SOME RESULTS ON SEMI-STRATIFIABLE SPACES

Wei-Feng Xuan, Yan-Kui Song, Nanjing

Received May 9, 2017. Published online April 12, 2018.
Communicated by Pavel Pyrih

Abstract. We study relationships between separability with other properties in semi-stratifiable spaces. Especially, we prove the following statements:

(1) If X is a semi-stratifiable space, then X is separable if and only if X is $DC(\omega_1)$;

(2) If X is a star countable extent semi-stratifiable space and has a dense metrizable subspace, then X is separable;

(3) Let X be a ω-monolithic star countable extent semi-stratifiable space. If $t(X)=\omega$ and $d(X)\leqslant\omega_1$, then X is hereditarily separable.

Finally, we prove that for any T_1-space X, $|X|\leqslant L(X)^{\Delta(X)}$, which gives a partial answer to a question of Basile, Bella, and Ridderbos (2011). As a corollary, we show that $|X|\leqslant e(X)^{\omega}$ for any semi-stratifiable space X.

Keywords: semi-stratifiable space; separable space; dense subset; feebly compact space; ω-monolithic space; property $DC(\omega_1)$; star countable extent space; cardinal equality; countable chain condition; perfect space; G^*_δ-diagonal

MSC 2010: 54D20, 54E35

1. Introduction

All topological spaces in this paper are assumed to be T_1-spaces unless stated otherwise. The notation of semi-stratifiable spaces was first introduced in [5] by Creede in 1970.

Definition 1.1. A space X is called semi-stratifiable (see [5]) if there is a function G which assigns to each $n\in\omega$ and a closed set $H\subset X$, an open set $G(n,H)$ containing H such that

(1) $H=\bigcap\limits_n G(n,H)$;

(2) $H\subset K\Rightarrow G(n,H)\subset G(n,K)$.

The paper is supported by NSFC project 11626131 and 11771029.

DOI: 10.21136/MB.2018.0043-17

It is well known that the class of semi-stratifiable spaces can be characterized by a g-function.

Lemma 1.2 ([5]). *A topological space (X, τ) is semi-stratifiable if there exists a function $g\colon \omega \times X \to \tau$ such that:*

(1) $\{x\} = \bigcap_{n\in\omega} g(n, x)$ *for any* $x \in X$;

(2) *if* $x \in g(n, x_n)$ *for each* n, *then* $x_n \to x$.

This class of spaces lies between the class of semi-metric spaces and the class of spaces in which closed sets are G_δ (i.e. perfect spaces). It turns out that a T_1-space is semi-metric if and only if it is first countable and semi-stratifiable. A completely regular space is a Moore space if and only if it is a semi-stratifiable p-space.

In this paper, we study the relationships between separability with other properties in semi-stratifiable spaces. In Section 3, we prove the following statements:

(1) If X is a semi-stratifiable space, then X is separable if and only if X is $DC(\omega_1)$ (see Theorem 3.6);

(2) If X is a star countable extent semi-stratifiable space and has a dense metrizable subspace, then X is separable (see Theorem 3.12);

(3) Let X be a ω-monolithic star countable extent semi-stratifiable space. If $t(X) = \omega$ and $d(X) \leqslant \omega_1$, then X is hereditarily separable (see Theorem 3.17).

In Section 4, we prove that for any T_1-space X, $|X| \leqslant L(X)^{\Delta(X)}$ (see Theorem 4.2), which gives a partial answer to a question of [4]. As a corollary, we show that $|X| \leqslant e(X)^\omega$ for any semi-stratifiable space X (see Corollary 4.5).

2. Notation and terminology

The cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal and ω_1 the first uncountable cardinal. We also write 2^ω for the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals.

If X is a space and $\mathcal{U}$ is a family of subsets of X, then the star of a subset $A \subset X$ with respect to $\mathcal{U}$ is the set

$$\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon\ U \cap A \neq \emptyset\}.$$

Definition 2.1 ([14]). Let $\mathcal{P}$ be a topological property. A space X is said to be *star* $\mathcal{P}$ if for any open cover $\mathcal{U}$ of X there is a subset $A \subset X$ with property $\mathcal{P}$ such that $\mathrm{St}(A, \mathcal{U}) = X$. The set A will be called a *star kernel* of the cover $\mathcal{U}$.

Therefore, a space X is said to be *star countable extent* (SCE) (see [12]) if for any open cover $\mathcal{U}$ of X there is a subspace $A \subset X$ of countable extent such that $\mathrm{St}(A, \mathcal{U}) = X$. We have the well-known implications:

$$\text{separable} \Rightarrow \text{star countable} \Rightarrow \text{star Lindelöf} \Rightarrow \text{SCE}.$$

In general, none of the implications can be reversed (see [2], [12]).

Definition 2.2 ([10])**.** We say that a space X has *property* $DC(\omega_1)$ if it has a dense subspace every uncountable subset of which has a limit point in X.

Definition 2.3. The *density* of a space X is defined as the smallest cardinal number of the form $|A|$, where A is a dense subset of X; this cardinal number is denoted by $d(X)$.

Definition 2.4. We say that X has *countable tightness* if for any $x \in \bar{A}$ for any A of X there exists a countable subset A_0 of A such that $x \in \overline{A_0}$; it is denoted by $t(X) = \omega$.

Definition 2.5 ([9])**.** The *extent* of a topological space X, denoted by $e(X)$, is the supremum of the cardinalities of closed discrete subsets of X.

Definition 2.6. The *Lindelöf number* is defined in the following way: $L(X) = \min\{\tau\colon$ for any open cover γ there exists a subcover γ' such that $|\gamma'| \leqslant \tau\}$.

Definition 2.7 ([18])**.** We say that a space X has a G_δ*-diagonal* if there is a countable family $\{U_n\colon n \in \omega\}$ of open neighbourhoods of the diagonal Δ_X in the square $X \times X$ such that $\Delta_X = \bigcap\{U_n\colon n \in \omega\}$.

Definition 2.8 ([3])**.** A space X has a *strong rank* 1*-diagonal* or G_δ^**-diagonal* if there exists a sequence $\{\mathcal{U}_n\colon n \in \omega\}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap\{\overline{\mathrm{St}(x, \mathcal{U}_n)}\colon n \in \omega\}$.

Definition 2.9. A topological space X is called *perfect* if every closed subset of X is a G_δ-set.

Definition 2.10. A space X is *subparacompact* if every open cover of X has a σ-discrete closed refinement.

Definition 2.11 ([15])**.** A space X has *countable chain condition* (abbreviated as CCC) if any disjoint family of open sets in X is countable, that is, the Souslin number (or cellularity) of X is at most ω.

All notations and terminology not explained in the paper are given in [6].

With the aid of the following lemma, we can deduce Proposition 3.2.

Lemma 3.1 ([8]). *Every semi-stratifiable space is perfect, subparacompact and has a G_δ-diagonal. Moreover, if the space is regular, then it has a G_δ^*-diagonal.*

Proposition 3.2. *Every Tychonoff pseudocompact semi-stratifiable space is separable.*

Proof. Since every regular semi-stratifiable space has a G_δ^*-diagonal (i.e. strong rank 1-diagonal) by Lemma 3.1, the conclusion is an easy corollary of [3], Theorem 3.12. □

Theorem 3.3 ([5]). *In a semi-stratifiable space X, the following statements are equivalent:*

(1) *X is Lindelöf;*
(2) *X is hereditarily separable;*
(3) *X has countable extent.*

Lemma 3.4 ([5]). *A semi-stratifiable space is hereditarily semi-stratifiable.*

Lemma 3.5. *If X is a perfect space and D is an uncountable discrete subset of X, then there exists an uncountable subset $E \subset D$ which is closed and discrete in X.*

Proof. Let $\mathcal{U} = \{U(d)\colon d \in D\}$ be an uncountable family of open subsets of X such that $U(d) \cap D = \{d\}$ for each $d \in D$. Since X is perfect, there are closed subsets F_n for $n \in \omega$ such that

$$\bigcup_{d\in D} U_d = \bigcup_{n\in\omega} F_n.$$

It is evident that there is an uncountable subset $E = D \cap F_{n_0} \subset X$ for some $n_0 \in \omega$. Now we show that E is closed and discrete in X. Suppose it is not, then there is an accumulation point ξ for E. Since F_{n_0} is closed, we have

$$\xi \in F_{n_0} \subset \bigcup_{n\in\omega} F_n = \bigcup_{d\in D} U_d.$$

Therefore there exists $d' \in D$ such that $\xi \in U(d')$, and hence $U(d')$ shall contain infinite points of E, which contradicts with the choice of $\mathcal{U}$. This completes the proof. □

Theorem 3.6. *If X is a semi-stratifiable space, then X is separable if and only if X is $DC(\omega_1)$.*

P r o o f. The necessity yields immediately from the definition of $DC(\omega_1)$. Now we prove the sufficiency. Assume that Y is the dense subspace of X which witnesses that X is $DC(\omega_1)$. We claim that Y is Lindelöf. Suppose it is not. Let $\mathcal{U}$ be an open cover of Y and suppose that $\mathcal{U}$ has no countable subcover. Since Y is semi-stratifiable (and hence subparacompact) by Lemma 3.4, $\mathcal{U}$ has a closed refinement $\mathcal{F} = \bigcup\limits_{n=1}^{\infty} \mathcal{F}_n$, where each $\mathcal{F}_n$ is discrete in Y. Since $\mathcal{U}$ has no countable subcover, there is an n such that $\mathcal{F}_n$ is uncountable. Let D be a subset of Y consisting of exactly one point of each nonempty element of $\mathcal{F}_n$. It is evident that D is uncountable and discrete in Y. Since X is perfect (Lemma 3.1), there exists an uncountable subset $E \subset D \subset Y$ which is closed and discrete in X by Lemma 3.5, which contradicts the hypothesis on Y. It follows from Theorem 3.3 that Y is hereditarily separable, so X is separable since Y is dense in X. □

Corollary 3.7. *Every $DC(\omega_1)$ Moore space is separable.*

P r o o f. Immediately follows from the fact that a Moore space is always semi-stratifiable (see [8], page 484). □

Corollary 3.8. *If a semi-stratifiable space X has a dense subspace of countable extent, then X is separable.*

P r o o f. Let Y be a dense subspace of X of countable extent, then every uncountable subset of Y has an accumulation point in Y. It remains to apply Theorem 3.6. (Note that Corollary 3.8 also follows directly from Theorem 3.3 and Lemma 3.4.) □

Corollary 3.9. *Each semi-stratifiable space with a dense Lindelöf subspace is separable.*

Corollary 3.10. *Each semi-stratifiable space with a dense σ-compact subspace is separable.*

Lemma 3.11 ([12])**.** *Let X be a semi-stratifiable space. The following statements are equivalent:*

(1) *X is star countable;*
(2) *X is star Lindelöf;*
(3) *X is* SCE.

Theorem 3.12. *Let X be a* SCE *semi-stratifiable space. If X has a dense metrizable subspace, then X is separable.*

P r o o f. We claim that X is CCC. Suppose it is not. Let $\mathcal{W} = \{U_\alpha\colon\ \alpha < \omega_1\}$ be an uncountable pairwise disjoint family of nonempty open sets of X. For each $\alpha < \omega_1$, pick a point $x_\alpha \in U_\alpha$ and let $D = \{x_\alpha\colon\ \alpha < \omega_1\}$. It follows from Lemma 3.5 that there exists an uncountable subset $E \subset D$ which is closed and discrete in X, since X is perfect (see Lemma 3.1). Let $\mathcal{U} = \{U_\alpha\colon\ x_\alpha \in E\} \cup \{X \setminus E\}$. Clearly, $\mathcal{U}$ is an open cover for which there is no countable subset A of X such that $\mathrm{St}(A, \mathcal{U}) = X$. This shows that X is not star countable, and therefore X is not SCE (see Lemma 3.11). A contradiction. Let Y be the dense metrizable subspace of X. Since X is CCC, Y is also CCC. Therefore Y and X are separable. $\square$

Corollary 3.13. *If X is a* SCE *semi-stratifiable space and has a dense paracompact subspace, then X is separable.*

P r o o f. Let Y be a dense paracompact subspace of X. Using the proof of Theorem 3.12, it can be shown that Y is CCC. Since every CCC paracompact space is Lindelöf, X has a dense Lindelöf subspace Y. Therefore, by Corollary 3.9, X is separable. $\square$

Corollary 3.14. *If X is a* SCE *semi-stratifiable space and has a dense subspace of isolated points, then X is separable.*

P r o o f. Note that every discrete space is metrizable. $\square$

Corollary 3.15. *If X is a* SCE *semi-stratifiable space and has a dense* GO-*subspace, then X is separable.*

P r o o f. Note that the property of being semi-stratifiable is equivalent to being metrizable for any GO-space. $\square$

Corollary 3.16. *If X is a Čech-complete,* SCE *semi-stratifiable space, then X is separable.*

P r o o f. Since X is Čech-complete, X contains a dense paracompact Čech-complete subspace Y (see [13]). Hence, Y is metrizable (see [6]). Therefore, by Theorem 3.12, X and Y are separable. (Since Y is paracompact, we also can get to the conclusion by Corollary 3.13.) $\square$

For any infinite cardinal κ, a space is called κ-monolithic if $nw(\bar{A}) \leqslant \kappa$ for any set $A \subset X$ with $|A| \leqslant \kappa$.

Theorem 3.17. *Let X be a ω-monolithic,* SCE *and semi-stratifiable space. Then X is hereditarily separable if X satisfies one of the following conditions:*

(1) *X is first countable;*

(2) *$|X| \leqslant \omega_1$;*

(3) *$t(X) = \omega$ and $d(X) \leqslant \omega_1$.*

Proof. (1) It was established in [17] that the extent of a ω-monolithic star countable W-space (see [17], Definition 1.8) is countable, so we have $e(X) = \omega$ since every first countable space is a W-space. Hence, by Theorem 3.3, X is hereditarily separable.

(2) It follows from Proposition 1.16 in [1] that if X is a star countable ω-monolithic space with $|X| = \omega_1$, then $e(X) \leqslant \omega$, so X has countable extent. Hence, by Theorem 3.3, X is hereditarily separable.

(3) Since $d(X) \leqslant \omega_1$, there exists a dense subset A of X with $|A| \leqslant \omega_1$. If $|A| < \omega_1$, it is obvious that X is separable. We assume that $|A| = \omega_1$. Enumerate A as $\{x_\alpha\colon \alpha < \omega_1\}$ and let $F_\alpha = \overline{\{x_\beta \in A\colon \beta < \alpha\}}$ for each $\alpha < \omega_1$. Then we have an ω_1-sequence $\mathcal{F} = \{F_\alpha\colon \alpha < \omega_1\}$ of increasing closed separable subsets of X.

Suppose that there exists a closed and discrete set $D \subset X$ with $|D| = \omega_1$. By ω-monolithity of X, for any subset $F_\alpha \subset X$ we have the inequality $|F_\alpha \cap D| \leqslant \omega < \omega_1$, so we can construct by induction a set $D' = \{d_\alpha\colon \alpha < \omega_1\} \subset D$ and an open expansion $\mathcal{U} = \{U_\alpha\colon \alpha < \omega_1\}$ of D' such that $\alpha \neq \beta$ implies $d_\alpha \neq d_\beta$ while $U_\alpha \cap D' = \{d_\alpha\}$ and $U_\alpha \cap F_\alpha = \emptyset$ for every $\alpha < \omega_1$.

Now we check that $\mathcal{U}$ is point-countable. For any point $x \in X$, $x \in \bar{A}$. Since $t(X) = \omega$, there exists a countable subset A_0 of A such that $x \in \overline{A_0}$, and hence there exists some F_α such that $x \in A_0 \subset F_\alpha$. By the construction of $\mathcal{F}$ and $\mathcal{U}$, it is not difficult to see that $x \in F_\beta$ and $F_\beta \cap U_\beta = \emptyset$ for any $\beta > \alpha$, which implies $x \notin U_\beta$ for any $\beta > \alpha$. This shows that $\mathcal{U}$ is point-countable.

Let $\mathcal{W} = \{U_\alpha\colon \alpha < \omega_1\} \cup \{X \setminus D'\}$. Clearly, $\mathcal{W}$ is an open cover of X. Since X is star countable (see Lemma 3.11), there is a countable subset C of X such that $\mathrm{St}(C, \mathcal{W}) = X$. It is evident that $|\{U_\alpha \in \mathcal{U}\colon U_\alpha \cap C \neq \emptyset\}| \leqslant \omega$, since $\mathcal{U}$ is point-countable. It follows that there exists $U_\beta \in \mathcal{U}$ such that $U_\beta \cap C = \emptyset$ and hence there is $d_\beta \in D'$ such that $d_\beta \notin \mathrm{St}(C, \mathcal{W}) = X$. A contradiction.

This proves that X has countable extent. Hence, by Theorem 3.3, X is hereditarily separable. □

Before giving the main results, let us recall some definitions from [4]. We say that a space X has a G_κ-diagonal if there is a family $\{G_\alpha\colon\ \alpha<\kappa\}$ of open sets in $X\times X$ such that $\Delta_X=\bigcap\limits_{\alpha<\kappa} G_\alpha$, where $\Delta_X=\{(x,x)\colon\ x\in X\}$. The diagonal degree of X, denoted by $\Delta(X)$, is the smallest infinite cardinal κ such that X has a G_κ-diagonal. Clearly, $\Delta(X)=\omega$ if and only if X has a G_δ-diagonal.

The following question was posted in [4] by Basile, Bella, and Ridderbos.

Question 4.1. Does the inequality $|X|\leqslant e(X)^{\Delta(X)}$ hold for any T_1-space X?

We will give a partial answer to this question by proving the following result.

Theorem 4.2. *For any T_1-space X, $|X|\leqslant L(X)^{\Delta(X)}$.*

Proof. Since X is T_1, Δ_X can be written as the intersection of some family of open sets of $X\times X$, so $\Delta(X)$ is well defined. Suppose that $\Delta(X)=\kappa$ and $L(X)=\tau$. Then X has a G_κ-diagonal, i.e. $\Delta_X=\bigcap\{G_\alpha\colon\ \alpha<\kappa\}$, where each G_α is open in $X\times X$. So for each $\alpha<\kappa$ and $x\in X$ there exists an open subset $B_\alpha(x)$ of X containing x, with $B_\alpha(x)\times B_\alpha(x)\subset G_\alpha$. For each $\alpha<\kappa$ let $\mathcal{V}_\alpha$ be a subcover of $\{B_\alpha(x)\colon\ x\in X\}$ such that $\mathcal{V}_\alpha\leqslant\tau$ and $X=\bigcup\{U\colon\ U\in\mathcal{V}_\alpha\}$.

Let $x\in X$. For each $\alpha<\kappa$ we fix $U_{x,\alpha}\in\mathcal{V}_\alpha$ such that $x\in U_{x,\alpha}$. Note that $U_{x,\alpha}$ may not be $B_\alpha(x)$. Now, let $y\in X\setminus\{x\}$. Then there is $\alpha<\kappa$ such that $(x,y)\notin G_\alpha$. Therefore $y\notin U_{x,\alpha}$; otherwise $(x,y)\in U_{x,\alpha}\times U_{x,\alpha}\subset G_\alpha$, a contradiction. This shows that $\{x\}=\bigcap\limits_{\alpha<\kappa} U_{x,\alpha}$.

Since each $U_{x,\alpha}$ could be chosen out of τ many sets, there are τ^κ such possible intersections. Therefore we conclude that $|X|\leqslant\tau^\kappa$. □

The referee reminded us that Theorem 4.2 should be compared to Theorem 4.18 of Gotchev (see [7]): If X is a Urysohn space, then $|X|\leqslant aL(X)^{\bar\Delta(X)}$, where $aL(X)$ is the almost Lindelöf number and $\bar\Delta(X)$ is the regular diagonal degree of a Urysohn space X, i.e. the smallest infinite cardinal κ such that X has a regular G_κ-diagonal, i.e. there is a family $\{G_\alpha\colon\ \alpha<\kappa\}$ of open sets in X^2 such that $\Delta_X=\bigcap\limits_{\alpha<\kappa}\overline{G}_\alpha$. The referee also pointed out that by applying the method of proof in Theorem 4.2, we can also prove Gotchev's result.

For the reader's convenience, we give its new proof: Suppose $\bar\Delta(X)=\kappa$ and $aL(X)=\tau$. Then X has a regular G_κ-diagonal, i.e. $\Delta_X=\bigcap\{\overline{G}_\alpha\colon\ \alpha<\kappa\}$, where each G_α is open in X^2. So for each $\alpha<\kappa$ and $x\in X$ there exists an open subset $B_\alpha(x)$ of X containing x, with $B_\alpha(x)\times B_\alpha(x)\subset G_\alpha$. For each $\alpha<\kappa$ let $\mathcal{V}_\alpha$ be a subcover of $\{B_\alpha(x)\colon\ x\in X\}$ such that $\mathcal{V}_\alpha\leqslant\tau$ and $X=\bigcup\{\overline{U}\colon\ U\in\mathcal{V}_\alpha\}$. Let $x\in X$. For each $\alpha<\kappa$ we fix $U_{x,\alpha}\in\mathcal{V}_\alpha$ such that $x\in\overline{U}_{x,\alpha}$. Now let $y\in X\setminus\{x\}$.

Then there is $\alpha < \kappa$ such that $(x,y) \notin \overline{G}_\alpha$. Therefore $y \notin \overline{U}_{x,\alpha}$; otherwise $(x,y) \in \overline{U}_{x,\alpha} \times \overline{U}_{x,\alpha} \subset \overline{G}_\alpha$, a contradiction. This shows that $\{x\} = \bigcap_{\alpha<\kappa} \overline{U}_{x,\alpha}$. Since each $U_{x,\alpha}$ could be chosen out of τ many sets, there are τ^κ such possible intersections. Therefore we conclude that $|X| \leqslant \tau^\kappa$. The proof is complete. □

Corollary 4.3. *If X is a space with a G_δ-diagonal and $L(X) \leqslant 2^\omega$, then $|X| \leqslant 2^\omega$.*

Since $e(X) = L(X)$ for any D-space X, we have the following corollary by Theorem 4.2.

Corollary 4.4. *If X is a D-space, then $|X| \leqslant e(X)^{\Delta(X)}$.*

Since every semi-stratifiable space is a D-space and has a G_δ-diagonal, we have the following corollary by Theorem 4.2 and Corollary 4.4.

Corollary 4.5. *If X is a semi-stratifiable space, then $|X| \leqslant e(X)^\omega$.*

Proposition 4.6. *If X is a regular semi-stratifiable space, then $|X| \leqslant 2^{d(X)}$.*

Proof. Since a regular and semi-stratifiable space has a strong rank 1-diagonal by Lemma 3.1, it follows that $s\Delta(X) = \omega$ (see [4], page 2). It has been established in [4], Proposition 4.1, that $|X| \leqslant 2^{d(X)s\Delta(X)}$ for any Hausdorff space X, so we have $|X| \leqslant 2^{d(X)\cdot\omega} = 2^{d(X)}$. □

Corollary 4.7. *If X is a regular separable semi-stratifiable space, then $|X| \leqslant 2^\omega$.*

Note that the regularity is necessary in Corollary 4.7, which can be seen in the following example.

Example 4.8 ([11], page 64). Let $\kappa\omega$ denote the Katětov's extension of ω with the discrete topology. Recall that $\kappa\omega = \omega \cup T$, where T is a set of cardinality 2^{2^ω} that indexes the collection of all free ultrafilters on ω. For $t \in T$ let $\mathcal{U}_t$ be the ultrafilter indexed by t; a local base for t is the collection $\{\{t\} \cup U \colon U \in \mathcal{U}_t\}$. The space $\kappa\omega$ has the following properties:

(1) $\kappa\omega$ is Hausdorff and non-regular;
(2) $\kappa\omega$ is separable;
(3) $\kappa\omega$ is semi-stratifiable;
(4) $\kappa\omega = 2^{2^\omega}$.

P r o o f. Points (1), (2) and (4) are obvious. It suffices to prove that $\kappa\omega$ is semi-stratifiable. To see it, define a function $g\colon \omega \times \kappa\omega \to \tau$ such that

$$g(n,x) = \begin{cases} \{x\}, & x \in \omega; \\ \{x\} \cup (\omega \setminus n), & x \in T. \end{cases}$$

Clearly, $\{x\} = \bigcap_{n\in\omega} g(n,x)$ holds for any $x \in \kappa\omega$. Now suppose that $x \in g(n, x_n)$ for every $n \in \omega$. It is not difficult to see that there exists $n_0 \in \omega$ such that $x = x_n$ for any $n \geqslant n_0$ by the definition of g. Hence, we have $x_n \to x$. Therefore, by Lemma 1.2, the space $\kappa\omega$ is semi-stratifiable. This completes the proof. □

We say that a space X satisfies the discrete countable chain condition (DCCC for short) if every discrete family of nonempty open subsets of X is countable.

E x a m p l e 4.9 ([16], Proposition 3.10). For any cardinal κ there exists a regular DCCC and semi-stratifiable space whose cardinality is greater than κ.

Proposition 4.10. *Let X be a semi-stratifiable space and let g be the function which witnesses that X is semi-stratifiable. If $X = \bigcup\{g(n,x)\colon x \in Y\}$ for each $n \in \omega$, then $|X| \leqslant |Y|^\omega$.*

P r o o f. To see it, fix any $x \in X$. For each $n \in \omega$ there exists $x_n \in Y$ such that $x \in g(n, x_n)$ since $X = \bigcup\{g(n,x)\colon x \in Y\}$. It follows from Lemma 1.2 that x is the limit point of the sequence $\{x_n\} \subset Y$. Therefore we have $|X| \leqslant |Y|^\omega$. □

We finish this section with the following questions.

Q u e s t i o n 4.11. Is the cardinality of a regular CCC semi-stratifiable space at most 2^ω?

Q u e s t i o n 4.12. Is the cardinality of a regular SCE and semi-stratifiable space at most 2^ω?

A c k n o w l e d g e m e n t. We would like to thank the referee for his or her valuable remarks and suggestions which greatly improved the paper.

References

[1] *O. T. Alas, L. R. Junqueira, J. van Mill, V. V. Tkachuk, R. G. Wilson*: On the extent of star countable spaces. Cent. Eur. J. Math. *9* (2011), 603–615. zbl MR doi

[2] *O. T. Alas, L. R. Junqueira, R. G. Wilson*: Countability and star covering properties. Topology Appl. *158* (2011), 620–626. zbl MR doi

[3] *A. A. Arhangel'skii, R. Z. Buzyakova*: The rank of the diagonal and submetrizability. Commentat. Math. Univ. Carol. *47* (2006), 585–597. zbl MR

[4] *D. Basile, A. Bella, G. J. Ridderbos*: Weak extent, submetrizability and diagonal degrees. Houston J. Math. *40* (2014), 255–266. zbl MR

[5] *G. D. Creede*: Concerning semi-stratifiable spaces. Pac. J. Math. *32* (1970), 47–54. zbl MR doi

[6] *R. Engelking*: General Topology. Sigma Series in Pure Mathematics 6. Heldermann, Berlin, 1989. zbl MR

[7] *I. S. Gotchev*: Cardinalities of weakly Lindelöf spaces with regular G_κ-diagonals. Avaible at `https://scirate.com/arxiv/1504.01785`.

[8] *G. Gruenhage*: Generalized metric spaces. Handbook of Set-Theoretic Topology (K. Kunen et al., eds.). North-Holland, Amsterdam, 1984, pp. 423–501. zbl MR doi

[9] *R. Hodel*: Cardinal functions. I. Handbook of Set-Theoretic Topology (K. Kunen et al., eds.). North-Holland, Amsterdam, 1984, pp. 1–61. zbl MR

[10] *S. Ikenaga*: Topological concept between Lindelöf and Pseudo-Lindelöf. Research Reports of Nara National College of Technology *26* (1990), 103–108. (In Japanese.)

[11] *I. Juhász*: Cardinal Functions in Topology. Mathematical Centre Tracts 34. Mathematisch Centrum, Amsterdam, 1971. zbl MR

[12] *A. D. Rojas-Sánchez, Á. Tamariz-Mascarúa*: Spaces with star countable extent. Commentat. Math. Univ. Carol. *57* (2016), 381–395. zbl MR doi

[13] *B. E. Šapirovskij*: On separability and metrizability of spaces with Souslin's condition. Sov. Math. Dokl. *13* (1972), 1633–1638; translation from Dokl. Akad. Nauk SSSR *207* (1972), 800–803. zbl

[14] *E. K. van Douwen, G. M. Reed, A. W. Roscoe, I. J. Tree*: Star covering properties. Topology Appl. *39* (1991), 71–103. zbl MR doi

[15] *M. R. Wiscamb*: The discrete countable chain condition. Proc. Am. Math. Soc. *23* (1969), 608–612. zbl MR doi

[16] *W. F. Xuan*: Symmetric g-functions and cardinal inequalities. Topology Appl. *221* (2017), 51–58. zbl MR doi

[17] *Z. Yu*: A note on the extent of two subclasses of star countable spaces. Cent. Eur. J. Math. *10* (2012), 1067–1070. zbl MR doi

[18] *P. Zenor*: On spaces with regular G_δ-diagonal. Pac. J. Math. *40* (1972), 759–763. zbl MR doi

Authors' addresses: *Wei-Feng Xuan*, School of Statistics and Mathematics, Nanjing Audit University, Nanjing, China, 211815, e-mail: `wfxuan@nau.edu.cn`; *Yan-Kui Song*, Institute of Mathematics, School of Mathematical Science, Nanjing Normal University, Nanjing, China, 210046, e-mail: `songyankui@njnu.edu.cn`.