A USEFUL ALGEBRA FOR FUNCTIONAL CALCULUS

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Remembering my Professor L. Waelbroeck 1929–2009

Abstract. We show that some unital complex commutative LF-algebra of $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered functions on $\mathbb{R}^+$ (M. Hemdaoui, 2017) equipped with its natural convex vector bornology is useful for functional calculus.

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1. Introduction

Let $(\mathbf{A}, \|\cdot\|)$ be a unital commutative complex Banach algebra, $a \in \mathbf{A}$ and $\operatorname{sp}(a)$ be the spectrum of the element a. The positive function defined as

$$\delta(t) = \begin{cases} \|(a-t)^{-1}\|^{-1} & \text{if } t \notin \operatorname{sp}(a), \\ 0 & \text{if } t \in \operatorname{sp}(a) \end{cases}$$

is a Lipschitz function and it satisfies $\lim\limits_{|t|\to\infty} \delta(t)/|t| = 1$.

The set $\{\delta(t)(a-t)^{-1};\ t \notin \operatorname{sp}(a)\}$ is obviously bounded in $\mathbf{A}$.

More generally, Waelbroeck (see [16]) considered a unital locally convex complete commutative complex algebra $\mathbf{A}$ and a positive Lipschitz function δ on $\mathbb{C}$ satisfying as $|t| \to \infty$,

$$\varepsilon \leqslant \frac{\delta(t)}{|t|} \leqslant M, \tag{1.1}$$

where $\varepsilon > 0$, $M > 0$ are two positive real numbers independent of the variable t.

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Let $a \in \mathbf{A}$ satisfy the following conditions:

▷ $\mathrm{sp}(a) = \delta^{-1}\{0\}$;
▷ The set $\{\delta(t)(a-t)^{-1};\ t \notin \delta^{-1}\{0\}\}$ is bounded in $\mathbf{A}$.

$\mathrm{Sp}(a)$ is obviously compact. Waelbroeck in [16] showed that the closed subalgebra generated by a and its resolvent function $(a-t)^{-1}$, $t \notin \delta^{-1}\{0\}$ is a unital complex Banach algebra. He gave two different descriptions of this Banach algebra and studied the extension of the holomorphic functional calculus.

In this note, let δ be a positive Lipschitz function on $\mathbb{C}$ and a be an element of a unital complex complete locally convex algebra (or a unital complex complete convex bornological algebra) $\mathbf{A}$ satisfying the following conditions:

▷ $\mathrm{sp}(a) = \delta^{-1}\{0\}$;
▷ The set $\{\delta(t)(a-t)^{-1};\ t \notin \delta^{-1}\{0\}\}$ is bounded in $\mathbf{A}$.

Two questions come to mind:

(1) What can we say about the spectrum of a?
(2) Can we describe the closed subalgebra generated by a and its resolvent function $(a-t)^{-1}$, $t \notin \delta^{-1}\{0\}$?

The spectrum of the element a may be unbounded in $\mathbb{C}$. In this case, the closed subalgebra generated by a and its resolvent function $(a-t)^{-1}$, $t \notin \delta^{-1}\{0\}$ need not to be a Banach algebra. My goal is to describe this closed subalgebra and see what it looks like. Fearing of falling into a purely abstract study since the spectrum is unknown in advance, I think it is wise and prudent to treat this study on a concrete example that checks the above assumptions that surely makes this subject more interesting.

In this work bornological language is used (see [3], [4], [8], [9], [10], [15], [17]).

Consider the Laplace operator $-\Delta$ acting on $\mathcal{S}'(\mathbb{R}^n)$, the vector space of tempered distributions on $\mathbb{R}^n$ (see [12]), such as

$$T \in \mathcal{S}'(\mathbb{R}^n) \to \mathfrak{F}(-\Delta T) = |\xi|^2 \mathfrak{F}(T) \in \mathcal{S}'(\mathbb{R}_n),$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}_n$, the dual space of $\mathbb{R}^n$.

For $t \in \mathbb{C} \setminus \mathbb{R}^+$ the operator $(-\Delta - t)^{-1}$ also acts on $\mathcal{S}'(\mathbb{R}^n)$ and is defined as

$$T \in \mathcal{S}'(\mathbb{R}^n) \to \mathfrak{F}((-\Delta - t)^{-1} T) = (|\xi|^2 - t)^{-1} \mathfrak{F}(T) \in \mathcal{S}'(\mathbb{R}_n),$$

where $\mathfrak{F}(T)$ is the Fourier transform of the tempered distribution T. The spectrum of the Laplace operator $-\Delta$ is $\mathbb{R}^+$. It is closed and unbounded in $\mathbb{C}$.

Let δ be positive Lipschitz function on $\mathbb{C}$ satisfying

$$\delta(t) = \inf\{1, d^n(t, \mathbb{R}^+), \forall\, n \in \mathbb{N}\},$$

where $d(t, \mathbb{R}^+)$ is the distance from the point $t \in \mathbb{C}$ to the set $\mathbb{R}^+$.

Function δ is very flat around the positive half of the axis $(\mathbb{R}^+)$, for example one considers function

$$\delta(t) = \mathrm{e}^{-1/d^2(t,\mathbb{R}^+)} \quad \forall\, t \in \mathbb{C}.$$

The function δ satisfies

$$\frac{\delta(t)}{|t|} \leqslant 1 \quad \forall\, t \in \mathbb{C}.$$

$\mathbb{C}_\delta(-\Delta)$ denotes the complex unital commutative algebra generated by $-\Delta$ and its resolvent function $(-\Delta - t)^{-1}$, $t \notin \delta^{-1}\{0\} = \mathbb{R}^+$. An element of this algebra is an operator of the form

$$\sum_{\substack{0\leqslant i\leqslant m\\ 0\leqslant j\leqslant n}} C_{k_i,k'_j}(-\Delta)^{k_i}(-\Delta - t)^{-k'_j}, \quad t \in \mathbb{C} \setminus \mathbb{R}^+,\ C_{k_i,k'_j} \in \mathbb{C},\ k_i, k'_j, m, n \in \mathbb{N}.$$

Let $(M, m, n) \in \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N}$. Consider the convex and balanced subset in $\mathbb{C}_\delta(-\Delta)$ defined as

$$B_{(M,m,n)} = \left\{ \sum_{\substack{0\leqslant i\leqslant m\\ 0\leqslant j\leqslant n}} C_{k_i,k'_j}\delta(t)(-\Delta)^{k_i}(-\Delta - t)^{-k'_j},\ t \notin \delta^{(-1)}\{0\},\ \sum_{\substack{0\leqslant i\leqslant m\\ 0\leqslant j\leqslant n}} |C_{k_i,k'_j}| \leqslant M \right\}.$$

These subsets $(B_{(M,m,n)})_{(M,m,n)\in\mathbb{R}^+\times\mathbb{N}\times\mathbb{N}}$ satisfy

(1) $B_{(M,m,n)} + B_{(M',m',n)} \subseteq B_{(M'',m'',n'')}$, where $M'' = M + M'$, $m'' = \sup\{m, m'\}$, $n'' = \sup\{n, n'\}$,

(2) $\lambda \cdot B_{(M,m,n)} \subseteq B_{(|\lambda|M,m,n)}$ for all $\lambda \in \mathbb{C}$,

(3) $B_{(M,m,n)} \cdot B_{(M',m',n')} \subseteq B_{(M'',m'',n'')}$, where $M'' = M \cdot M'$, $m'' = m + m'$, $n'' = n + n'$.

This family of subsets $(B_{(M,m,n)})_{(M,m,n)\in\mathbb{R}^+\times\mathbb{N}\times\mathbb{N}}$ is a filter basis compatible with the algebraic structure of algebra $\mathbb{C}_\delta(-\Delta)$. It is also a convex vector bornology basis (see [9], [10]). A subset $B \subset \mathbb{C}_\delta(-\Delta)$ is said to be bounded if it is contained in some $B_{(M,m,n)}$. This natural convex vector bornology on algebra $\mathbb{C}_\delta(-\Delta)$ is separated since no one-dimensional vector subspace of algebra $\mathbb{C}_\delta(-\Delta)$ is bounded (see [9], [10]).

The positive Lipschitz function δ and the operator $-\Delta$ satisfy

(1) $(-\Delta - t)^{-1}$ exists for $t \notin \delta^{-1}\{0\}$ in $\mathbb{C}_\delta(-\Delta)$;

(2) The set $\{\delta(t)(-\Delta - t)^{-1}, t \notin \delta^{-1}\{0\}\} \subseteq B_{(1,0,1)}$ is bounded in $\mathbb{C}_\delta(-\Delta)$.

A bounded multiplicative linear functional φ on $\mathbb{C}_\delta(-\Delta)$ maps bounded sets in $\mathbb{C}_\delta(-\Delta)$ into bounded sets in $\mathbb{C}$. Functional φ is defined if and only if its value $\varphi(-\Delta)$ is defined. But $(-\Delta - t)^{-1}$ exists in $\mathbb{C}_\delta(-\Delta)$ if and only if $t \notin \delta^{-1}(0) = \mathbb{R}^+$. So, $\varphi((-\Delta - t)^{-1}) = (-\varphi(\Delta) - t)^{-1}$ exists in $\mathbb{C}$ if and only if $t \notin \delta^{-1}(0) = \mathbb{R}^+$. This happens if and only if $-\varphi(\Delta) \in \mathbb{R}^+$.

The set $\Phi_{\mathbb{C}_\delta(-\Delta)}$ of bounded multiplicative linear functionals on $\mathbb{C}_\delta(-\Delta)$ is identified with $\mathbb{R}^+$ in the sense

$$\varphi \in \Phi_{\mathbb{C}_\delta(-\Delta)} \Leftrightarrow \exists!\, x \in \mathbb{R}^+ \text{ such that } \varphi(-\Delta) = x.$$

The separated convex bornological algebra $\mathbb{C}_\delta(-\Delta)$ is not necessarily complete. Its completion convex bornological algebra is $\widehat{\mathbb{C}_\delta(-\Delta)}$ (see [8], [9], [10]).

Our goal is to show that the completion convex bornological algebra $\widehat{\mathbb{C}_\delta(-\Delta)}$ is bornologically isomorphic to some unital complex commutative LF-algebra of class $\mathcal{C}^{(\infty)}$ functions $\mathbb{N}$-tempered on $\mathbb{R}^+$ (see [7]) equipped with its natural convex vector bornology. This bornological isomorphism is obtained by using the functional calculus theorem (see [6]) and Gelfand transform.

2. Functional calculus theorem

For $\varepsilon \in]\frac{1}{2}, 1[$ consider the closed sector in the complex plan defined by

$$T_\varepsilon = \{z \in \mathbb{C}\colon\ \Re e(z) \geqslant -\varepsilon \text{ and } |\arg(z+\varepsilon)| \leqslant \varepsilon\}.$$

$\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ denotes the unital commutative algebra of complex continuous functions f on the closed sector T_ε and of class $\mathcal{C}^{(1)}$ in $\mathring{T}_\varepsilon$ satisfying for a certain positive integer $N \in \mathbb{N}$

$$\|f\|_N = \sup_{t \in T_\varepsilon} \delta_0^N(t)|f(t)| + \frac{1}{2\pi}\int_{T_\varepsilon} \delta_0^N(t)\delta^{-1}(t)|\overline{\partial} f(t)\,\mathrm{d}t| < \infty, \tag{2.1}$$

where $\delta_0(t) = 1/\sqrt{1+|t|^2}$, $\overline{\partial} f(t) = \partial(f(t)\,\mathrm{d}\overline{t})/\partial\overline{t}$ and δ is the positive Lipschitz function defined in the first paragraph.

$\mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ denotes the vector subspace of $\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ of functions f satisfying equation (2.1) for the same positive integer N, which normed by (2.1) is a complex Banach space.

For any pair $(N, N') \in \mathbb{N} \times \mathbb{N}$ such that $N \leqslant N'$ the canonical injection

$$i_{N,N'}\colon\ \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \hookrightarrow \mathcal{C}_{N'}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$$

is continuous since it satisfies

$$\| i_{N,N'}(f)\|_{N'} = \|f\|_{N'} \leqslant \|f\|_N.$$

For any pair $(N, N') \in \mathbb{N} \times \mathbb{N}$ the bilinear mapping

$$(f, g) \in \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \times \mathcal{C}_{N'}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to f \cdot g \in \mathcal{C}_{N+N'}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$$

is continuous since it satisfies

$$\|f \cdot g\|_{N+N'} \leqslant \|f\|_N \|g\|_{N'}.$$

The unital commutative algebra $\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) = \bigcup\limits_{N\in\mathbb{N}} \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ is equipped with the inductive limit topology of Banach spaces $\mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$.

A set $B \subset \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ is bounded if there exists a pair $(N', M') \in \mathbb{N} \times \mathbb{R}^+$ such that $B \subset \mathcal{C}_{N'}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ and

$$\|f\|_{N'} = \sup_{t\in T_\varepsilon} \delta_0^{N'}(t)|f(t)| + \frac{1}{2\pi}\int_{T_\varepsilon} \delta_0^{N'}(t)\delta^{-1}(t)\big|\overline{\partial} f(t)\,\mathrm{d}t\big| \leqslant M' \quad \forall\, f \in B.$$

The family of convex and balanced sets $(B_{(N',M')})_{(N',M')\in\times\mathbb{R}^+}$, where

$$B_{(N',M')} = \{f \in \mathcal{C}_{N'}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})\colon\ \|f\|_{N'} \leqslant M'\},$$

is a natural convex vector bornology basis (see [9] and [10]) on the algebra $\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$.

The complete convex bornological algebra $\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) = \varinjlim\limits_{N} \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ is a b-algebra (see [15]).

Theorem 2.1 ([6])**.** *There exists a bounded algebra homomorphism*

$$T\colon\ f \in \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to T(f) = f[-\Delta] \in \widehat{\mathbb{C}_\delta(-\Delta)}.$$

The proof of Theorem 2.1 is very technical, so we omit it (see [5], [11], [13], [14] and [18]).

Corollary 2.2 ([6])**.** *The kernel of the bounded algebra homomorphism*

$$T\colon\ f \in \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to T(f) = f[-\Delta] \in \widehat{\mathbb{C}_\delta(-\Delta)}$$

is the closed ideal $\{f \in \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})\colon\ f(x) = 0$ *for all* $x \in \mathbb{R}^+\}$.

Theorem 2.3. *Let $f \in \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$. Then its restriction $f|_{\mathbb{R}^+}$ to $\mathbb{R}^+$ satisfies*

(1) *$f|_{\mathbb{R}^+}$ is of class $\mathcal{C}^{(\infty)}$ on $\mathbb{R}^+$;*

(2) *for all $n \in \mathbb{N}$, $\sup\limits_{x\in\mathbb{R}^+} \delta_0^N(x)|f^{(n)}(x)| < \infty$,*

where $\delta_0(x) = 1/\sqrt{1+x^2}$ and $f^{(n)}$ is the nth derivative of f.

Proof. For $f \in \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ we have

$$\|f\|_N = \sup_{t\in T_\varepsilon} \delta_0^N(t)|f(t)| + \frac{1}{2\pi}\int_{T_\varepsilon} \delta_0^N(t)\delta^{-1}(t)\big|\overline{\partial} f(t)\,\mathrm{d}t\big| < \infty.$$

Fix $\varepsilon \in \,]0,1[$ and x in $\mathbb{R}^+$. Put $\varrho = (x+\varepsilon)\sin\varepsilon > 0$. Consider the closed discus centered at x with radius $\varrho > 0$:

$$D(x,\varrho) = \{z \in T_\varepsilon\colon\ |z-x| \leqslant \varrho\}.$$

Function f is continuous on the sector T_ε and of class $\mathcal{C}^{(1)}$ in $\mathring{T}_\varepsilon$, using Pompieu formula (generalized Cauchy formula) on the closed discus $D(x,\varrho)$. We have

$$f(x) = \frac{1}{2\pi\mathrm{i}}\int_{\partial D(x,\varrho)} \frac{f(t)}{t-x}\,\mathrm{d}t + \frac{1}{2\pi\mathrm{i}}\int_{D(x,\varrho)} \frac{1}{t-x}\overline{\partial} f(t)\,\mathrm{d}t,$$

where $\overline{\partial} f(t) = \partial(f(t)\,\mathrm{d}\overline{t})/\partial\overline{t}$.

For all $n \in \mathbb{N}$ function $t \in \partial D(x,\varrho) \to f(t)(t-x)^{-n}$ is continuous on the compact boundary. So the integral $\int_{\partial D(x,\varrho)} f(t)(t-x)^{-n}\,\mathrm{d}t$ exists for all $n \in \mathbb{N}$.

The differential form $t \in D(x,\varrho) \to (t-x)^{-n}\overline{\partial} f(t)$ is integrable on the compact discus $D(x,\varrho)$ since

$$\frac{\delta(t)}{|t-x|^n} \leqslant 1 \quad \forall\, n \in \mathbb{N}\ \forall\, t \in D(x,\varrho)$$

and

$$\int_{\partial D(x,\varrho)} \delta_0^N(t)\delta^{-1}(t)\frac{\delta(t)}{|t-x|^n}|\overline{\partial} f(t)\,\mathrm{d}t| \leqslant \int_{T_\varepsilon} \delta_0^N(t)\delta^{-1}(t)|\overline{\partial} f(t)\,\mathrm{d}t| < \infty.$$

Thus, for every positive integer $n \in \mathbb{N}$ we have

$$f^{(n)}(x) = \frac{n!}{2\pi\mathrm{i}}\int_{\partial D(x,\varrho)} \frac{f(t)}{(t-x)^{n+1}}\,\mathrm{d}t + \frac{n!}{2\pi\mathrm{i}}\int_{D(x,\varrho)} \frac{1}{(t-x)^{n+1}}\overline{\partial} f(t)\,\mathrm{d}t.$$

So $f|_{\mathbb{R}^+}$ is of class $\mathcal{C}^{(\infty)}$ on $\mathbb{R}^+$.

Now, we show for all $n \in \mathbb{N}$, $\sup\limits_{x\in\mathbb{R}^+} \delta_0^N(x)|f^{(n)}(x)| < \infty$. In fact:

$$\delta_0^N(x)f^{(n)}(x) = \frac{n!}{2\pi\mathrm{i}}\int_{\partial D(x,\varrho)} \delta_0^N(x)\frac{f(t)}{(t-x)^{n+1}}\,\mathrm{d}t + \frac{n!}{2\pi\mathrm{i}}\int_{D(x,\varrho)} \delta_0^N(x)\frac{\overline{\partial} f(t)}{(t-x)^{n+1}}\,\mathrm{d}t.$$

For the first integral we have

$$\left|\frac{n!}{2\pi\mathrm{i}}\int_{\partial D(x,\varrho)}\delta_0^N(x)\frac{f(t)}{(t-x)^{n+1}}\,\mathrm{d}t\right| \leqslant \frac{n!}{2\pi\varrho^n}\sup_{t\in\partial B(x,\varrho)}\Big(\frac{\delta_0(x)}{\delta_0(t)}\Big)^N\sup_{t\in T_\varepsilon}\delta_0^N(t)|f(t)|.$$

For the second integral, since $\delta(t)/|t-x|^n \leqslant 1$ for all $n\in\mathbb{N}$, we get

$$\begin{aligned}&\left|\frac{n!}{2\pi\mathrm{i}}\int_{D(x,\varrho)}\delta_0^N(x)\frac{\overline{\partial}f(t)}{(t-x)^{n+1}}\,\mathrm{d}t\right|\\ &\qquad\leqslant \frac{n!}{2\pi}\sup_{t\in B(x,\varrho)}\Big(\frac{\delta_0(x)}{\delta_0(t)}\Big)^N\int_{T_\varepsilon}\delta_0^N(t)\delta^{-1}(t)|\overline{\partial}f(t)\,\mathrm{d}t|<\infty.\end{aligned}$$

Since $|t-x|=\varrho$, we have $t=x+\varrho\mathrm{e}^{\theta\mathrm{i}}=x+\varrho\cos\theta+\varrho\sin\theta\mathrm{i}$. So

$$|t|^2=(x+\varrho\cos\theta)^2+(\varrho\sin\theta)^2\leqslant(x+\varrho)^2.$$

But $\varrho=(x+\varepsilon)\sin\varepsilon$, where $\varrho\in\,]0,1[$. We have $|t|^2\leqslant(2x+1)^2$. So

$$\Big(\frac{\delta_0(x)}{\delta_0(t)}\Big)^N=\Big(\frac{1+|t|^2}{1+x^2}\Big)^{N/2}\leqslant\Big(\frac{1+(2x+1)^2}{1+x^2}\Big)^{N/2}=\sqrt[N]{\frac{4x^2+4x+2}{1+x^2}}.$$

The positive continuous function of one real variable

$$x\in[0,\infty[\,\to\,\sqrt[N]{\frac{4x^2+4x+2}{1+x^2}}$$

is bounded on $\mathbb{R}^+$. There exists a positive real number $C_N>0$ such that

$$\sqrt[N]{\frac{4x^2+4x+2}{1+x^2}}\leqslant C_N\quad\forall\,x\geqslant 0.$$

So, for every positive integer n and ϱ fixed in $]0,1[$ we get

$$\left|\frac{n!}{2\pi\mathrm{i}}\int_{\partial D(x,\varrho)}\delta_0^N(x)\frac{f(t)}{(t-x)^{n+1}}\,\mathrm{d}t\right|\leqslant\frac{n!C_N}{2\pi(\varepsilon\sin\varepsilon)^n}\sup_{t\in T_\varepsilon}\delta_0^N(t)|f(t)|<\infty$$

and

$$\left|\frac{n!}{2\pi\mathrm{i}}\int_{D(x,\varrho)}\delta_0^N(x)\frac{\overline{\partial}f(t)}{(t-x)^{n+1}}\,\mathrm{d}t\right|\leqslant\frac{n!C_N}{2\pi(\varepsilon\sin\varepsilon)^n}\int_{T_\varepsilon}\delta_0^N(t)\delta^{-1}(t)|\overline{\partial}f(t)\,\mathrm{d}t|<\infty.$$

Finally, for all $n\in\mathbb{N}\Rightarrow\exists\,M_{(\varepsilon,n,N)}>0$ such that $\delta_0^N(x)|f^{(n)}(x)|\leqslant M_{(\varepsilon,n,N)}$,

$$\sup_{t\in T_\varepsilon}\delta_0^N(t)|f(t)|+\frac{1}{2\pi}\int_{T_\varepsilon}\delta_0^N(t)\delta^{-1}(t)|\overline{\partial}f(t)\,\mathrm{d}t|<\infty.$$

So

$$\forall\,n\in\mathbb{N},\quad\sup_{x\in\mathbb{R}^+}\delta_0^N(x)|f^{(n)}(x)|\leqslant M_{(\varepsilon,n,N)}\|\!|f|\!\|_N<\infty,$$

where $M_{(\varepsilon,n,N)}=n!\,C_N/2\pi(\varepsilon\sin\varepsilon)^n$. □

Definition 2.4 ([7]). Let N be a positive integer. A $\mathcal{C}^{(\infty)}$ complex function f on $\mathbb{R}^+$ is N-tempered if for any positive integer $n \in \mathbb{N}$ the nth derivative $f^{(n)}$ of f satisfies

$$\sup_{x\in\mathbb{R}^+} \delta_0^N(x)|f^{(n)}(x)| < \infty,$$

where $\delta_0(x) = 1/\sqrt{1+x^2}$.

For every positive integer $N \in \mathbb{N}$, $\mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ denotes the vector space of functions of class $\mathcal{C}^{(\infty)}$ N-tempered on $\mathbb{R}^+$ equipped with the family of norms

$$f \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \to \|f\|_{N,n} = \sup_{0\leqslant k\leqslant n} \sup_{x\in\mathbb{R}^+} \delta_0^N(x)|f^{(k)}(x)|, \quad n \in \mathbb{N}.$$

$(\mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}), (\|\cdot\|_{N,n})_{n\in\mathbb{N}})$ is a Fréchet space (see [12]).

For any pair (N, N') of positive integers such that $N \leqslant N'$ the canonical injection

$$i_{N,N'}\colon \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \hookrightarrow \mathcal{C}_{N'}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$$

is obviously continuous since

$$\|i_{N,N'}(f)\|_{N',n} = \|f\|_{N',n} \leqslant \|f\|_{N,n} \quad \forall\, f \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}).$$

Consider the bilinear mapping

$$(f, g) \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \times \mathcal{C}_{N'}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \to f \cdot g \in \mathcal{C}_{N+N'}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}).$$

Let $n \in \mathbb{N}$, the nth derivative $(f \cdot g)^{(n)}$ of the function $f \cdot g$ at any point $x \in \mathbb{R}^+$ by Leibniz formula is

$$(f(x)g(x))^{(n)} = \sum_{p=0}^{n} \binom{n}{p} f^{(n-p)}(x) \cdot g^{(p)}(x),$$

where $\binom{n}{p} = n!/(n-p)!\,p!$ is the binomial coefficient.

So

$$\delta_0^{N+N'}(x)|(f(x)g(x))^{(n)}| \leqslant 2^n \sup_{x\in\mathbb{R}^+} \sup_{0\leqslant k\leqslant n} \delta_0^N(x)|f^{(k)}(x)| \sup_{x\in\mathbb{R}^+} \sup_{0\leqslant p\leqslant n} \delta_0^{N'}(x)|g^{(p)}(x)|.$$

Then

$$\forall\, n \in \mathbb{N}, \quad \|f \cdot g\|_{N+N',n} \leqslant 2^n \|f\|_{N,n} \|g\|_{N',n}.$$

This inequality shows the continuity of the bilinear mapping

$$(f, g) \in \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \times \mathcal{C}_{N'}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \to f \cdot g \in \mathcal{C}_{N+N'}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}).$$

The vector space $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}) = \varinjlim_{N} \mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}) = \bigcup_{N\in\mathbb{N}} \mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$ of functions of class $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered on $\mathbb{R}^+$ equipped with the inductive limit topology of Fréchet spaces $\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$ is a unital complex commutative topological algebra since the bilinear mapping

$$(f,g)\in\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})\times\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})\to f\cdot g\in\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$$

is continuous.

Let $(N,(M_{N,n})_{n\in\mathbb{N}})\in\mathbb{N}\times{\mathbb{R}^+}^{\mathbb{N}}$ and the convex and balanced subset be defined as

$$B_{(N,(M_{N,n})_{n\in\mathbb{N}})}=\{f\in\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})\colon\ \forall\, n\in\mathbb{N},\ \|f\|_{N,n}\leqslant M_{N,n}\}.$$

The family $(B_{(N,(M_{N,n})_{n\in\mathbb{N}})})_{(N,(M_{N,n})_{n\in\mathbb{N}})\in\mathbb{N}\times{\mathbb{R}^+}^{\mathbb{N}}}$ of subsets satisfies

(1) $B_{(N,(M_N,n)_{n\in\mathbb{N}})}+B_{(N',(M'_{N'},n)_{n\in\mathbb{N}})}\subseteq B_{(N'',(M''_{N''},n)_{n\in\mathbb{N}})}$, where for all $n\in\mathbb{N}$, $M''_{N'',n}=M_{N,n}+M'_{N',n}$ and $N''=\sup(N,N')$,

(2) $\lambda\cdot B_{(N,(M_N,n)_{n\in\mathbb{N}})}\subseteq B_{(N,(|\lambda|\cdot M_N,n)_{n\in\mathbb{N}})}$ for all $\lambda\in\mathbb{C}$,

(3) $B_{(N,(M_N,n)_{n\in\mathbb{N}})}\cdot B_{(N',(M'_{N'},n)_{n\in\mathbb{N}})}\subseteq B_{(N'',(M''_{N''},n)_{n\in\mathbb{N}})}$, where for all $n\in\mathbb{N}$, $M''_{N'',n}=\sum\limits_{p=0}^{n}M_{N,n-p}\cdot M'_{N',p}$.

This family $(B_{(N,(M_{N,n})_{n\in\mathbb{N}})})_{(N,(M_{N,n})_{n\in\mathbb{N}})\in\mathbb{N}\times{\mathbb{R}^+}^{\mathbb{N}}}$ of subsets is a natural convex vector bornology basis (see [9], [10]) compatible with the vector structure of the algebra $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$.

A set $B\subset\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$ is bounded if there exists a pair $(N,(M_{N,n})_{n\in\mathbb{N}})$ in $\mathbb{N}\times{\mathbb{R}^+}^{\mathbb{N}}$ such that $B\subset B_{(N,(M_{N,n})_{n\in\mathbb{N}})}\subset\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$.

$\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$ equipped with this family of bounded sets becomes a unital complex complete commutative convex bornological algebra (see [9], [10]). $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$ is a unital complex commutative LF-algebra.

Theorem 2.3 ensures that the restriction mapping

$$\mathcal{R}_N\colon\ \mathcal{C}_N^{(1)}(T_\varepsilon,\delta,\delta_0,\mathbb{C})\to\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$$

is continuous. Since

$$\mathcal{C}^{(1)}(T_\varepsilon,\delta,\delta_0,\mathbb{C})=\varinjlim_{N}\mathcal{C}_N^{(1)}(T_\varepsilon,\delta,\delta_0,\mathbb{C})\text{ and }\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})=\varinjlim_{N}\mathcal{C}_N^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C}),$$

(see [1], [2], [12]) there exists a unique continuous restriction mapping

$$\mathcal{R}\colon\ f\in\mathcal{C}^{(1)}(T_\varepsilon,\delta,\delta_0,\mathbb{C})\to\mathcal{R}(f)=f|_{\mathbb{R}^+}\in\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+,\delta_0,\mathbb{C})$$

satisfying for every positive integer $N \in \mathbb{N}$ the commutative diagram

$$\begin{tikzcd} \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \arrow[r, "\mathcal{R}_N"] \arrow[d, "j_N"'] \arrow[dr, "\mathcal{P}_N"] & \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \arrow[d, "i_N"] \\ \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \arrow[r, "\mathcal{R}"] & \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}), \end{tikzcd} \tag{2.2}$$

where

$$\mathcal{P}_N = i_N \circ \mathcal{R}_N = \mathcal{R} \circ j_N \quad \forall\, N \in \mathbb{N}$$

and

$i_N\colon\ \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \to \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ is the continuous canonical injection,
$j_N\colon\ \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ is the continuous canonical injection,
$\mathcal{R}_N\colon\ \mathcal{C}_N^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to \mathcal{C}_N^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ is the continuous restriction mapping.

3. Bornological isomorphism between $\widehat{\mathbb{C}_\delta(-\Delta)}$ and $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$

In Section 1, we considered the Laplace operator acting on $\mathcal{S}'(\mathbb{R}^n)$ (the space of tempered distributions on $\mathbb{R}^n$) and δ being the Lipschitz function considered in the first paragraph.

The spectrum of the Laplace operator is $\mathbb{R}^+$.

$\mathbb{C}_\delta(-\Delta)$ denotes the unital commutative complex algebra generated by the Laplace operator $-\Delta$ and the resolvent function $(-\Delta - t)^{-1}$, $t \notin \delta^{-1}(0) = \mathbb{R}^+$ equipped with the described convex vector bornology. Its completion convex bornological algebra is $\widehat{\mathbb{C}_\delta(-\Delta)}$ (see [8], [9], [10]).

$\Phi_{\mathbb{C}_\delta(-\Delta)} = (\varphi_x)_{x \in \mathbb{R}^+}$ denotes the set of all bounded multiplicative linear functionals on the convex bornological algebra $\mathbb{C}_\delta(-\Delta)$ identified with $\mathbb{R}^+$.

The Gelfand transform is the algebra homomorphism defined by

$$u \in \mathbb{C}_\delta(-\Delta) \to \widehat{G}(u) = \widehat{u}\colon\ \mathbb{R}^+ \to \mathbb{C},$$
$$\widehat{u}(\varphi_x) = \varphi_x(u) = u(x) \in \mathbb{C}, \quad x \in \mathbb{R}^+.$$

But $u \in \mathbb{C}_\delta(-\Delta)$, so it is an operator of the form

$$u = \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} C_{k_i, k'_j} (-\Delta)^{k_i} (-\Delta - t)^{-k'_j}, \quad t \in \mathbb{C} \setminus \mathbb{R}^+,\ C_{k_i, k'_j} \in \mathbb{C},\ k_i, k'_j, m, n \in \mathbb{N}.$$

So,

$$u(x) = \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} C_{k_i, k'_j} \frac{x^{k_i}}{(x - t)^{k'_j}} \quad \forall\, t \in \mathbb{C} \setminus \mathbb{R}^+.$$

For a fixed t the function of variable $x \in \mathbb{R}^+$

$$\delta(t)u(x) = \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} C_{k_i,k'_j}\delta(t)\frac{x^{k_i}}{(x-t)^{k'_j}} \quad \forall\, t \in \mathbb{C} \setminus \mathbb{R}^+$$

is of class $\mathcal{C}^{(\infty)}$ and k-tempered on $\mathbb{R}^+$ since by Leibniz formula, the derivative of order n of the function u with respect to the variable x is

$$\delta(t)u^{(n)}(x) = \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} C_{k_i,k'_j}\delta(t)\sum_{p=0}^{n}\binom{n}{p}(x^{k_i})^{(n-p)}\Big(\frac{1}{(x-t)^{k'_j}}\Big)^{(p)}.$$

It is a finite sum of terms each one of which is at most k-tempered on $\mathbb{R}^+$ since we have

$$\frac{\delta(t)}{|x-t|^l} \leqslant 1 \quad \forall\, l \in \mathbb{N},\ \forall\, x \in \mathbb{R}^+,\ \forall\, t \in \mathbb{C} \setminus \mathbb{R}^+.$$

Thus, we have

$$\forall\, n \in \mathbb{N}, \quad \sup_{x \in \mathbb{R}^+} \delta_0^k(x)\delta(t)|u^{(n)}(x)| < \infty,$$

where $k = \sup\limits_{0 \leqslant i \leqslant m} k_i \in \mathbb{N}$.

Obviously, the Gelfand transform $\widehat{G}\colon \mathbb{C}_\delta(-\Delta) \to \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ is injective and bounded algebra homomorphism. It can be extended (see [8], [9], [10]) as a unique bounded algebra homomorphism $\widetilde{\widehat{G}}\colon \widehat{\mathbb{C}_\delta(-\Delta)} \to \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ satisfying the diagram

(3.1)
$$\begin{tikzcd} \mathbb{C}_\delta(-\Delta) \arrow[r, "\mathcal{J}"] \arrow[dr, "\widehat{G}"'] & \widehat{\mathbb{C}_\delta(-\Delta)} \arrow[d, "\widetilde{\widehat{G}}"] \\ & \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}), \end{tikzcd}$$

where $\widetilde{\widehat{G}} \circ \mathcal{J} = \widehat{G}$ and $\mathcal{J}\colon \mathbb{C}_\delta(-\Delta) \to \widehat{\mathbb{C}_\delta(-\Delta)}$ is the canonical bounded algebra homomorphism.

Theorem 3.1. *The continuous restriction mapping*

$$\mathcal{R}\colon \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \to \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$$

is surjective.

P r o o f. Let $f \in \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \Rightarrow \exists\,(N, (M_N, n)_{n\in\mathbb{N}}) \in \mathbb{N} \times {\mathbb{R}^+}^{\mathbb{N}}$ such that

$$\forall\, n \in \mathbb{N}, \quad \sup_{x\in\mathbb{R}^+} \delta_0^N(x)|f^{(n)}(x)| = M_{N,n} < \infty.$$

Applying Borel theorem and Lemma 3.1 of [7] for all $\varepsilon > 0$, f can be extended as a $\mathcal{C}^{(1)}$ function $\widetilde{f}_\varepsilon$ defined in $[-\varepsilon, \infty[+ \mathbb{R}\mathrm{i}$ by a locally finite series

$$\widetilde{f}_\varepsilon(x + y\mathrm{i}) = \sum_{n=0}^{\infty} \frac{1}{n!} f_\varepsilon^{(n)}(x)\varphi_n(y)(y\mathrm{i})^n$$

such that

$$\frac{\partial}{\partial \overline{z}} \widetilde{f}_\varepsilon(z)|_{\{z=x\}} = 0 \quad \forall\, x \in [-\varepsilon, \infty[,$$

where $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{C}^{(\infty)}$ positive functions with compact support in $]-1, 1[$ satisfying

(1) $\varphi_n(y) = 1$ if $|y| \leqslant \frac{1}{2}\alpha_n^{-1}$ for all $n \in \mathbb{N}$,
(2) $\operatorname{supp}(\varphi_{n+1}) \subset \operatorname{supp}(\varphi_n)$ for all $n \in \mathbb{N}$,

where

$$\forall\, n \in \mathbb{N}, \quad M_{N,n} = \sup_{x\in[-\varepsilon,\infty[} \delta_0^N(x)|f_\varepsilon^{(n)}(x)| \quad \text{and} \quad \alpha_n = \sum_{p=0}^{n+1}(M_{N,p} + 1).$$

$\widetilde{f}_\varepsilon$ is defined on the sector T_ε since $T_\varepsilon \subset [-\varepsilon, \infty[+ \mathbb{R}\mathrm{i}$.

A calculation leads to

$$\|\widetilde{f}_\varepsilon\|_{N+2} = \sup_{z\in T_\varepsilon} \delta_0^{N+2}(z)|\widetilde{f}_\varepsilon(z)| + \frac{1}{2\pi}\int_{T_\varepsilon} \delta_0^{N+2}(t)\delta^{-1}(t)\Big|\frac{\partial}{\partial \overline{t}}\widetilde{f}_\varepsilon(t)\,\mathrm{d}\overline{t}\,\mathrm{d}t\Big| < \infty,$$

where $\delta_0(z) = 1/\sqrt{1 + |z|^2}$ and $\delta(z) = \inf\{1, d^n(z, \mathbb{R}^+)$ for all $n \in \mathbb{N}\}$.

Thus $\widetilde{f}_\varepsilon \in \mathcal{C}_{N+2}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \subset \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$. □

Since the bounded restriction algebra homomorphism $\mathcal{R}$ is surjective (Theorem 3.1) there exists a bounded algebra homomorphism

$$\widetilde{T}\colon\ \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}) \to \widehat{\mathbb{C}_\delta(-\Delta)}$$

satisfying the commutative diagram

$$\begin{tikzcd} \mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C}) \arrow[r, "T"] \arrow[dr, "\mathcal{R}"'] & \widehat{\mathbb{C}_\delta(-\Delta)} \\ & \mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C}), \arrow[u, "\widetilde{T}"'] \end{tikzcd} \tag{3.2}$$

where $\widetilde{T} \circ \mathcal{R} = T$, and $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ is the algebra of $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered functions on $\mathbb{R}^+$.

$\mathcal{C}^{(1)}(T_\varepsilon, \delta, \delta_0, \mathbb{C})$ is the algebra of functions defined in Section 2.

From diagrams 3.1 and 3.2 we get the following equations

$$\widetilde{\widehat{G}} \circ \mathcal{J} = \widehat{G} \quad \text{and} \quad \widetilde{T} \circ \mathcal{R} = T, \tag{3.3}$$

$$\widetilde{T} \circ \widehat{G} = \mathcal{J} \quad \text{and} \quad \widetilde{\widehat{G}} \circ T = \mathcal{R}. \tag{3.4}$$

From equations (3.3) and (3.4) we get the following new equations

$$\widetilde{\widehat{G}} \circ \widetilde{T} = 1_{\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+\delta_0, \mathbb{C})} \quad \text{and} \quad \widetilde{T} \circ \widetilde{\widehat{G}} = 1_{\widehat{\mathbb{C}_\delta(-\Delta)}}, \tag{3.5}$$

where $1_{\widehat{\mathbb{C}_\delta(-\Delta)}}$ and $1_{\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})}$ are, respectively, the identity operators on $\widehat{\mathbb{C}_\delta(-\Delta)}$ and $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$.

The algebras $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ and $\widehat{\mathbb{C}_\delta(-\Delta)}$ are bornologically isomorphic.

The complete convex bornological algebra $\mathcal{C}_{\mathbb{N}}^{(\infty)}(\mathbb{R}^+, \delta_0, \mathbb{C})$ of $\mathcal{C}^{(\infty)}$ $\mathbb{N}$-tempered functions on $\mathbb{R}^+$ is then useful for functional calculus.

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