ON A DIVISIBILITY PROBLEM

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Abstract. Let $p_1, p_2, \ldots$ be the sequence of all primes in ascending order. Using explicit estimates from the prime number theory, we show that if $k \geqslant 5$, then

$$(p_{k+1}-1)! \mid (\tfrac{1}{2}(p_{k+1}-1))!\, p_k!,$$

which improves a previous result of the second author.

Keywords: prime; divisibility; exponent; Sándor-Luca's theorem

MSC 2010: 11A25, 11B83

1. Introduction

Let n be a positive integer. The sequence of products of n consecutive positive integers is an important arithmetic sequence in number theory. There are many interesting related problems (see for example [6]–[9]). In 1975, Erdős and Selfridge in [9] proved that the product of two or more consecutive integers is never a power. One can also refer to the results on the index decomposition of prime numbers of $n!$ obtained by Erdős and Graham (see [7], [8]). For the past ten years, many scholars such as Berend in [2], Chen and Zhu in [3], Le in [11], Luca and Stănică in [13], Moree and Roskam in [14], Sándor in [17]–[18] and others studied the arithmetic of $n!$ obtaining many important results.

Let p_k be the kth prime. In 1999, Sándor in [17]–[18] set forward the following conjecture:

Conjecture 1.1 ([17]). *For every $k \geqslant 5$, the integer $(p_k-2)!\, p_k!$ is divisible by $(p_{k+1}-1)!$.*

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In 2000, Luca in [12] confirmed Sándor's conjecture and proved the following theorem.

Theorem 1.2 (Sándor-Luca theorem)**.** *When* $k \geqslant 5$, *we have*

$$(p_{k+1} - 1)! \mid (p_k - 2)!\, p_k!. \tag{1}$$

In 2002, Atanassov in [1] gave a further strengthening of Theorem 1.2 by proving the following result.

Theorem 1.3. *If* $k \geqslant 5$, *then*

$$(p_{k+1} - 1)! \mid p_{k-1}!\, p_k!. \tag{2}$$

In this paper, using explicit estimates from the prime number theory we prove the following result.

Theorem 1.4. *If* $k \geqslant 5$, *we have*

$$(p_{k+1} - 1)! \mid (\tfrac{1}{2}(p_{k+1} - 1))!\, p_k!. \tag{3}$$

R e m a r k 1.5. When $k \geqslant 5$, we have $\frac{1}{2}(p_{k+1} - 1) < p_{k-1}$. Therefore, Theorem 1.4 improves Theorem 1.2 and Theorem 1.3. If $\frac{1}{2}(p_{k+1} - 1) = p$, where p is a prime, then $p \parallel (\frac{1}{2}(p_{k+1} - 1))!$, where we use the notation $d^k \parallel n$ to mean that $d^k \mid n$ but $d^{k+1} \nmid n$. Then, we get

$$(p_{k+1} - 1)! \parallel (\tfrac{1}{2}(p_{k+1} - 1))!\, p_k!.$$

Therefore, Theorem 1.4 shows that the best (smallest) answer to the question of what is m such that $(p_{k+1}-1)! \mid m!\, p_k!$ is $m = \frac{1}{2}(p_{k+1}-1)$. For example, taking $k = 22$, we have $82! \mid 41!\, 79!$. Obviously, 41 cannot be replaced by any smaller positive integer in the above divisibility relation.

We organize this paper as follows. In Section 2, we prove Theorem 1.4. In Section 3, we prove a corollary to Theorem 1.4. Furthermore, setting

$$Q_k = \frac{(p_{k+1} - 1)!}{p_k!},$$

we propose a problem concerning numbers k for which the prime factors of Q_k occupy an initial interval of primes. We suggest a conjectural answer and provide some heuristic and numerical evidence in order to support it.

2. Proof of Theorem 1.4

When $k \geqslant 5$, formula (3) is equivalent to

$$(p_k+1)(p_k+2)\dots(p_{k+1}-1) \mid (\tfrac{1}{2}(p_{k+1}-1))!. \tag{4}$$

If p_k, p_{k+1} are two adjacent primes, then $p_k+1, p_k+2, \dots, p_{k+1}-1$ are all composite numbers and each of their prime factors p satisfies $p \leqslant \frac{1}{2}(p_{k+1}-1)$. Thus, for every prime $2 \leqslant p \leqslant \frac{1}{2}(p_{k+1}-1)$, we just need to prove that

$$v_p((p_k+1)(p_k+2)\dots(p_{k+1}-1)) \leqslant v_p((\tfrac{1}{2}(p_{k+1}-1))!), \tag{5}$$

where $v_p(n)$ denotes the p-adic valuation of a positive integer n.

We assume that n, l are positive integers such that $l < \frac{1}{2}n$. Let p be a prime satisfying $2 \leqslant p \leqslant n$. Next, we prove that

$$v_p((2n-l+2)(2n-l+3)\dots(2n-1)(2n)) \leqslant v_p(n!), \quad n > 30. \tag{6}$$

Writing

$$n = a_0 + a_1 p + \dots + a_{s-1}p^{s-1} + a_s p^s, \quad 0 \leqslant a_0, a_1, \dots, a_s < p,\ a_s \geqslant 1, \tag{7}$$

we get

$$\begin{aligned} 2n &= 2a_0 + 2a_1 p + \dots + 2a_{s-1}p^{s-1} + 2a_s p^s \\ &= b_0 + b_1 p + \dots + b_{r-1}p^{r-1} + b_r p^r, \end{aligned} \tag{8}$$

where $0 \leqslant b_0, b_1, \dots, b_r < p$ and $b_r \geqslant 1$. From (8), we have $s \leqslant r \leqslant s+1$ and then inequality (6) is equivalent to

$$\sum_{j=1}^{s+1} \Big\lfloor \frac{2n}{p^j} \Big\rfloor - \sum_{j=1}^{s+1} \Big\lfloor \frac{2n-l}{p^j} \Big\rfloor \leqslant \sum_{j=1}^{s+1} \Big\lfloor \frac{n}{p^j} \Big\rfloor, \tag{9}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Since $l < \frac{1}{2}n$, and $\lfloor x \rfloor + \lfloor y \rfloor \geqslant \lfloor x+y \rfloor - 1$, it follows that

$$\begin{aligned} \Big\lfloor \frac{n}{p^j} \Big\rfloor + \Big\lfloor \frac{2n-l}{p^j} \Big\rfloor - \Big\lfloor \frac{2n}{p^j} \Big\rfloor &\geqslant \Big\lfloor \frac{\frac{5}{2}n}{p^j} \Big\rfloor - 1 - \Big\lfloor \frac{2n}{p^j} \Big\rfloor \\ &= \big\lfloor \tfrac{5}{2}(a_s p^{s-j} + \dots + a_{s-j+1}p + a_{s-j}) \big\rfloor \\ &\quad - 2(a_s p^{s-j} + \dots + a_{s-j+1}p + a_{s-j}) - 1 \\ &=: A_j. \end{aligned} \tag{10}$$

Suppose that $s \geqslant 4$. When $j \leqslant s-2$, we have

$$A_j \geqslant \lfloor \tfrac{5}{2}(a_s p^2 + a_{s-1}p) \rfloor - 2(a_s p^2 + a_{s-1}p) - 1 \geqslant 2.$$

When $j = s-1$, we obtain

$$A_j \geqslant \lfloor \tfrac{5}{2} a_s p \rfloor - 2a_s p - 1 \geqslant 0.$$

When $j = s$, one can see that

$$A_j \geqslant \lfloor \tfrac{5}{2} a_s \rfloor - 2a_s - 1 \geqslant -1.$$

When $j = s+1$, we deduce that

$$A_j \geqslant \left\lfloor \frac{\frac{5}{2} a_s}{p} \right\rfloor - \frac{2a_s}{p} - 1 \geqslant -1.$$

Therefore, from (10) we get

$$\sum_{j=1}^{s+1} \left\lfloor \frac{n}{p^j} \right\rfloor + \sum_{j=1}^{s+1} \left\lfloor \frac{2n-l}{p^j} \right\rfloor - \sum_{j=1}^{s+1} \left\lfloor \frac{2n}{p^j} \right\rfloor \geqslant 2 + 0 - 1 - 1 = 0.$$

Thus, we conclude that inequality (9) holds.

If $s = 3$, then $A_j \geqslant -1$ and further

$$\begin{aligned} A_1 &+ A_2 + A_3 + A_4 \\ &\geqslant \left\lfloor \tfrac{5}{2}(a_3 p^2 + a_2 p + a_1) \right\rfloor - 2(a_3 p^2 + a_2 p + a_1) - 1 \\ &\quad + \left\lfloor \tfrac{5}{2}(a_3 p + a_2) \right\rfloor - 2(a_3 p + a_2) - 1 - 1 - 1 \\ &\geqslant \left\lfloor \tfrac{5}{2}(4a_3 + 2a_2) \right\rfloor - 2(4a_3 + 2a_2) + \left\lfloor \tfrac{5}{2} \cdot 2a_3 \right\rfloor - 2 \cdot 2a_3 - 4 \geqslant 0. \end{aligned} \tag{11}$$

So, (9) is verified for $s = 3$ also.

If $s = 2$, when $p \geqslant 7$, we have

$$\begin{aligned} A_1 + A_2 + A_3 &\geqslant \left\lfloor \tfrac{5}{2}(a_2 p + a_1) \right\rfloor - 2(a_2 p + a_1) - 1 - 1 - 1 \\ &\geqslant \left\lfloor \tfrac{35}{2} a_2 \right\rfloor - 14a_2 - 3 \geqslant 0, \end{aligned}$$

so (9) holds. When $p \leqslant 5$, from (7) we get $n < 125$. For $p = 2, 3, 5$, one can directly verify the validity of (9).

If $s = 1$, by (7), we get $n = a_1 p + a_0$. When $a_1 \geqslant 4$, since

$$A_1 + A_2 \geqslant \left\lfloor \tfrac{5}{2} a_1 \right\rfloor - 2a_1 - 1 - 1 \geqslant 10 - 8 - 2 = 0,$$

it follows that formula (9) holds. When $a_1 \leqslant 3$, if $p \geqslant 7$, then (8) implies $2n = 2a_1p + 2a_0$ and $r = s = 1$. As $a_0 < p$, we then get

$$\left\lfloor \frac{2n}{p} \right\rfloor - \left\lfloor \frac{2n-l}{p} \right\rfloor - \left\lfloor \frac{n}{p} \right\rfloor \leqslant \left\lfloor \frac{l}{p} \right\rfloor + 1 - \left\lfloor \frac{n}{p} \right\rfloor \leqslant \left\lfloor \frac{\frac{1}{2}n}{p} \right\rfloor + 1 - \left\lfloor \frac{n}{p} \right\rfloor$$
$$= \left\lfloor \tfrac{1}{2}a_1 + \frac{\frac{1}{2}a_0}{p} \right\rfloor + 1 - a_1 \leqslant a_1 - 1 + 1 - a_1 = 0.$$

Thus, (9) also holds. When $a_1 \leqslant 3$ and $p \leqslant 5$, it follows from (7) that $n < 20$, which contradicts the condition $n > 30$. Thus, if $n > 30$, then (9) holds.

Next, we show that

$$p_{k+1} - p_k < \tfrac{1}{4}(p_{k+1} - 1) \quad \forall\, k \geqslant 7. \tag{12}$$

Notice that inequality (12) is equivalent to

$$p_{k+1} < \tfrac{4}{3}p_k - \tfrac{1}{3} \quad \forall\, k \geqslant 7. \tag{13}$$

We first check (13) numerically for $k \in \{7, 8, 9\}$, and then assume $k \geqslant 10$, so that $p_k > 25$. From [15], the interval $(p_k, \frac{6}{5}p_k)$ contains at least one prime number, the smallest of them being p_{k+1}. Consequently,

$$p_k + 1 < \tfrac{6}{5}p_k \leqslant \tfrac{4}{3}p_k - \tfrac{1}{3},$$

as required.

Thus, $p_{k+1} - p_k < \frac{1}{4}(p_{k+1} - 1)$. Therefore, we can take $n = \frac{1}{2}(p_{k+1} - 1)$ and $l = p_{k+1} - p_k$ and the inequality $l < \frac{1}{2}n$ is satisfied. If $k \geqslant 18$, then $n > 30$, so we can apply (5) and obtain

$$\begin{aligned} v_p((\tfrac{1}{2}(p_{k+1} - 1))!) &\geqslant v_p((p_{k+1} - 1)(p_{k+1} - 2)\dots(p_{k+1} - l + 2)(p_{k+1} - l + 1)) \\ &= v_p((p_{k+1} - 1)(p_{k+1} - 2)\dots(p_k + 2)(p_k + 1)). \end{aligned} \tag{14}$$

Thus, (4) is verified for $k \geqslant 18$. For $5 \leqslant k \leqslant 17$, we verified (5) using Maple. This completes the proof of Theorem 1.4. $\square$

3. A corollary and a problem

Let us recall the following result.

Lemma 3.1. *Let p_n be the nth prime. If $n \geqslant 198$, we have*

$$n(\log n + \log\log n - 1) < p_n < n\Big(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n}\Big). \tag{15}$$

P r o o f. See (2) and (3) of [4]. □

Using Theorem 1.4 and the above prime number estimate (15), we can deduce the following corollary.

Corollary 3.2. *If $k \geqslant 5$, we have*

$$(p_{k+1} - 1)! \mid p_{\lfloor \frac{2}{3}k \rfloor}!\, p_k!. \tag{16}$$

P r o o f. From (15), when $k \geqslant 198$, we have

$$\begin{aligned} p_{\lfloor \frac{2}{3}k \rfloor} &> (\tfrac{2}{3}k - 1)(\log(\tfrac{2}{3}k - 1) + \log\log(\tfrac{2}{3}k - 1) - 1) \\ &> \tfrac{1}{2}(k+1)(\log(k+1) + \log\log(k+1) - \tfrac{1}{2}) \\ &> \tfrac{1}{2}(p_{k+1} - 1). \end{aligned} \tag{17}$$

Then, from (3) we get (16).

For $7 \leqslant k < 198$, using Maple, we can directly verify (16). This completes the proof of Corollary 3.2. □

R e m a r k 3.3. The referee pointed out that for the proof of Corollary 3.2, the weaker upper bound $p_n \leqslant n(\log n + \log\log n)$ valid for $n \geqslant 6$ (see [16], Theorem 3) instead of the upper bound on p_n from (15) suffices.

Since for $k \geqslant 7$, $\lfloor \frac{2}{3}k \rfloor < k - 1$, we conclude that (16) improves (1) and (2).

Consider the standard decomposition

$$Q_k = \frac{(p_{k+1} - 1)!}{p_k!} = (p_k + 1)(p_k + 2)\ldots(p_{k+1} - 1).$$

Calculations show that there are only 35 values of k such that Q_k can be written as the product of the powers of consecutive prime numbers when $k < 1000$. For example, we have

$$\begin{aligned}
Q_8 &= 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11,\\
Q_{70} &= 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13,\\
Q_{85} &= 2^4 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17,\\
Q_{646} &= 2^6 \cdot 3 \cdot 5^2,\\
Q_{936} &= 2 \cdot 3 \cdot 5^2 \cdot 7^2.
\end{aligned}$$

Therefore, it is natural to ask the following question:

Problem 3.4. Are there infinitely many positive integers k such that

$$Q_k = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}, \tag{18}$$

where $s \geqslant 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_s \geqslant 1$?

We offer the following conjecture.

Conjecture 3.5. *There are infinitely many positive integers k verifying the condition of Problem 3.4 but only finitely many for which $p_{k+1} - p_k > 2$.*

Below we offer some heuristics towards Conjecture 3.5. In what follows, we use $c_1, c_2, \ldots$ for positive constants. Assume that k satisfies the condition of Problem 3.4. Crámer's conjecture asserts that $p_{k+1} - p_k = O((\log k)^2)$. We assume that it holds in order to justify our heuristic. Then

$$Q_k \leqslant p_{k+1}^{p_{k+1}-p_k} = \exp(O((\log k)^3)).$$

By the Prime Number Theorem

$$Q_k \geqslant \prod_{p \leqslant p_s} p \geqslant \exp((1+o(1))p_s)$$

as $s \to \infty$. Thus, $p_s \leqslant c_1(\log k)^3$. Letting $P(m)$ be the maximal prime factor of m, it follows that

$$P(m) \leqslant c_1(\log k)^3$$

holds for all $m \in [p_k + 1, \ldots, p_{k+1} - 1]$. Thus, all such m are $c_1(\log k)^3$-smooth. Let $\Psi(x, y)$ be the function that counts the number of $n \leqslant x$ with $P(n) \leqslant y$. By [10], (1.14), we know that for fixed $\alpha > 1$, the following estimate holds:

$$\Psi(x, (\log x)^\alpha) = x^{1-1/\alpha+o(1)} \quad \text{as } x \to \infty.$$

Taking $x = p_{k+1}$ and $\alpha = 3+\varepsilon$ for any $\varepsilon > 0$ small but fixed, and using the fact that $k < p_{k+1}$, we get

$$\Psi(p_{k+1}, c_1(\log k)^3) \leqslant \Psi(p_{k+1}, (\log p_{k+1})^{3+\varepsilon}) \leqslant p_{k+1}^{1-1/(3+\varepsilon)+o(1)}$$

for $k \geqslant c_2(\varepsilon)$. Making ε tend to 0 and k tend to infinity we get that

$$\Psi(p_{k+1}, c_1(\log k)^3) \leqslant p_{k+1}^{2/3+o(1)}$$

as $k \to \infty$. We interpret this as saying that the probability of a number $m \leqslant p_{k+1}$ to be $c_1(\log k)^3$-smooth is

$$\frac{p_{k+1}^{2/3+o(1)}}{p_{k+1}} = \frac{1}{p_{k+1}^{1/3+o(1)}} \quad \text{as } k \to \infty. \tag{19}$$

Assume now that $p_{k+1} - p_k \geqslant 6$. Then $p_k + 1$, $p_k + 2$, $p_k + 3$, $p_k + 4$ are all in $[p_k + 1, p_{k+1} - 1]$ and are all $c_1(\log k)^3$-smooth as $k \to \infty$. Assuming that the events "$n + i \leqslant x$ is y-smooth" are independent for $i = 1, 2, 3, 4$, it follows that the "probability" that $p_k + i$ are all $c_1(\log k)^3$-smooth for $i = 1, 2, 3, 4$ is at most

$$\left(\frac{1}{p_{k+1}^{1/3+o(1)}}\right)^4 = \frac{1}{p_{k+1}^{4/3+o(1)}} \quad \text{as } k \to \infty.$$

The above number is smaller than $p_{k+1}^{-5/4}$ for large k and since the series

$$\sum_{k\geqslant 1} \frac{1}{p_{k+1}^{5/4}} \tag{20}$$

is convergent, we infer that there should be only finitely many instances of k satisfying Problem 3.4 with $p_{k+1} - p_k \geqslant 6$. Assume next that $p_{k+1} - p_k = 4$. Then

$$Q_k = (p_k + 1)(p_k + 2)(p_k + 3) < p_{k+1}^3 = \exp(3 \log p_{k+1}).$$

Since $Q_k \geqslant \exp((1 + o(1))p_s)$, we get that $p_s \leqslant (3 + o(1)) \log k \leqslant (3 + o(1)) \log p_{k+1}$ as $k \to \infty$. By a result of de Bruijn (see [19], Theorem 2), if we put

$$Z = \frac{\log x}{\log y} \log\Big(1 + \frac{y}{\log x}\Big) + \frac{y}{\log y} \log\Big(1 + \frac{\log x}{y}\Big),$$

then

$$\log \Psi(x, y) = (1 + o(1))Z$$

uniformly as x and y tend to infinity. For us, taking $x = p_{k+1}$ and $y = (3 + o(1)) \times \log p_{k+1}$ as $k \to \infty$, a quick calculation shows that

$$Z = (c_3 + o(1))\frac{\log p_{k+1}}{\log\log k}, \quad \text{where } c_3 = \log(1+3) + 3\log(1+\tfrac{1}{3}) = \log\tfrac{256}{27}$$

as $k \to \infty$. In particular, $\Psi(p_{k+1}, (3+o(1))\log p_{k+1}) = p_{k+1}^{o(1)}$ as $k \to \infty$. We interpret this by saying that the probability that $m \leqslant p_{k+1}$ has $P(m) \leqslant (3 + o(1))\log p_{k+1}$ is

$$\frac{p_{k+1}^{o(1)}}{p_{k+1}} = \frac{1}{p_{k+1}^{1+o(1)}} \quad \text{as } k \to \infty.$$

Assuming again that $n + i \leqslant x$ being y-smooth are independent events for $i = 1, 2$ and taking $n = p_k$, we get that the probability that both $p_k + 1$ and $p_k + 2$ are $(3 + o(1))\log p_{k+1}$-smooth is at most

$$\left(\frac{1}{p_{k+1}^{1+o(1)}}\right)^2 = \frac{1}{p_{k+1}^{2+o(1)}}$$

as $k \to \infty$. By the heuristic used at the convergence of series (20), it follows that there should be only finitely many k with $p_{k+1} - p_k = 4$ satisfying Problem 3.4.

When $p_{k+1} - p_k = 2$, then $Q_k = p_k + 1$. So, instead of asking Problem 3.4, we can reformulate the problem by saying:

P r o b l e m 3.6. Are there infinitely many numbers of the form

$$M = p_1^{\alpha_1} \dots p_s^{\alpha_s}$$

for positive s and $\alpha_1, \dots, \alpha_s$ such that both $M - 1$ and $M + 1$ are primes?

If the answer to Problem 3.6 is affirmative, letting k be such that $M - 1 = p_k$, then certainly $M + 1 = p_{k+1}$, therefore for this k we have that $Q_k = M$ satisfies the requirement of Problem 3.4. To see why perhaps there are infinitely many solutions to Problem 3.6, we just take any s, let $p_s = p$ and search for numbers

$$M = p_1^{\alpha_1} \dots p_s^{\alpha_s}, \quad \alpha_1, \dots, \alpha_s \in \{1, 2, \dots, a\}$$

where a is some fixed number. By the Prime Number Theorem, $M \leqslant \exp((a+o(1))p)$ as $s \to \infty$, so

$$\log M \leqslant 2ap \tag{21}$$

as $s \to \infty$. Also by the Prime Number Theorem, the "probability" that a number n is prime is $(1 + o(1))/\log n$ as $n \to \infty$. Assuming that $M - 1$ and $M + 1$ being primes are "almost" independent events, that is, the probability of them being both prime is some constant times $((1+o(1))/\log M)^2$ (an assumption backed up by the Twin Prime Conjecture or the Bateman-Horn heuristics), we would get that the probability that both $M-1$ and $M+1$ are primes is greater than

$$c_4\Big(\frac{1}{\log M}\Big)^2 \geqslant \frac{c_5}{(ap)^2} \quad \text{as } p \to \infty,$$

with $c_5 = \frac{1}{4}c_4$ by (21). Now we vary $(\alpha_1, \ldots, \alpha_s) \in [1, a]$ obtaining

$$a^s = a^{\pi(p)} = \exp\Big((\log a + o(1))\frac{p}{\log p}\Big)$$

choices for M as $p \to \infty$. Thus, the number of "twin primes" obtainable in this way should be at least

$$\frac{c_5 \exp((\log a + o(1))p/\log p)}{(ap)^2} = c_5 \exp\Big((\log a + o(1))\frac{p}{\log p} - 2\log(ap)\Big),$$

an amount which tends to infinity with p. For fun, we took $p = 29$, and $a = 2$ obtaining 41 examples. The largest is

$$M = 181986654305686230 = 2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2.$$

When we increased $a = 2$ to $a = 4$ (keeping the same p), we got 13746 examples. The largest one is

$$\begin{aligned} M &= 93689980217671739074639839737664300 00\\ &= 2^4 \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^3 \cdot 13^4 \cdot 17^3 \cdot 19^4 \cdot 23^4 \cdot 29^4. \end{aligned}$$

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