap \bigcup\limits_{i=1}^{n} U(N_i) = \varphi$. For, if $P \in U(x1_M) \cap \bigcup\limits_{i=1}^{n} U(N_i)$, then $x1_M \nleqslant P$, which implies $(0_M : x) \leqslant P$. Therefore $N_i \leqslant P$ for $i = 1, 2, \ldots, n$, a contradiction as $P \in \bigcup\limits_{i=1}^{n} U(N_i)$ and so $N_k \nleqslant P$ for some k, $1 \leqslant k \leqslant n$. Thus, we have $\pi(M) = U(x1_M) \cup \bigcup\limits_{i=1}^{n} U(N_i)$ and $U(x1_M) \cap \bigcup\limits_{i=1}^{n} U(N_i) = \varphi$. Consequently, $(\{U(x1_M)\colon x \in L\}, \cup, \cap)$ is a Boolean lattice.

(2) $\Rightarrow$ (3): Suppose that the finite union of $\{U(x1_M)\colon x \in L\}$ forms a Boolean lattice and suppose that the complement of $U(x1_M)$ is $\bigcup\limits_{i=1}^{n} U(N_i)$. As $U(x1_M) \cap \bigcup\limits_{i=1}^{n} U(N_i) = \varphi$, we get $U(x1_M) \cap U(N_i) = \varphi$, $i = 1, 2, \ldots, n$. Therefore $\{P \in \pi(M)\colon xN_i \nleqslant P\} = \varphi$, $i = 1, 2, \ldots, n$, i.e. $U(xN_i) = \varphi$ for $i = 1, 2, \ldots, n$, which implies $xN_i = 0_M$ for $i = 1, 2, \ldots, n$. Thus $N_i \leqslant (0_M : x)$ for $i = 1, 2, \ldots, n$. Also, $\pi(M) = U(x1_M) \cup \bigcup\limits_{i=1}^{n} U(N_i)$ gives $\bigwedge(\pi(M)) = \bigwedge\Big(U(x1_M) \cup \bigcup\limits_{i=1}^{n} U(N_i)\Big)$, i.e. $0_M = \bigwedge(\pi(M)) = \bigwedge\Big(U(x1_M) \vee \bigvee\limits_{i=1}^{n} N_i\Big)$. Note that $\bigwedge\Big(U(x1_M) \vee \bigvee\limits_{i=1}^{n} N_i\Big) = \bigwedge(U(x1_M)) \wedge \bigwedge\limits_{i=1}^{n} (U(N_i))$. Then by Theorem 2.16 we have $(0_M : x) \wedge \bigwedge\limits_{i=1}^{n} (0_M : y_i) = 0_M$.

(3) $\Rightarrow$ (4): Suppose that (3) holds. Then for any $x \in L$ there exist $N_1 = y_1 1_M$, $N_2 = y_2 1_M$, …, $N_n = y_n 1_M \in M$ with $y_i 1_M = N_i \leqslant (0_M : x)$ for $i = 1, 2, \ldots, n$

and $(0_M : x) \wedge \bigwedge_{i=1}^{n} (0_M : y_i) = 0_M$. This implies $(0_M : x1_M) \bigwedge_{i=1}^{n} (0_M : y_i) = 0_M$, i.e. $\bigwedge_{i=1}^{n} (0_M : y_i) \leqslant (0_M : (0_M : x1_M))$. Also note that $(0_M : (0_M : x1_M)) \leqslant (0_M : y_i)$ for $i = 1, 2, \ldots, n$. Hence $(0_M : (0_M : x1_M)) \leqslant \bigwedge_{i=1}^{n} (0_M : y_i)$ and consequently, $(0_M : (0_M : x1_M)) = \bigwedge_{i=1}^{n} (0_M : y_i)$.

(4) $\Rightarrow$ (5): Let x be an element of L. By (4), there exist $N_1 = y_1 1_M, N_2 = y_2 1_M, \ldots, N_n = y_n 1_M \in M$ such that $(0_M : (0_M : x1_M)) = \bigwedge_{i=1}^{n} (0_M : y_i)$. Hence we have

$$V(0_M : (0_M : x1_M)) = V\left(\bigwedge_{i=1}^{n} (0_M : y_i)\right) = \bigcup_{i=1}^{n} V(0_M : y_i) = \bigcup_{i=1}^{n} U(y_i 1_M) = V(x1_M).$$

Taking complements in $\pi(M)$, we have $\pi(M) - V(x1_M) = \pi(M) - \bigcup_{i=1}^{n} U(y_i 1_M)$, i.e. $U(x1_M) = \bigcap_{i=1}^{n} V(y_i 1_M)$. It follows that $U(x1_M)$ is a finite intersection of open sets in dual topology τ^{d}. Hence, $U(x1_M)$ is open in τ^{d}, which implies τ^{d} is finer than τ, and τ is finer than τ^{d} follows by Theorem 2.23.

(5) $\Rightarrow$ (1): Suppose that $\tau = \tau^{\mathrm{d}}$. Then $\{U(x1_M) \colon x \in L\}$ is also a base for closed sets in $\pi(M)$. Let $\{U(y1_M) \colon y \in K\}$ be a family of closed sets with finite intersection property in $\pi(M)$, where $K \subseteq L$. Then $\bigcap_{i=1}^{n} U(y_i 1_M) = U(y_1 y_2 \ldots y_n 1_M) \neq \varphi$ and so $y_1 y_2 \ldots y_n 1_M \neq 0_M$ for any finite number of elements $y_1, y_2, \ldots, y_n \in K$. All the nonzero elements in K together with the finite multiplication of elements in K form a multiplicatively closed set not containing 0. This multiplicatively closed set is again contained in some maximal multiplicatively closed set S not containing 0. As proved in Lemma 2.12, $S = C(P) = C(p1_M)$, where $p = \bigvee(L - S)$ is a minimal prime element of L. Note that $K \subseteq C(P)$ and therefore $P \in U(y1_M)$ for all $y \in K$. Thus, $p \in \bigcap\{U(y1_M) \colon y \in K\} \neq \varphi$ and so $\pi(L)$ is compact.

(5) $\Rightarrow$ (6): The implication follows immediately as $\{V(x1_M) \colon x \in L\}$ is a basis for open sets in τ^{d}.

(6) $\Rightarrow$ (5): Let $\{U(x1_M) \colon x \in L\}$ be any basis for open sets in τ. Then we have $U(x1_M) = \bigcap_{i=1}^{n} V(x_i)$ as $\{V(x1_M) \colon x \in L\}$ is a subbasis for open sets in $\pi(M)$ with respect to τ. This implies that $\{U(x1_M) \colon x \in L\}$ is open in τ^{d} and hence $\tau \subseteq \tau^{\mathrm{d}}$ and the result follows by Theorem 2.23.

(6) $\Rightarrow$ (7): If $\{V(x1_M) \colon x \in L\}$ is a subbasis for open sets in τ, then $\{\pi(M) - V(x1_M) \colon x \in L\} = \{U(x1_M) \colon x \in L\}$ forms a subbasis for open sets in τ^{d} and conversely. $\square$

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References

[1] *F. Alarcon, D. D. Anderson, C. Jayaram*: Some results on abstract commutative ideal theory. Period. Math. Hung. *30* (1995), 1–26. zbl MR doi
[2] *E. A. Al-Khouja*: Maximal elements and prime elements in lattice modules. Damascus University Journal for Basic Sciences *19* (2003), 9–21.
[3] *S. Ballal, V. Kharat*: Zariski topology on lattice modules. Asian-Eur. J. Math. *8* (2015), Article ID 1550066, 10 pages. zbl MR doi
[4] *F. Calliap, U. Tekir*: Multiplication lattice modules. Iran. J. Sci. Technol., Trans. A, Sci. *35* (2011), 309–313. zbl MR doi
[5] *D. S. Culhan*: Associated Primes and Primal Decomposition in Modules and Lattice Modules, and Their Duals. Thesis (Ph.D.), University of California, Riverside, 2005. MR
[6] *R. P. Dilworth*: Abstract commutative ideal theory. Pac. J. Math. *12* (1962), 481–498. zbl MR doi
[7] *K. Keimel*: A unified theory of minimal prime ideals. Acta Math. Acad. Sci. Hung. *23* (1972), 51–69. zbl MR doi
[8] *C. S. Manjarekar, U. N. Kandale*: Baer elements in lattice modules. Eur. J. Pure Appl. Math. *8* (2015), 332–342. zbl MR
[9] *Y. S. Pawar, N. K. Thakare*: The space of minimal prime ideals in a 0-distributive semilattice. Period. Math. Hung. *13* (1982), 309–319. zbl MR doi
[10] *N. Phadatare, S. Ballal, V. Kharat*: On the second spectrum of lattice modules. Discuss. Math. Gen. Algebra Appl. *37* (2017), 59–74. MR doi
[11] *N. K. Thakare, C. S. Manjarekar*: Abstract spectral theory: Multiplicative lattices in which every character is contained in a unique maximal character. Algebra and Its Applications. Int. Symp. Algebra and Its Applications, 1981, New Delhi, India (H. L. Manocha et al., eds.). Lect. Notes Pure Appl. Math. 91. Marcel Dekker Inc., New York, 1984, pp. 265–276. zbl MR
[12] *N. K. Thakare, C. S. Manjarekar, S. Maeda*: Abstract spectral theory. II: Minimal characters and minimal spectrums of multiplicative lattices. Acta Sci. Math. *52* (1988), 53–67. zbl MR

Authors' address: *Sachin Ballal*, *Vilas Kharat*, Department of Mathematics, Savitribai Phule Pune University, Pune-411 007, India, e-mail: `ballalshyam@gmail.com`, `laddoo1@yahoo.com`.