NEW EXTENSION OF THE VARIATIONAL MCSHANE INTEGRAL OF VECTOR-VALUED FUNCTIONS

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Dedicated to the memory of Professor Štefan Schwabik (1941–2009)

Abstract. We define the Hake-variational McShane integral of Banach space valued functions defined on an open and bounded subset G of m-dimensional Euclidean space $\mathbb{R}^m$. It is a "natural" extension of the variational McShane integral (the strong McShane integral) from m-dimensional closed non-degenerate intervals to open and bounded subsets of $\mathbb{R}^m$. We will show a theorem that characterizes the Hake-variational McShane integral in terms of the variational McShane integral. This theorem reduces the study of our integral to the study of the variational McShane integral. As an application, a full descriptive characterization of the Hake-variational McShane integral is presented in terms of the cubic derivative.

Keywords: Hake-variational McShane integral; variational McShane integral; Banach space; m-dimensional Euclidean space

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1. Introduction and preliminaries

In paper [4], Fremlin studies, in a σ-finite outer regular quasi-Radon measure space, a method of integration of vector-valued functions, which is an essential generalization of the McShane process of integration (see [11]). The method involves (infinite) McShane partitions which are formed by sequences of disjoint measurable sets of finite measure. For a Banach-space valued function defined on a closed interval endowed with the Lebesgue measure, the variational McShane integral has been investigated in [17] by Wu and Xiaobo (who called the integral the strong McShane integral). Wu and Xiaobo showed that a Banach-space valued function is variationally McShane integrable if and only if it is Bochner integrable. Di Piazza and Musial

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have proved a surprising result that, in the case of an arbitrary (even finite) quasi-Radon measure space, the class of variationally McShane integrable functions can be significantly larger, see [2], Theorem 1.

In paper [8], the Hake-Henstock-Kurzweil and the Hake-McShane integrals are defined. These are extensions of the Henstock-Kurzweil and the McShane integrals from m-dimensional closed non-degenerate intervals to open and bounded subsets of $\mathbb{R}^m$. In this paper, we define the Hake-variational McShane integral which is an extension of the variational McShane integral from m-dimensional closed non-degenerate intervals to open and bounded subsets of $\mathbb{R}^m$. Our goal is not a generalization for the sake of generalization. Indeed, Theorem 2.1 reduces the study of our integral to the study of the variational McShane integral. As an application, a full descriptive characterization of the Hake-variational McShane integral is presented in terms of the cubic derivative, see Theorem 2.2.

Throughout this paper, X denotes a real Banach space with its norm $\|\cdot\|$. The Euclidean space $\mathbb{R}^m$ is equipped with the maximum norm. $B_m(t,r)$ denotes an open ball in $\mathbb{R}^m$ with center t and radius $r>0$. We denote by $\mathcal{L}(\mathbb{R}^m)$ the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^m$ and by λ the Lebesgue measure on $\mathcal{L}(\mathbb{R}^m)$. $|A|$ denotes the Lebesgue measure of $A\in\mathcal{L}(\mathbb{R}^m)$. G denotes an open and bounded subset of $\mathbb{R}^m$. We put

$$\mathcal{L}(A)=\{A\cap L\colon\ L\in\mathcal{L}(\mathbb{R}^m)\}$$

for any $A\in\mathcal{L}(\mathbb{R}^m)$.

Let $\alpha=(a_1,\ldots,a_m)$ and $\beta=(b_1,\ldots,b_m)$ with $-\infty<a_j<b_j<\infty$ for $j=1,\ldots,m$. The set $[\alpha,\beta]=\prod\limits_{j=1}^{m}[a_j,b_j]$ is called a *closed non-degenerate interval* in $\mathbb{R}^m$. In particular, if $b_1-a_1=\ldots=b_m-a_m$, then $I=[\alpha,\beta]$ is called a *cube*. Two closed non-degenerate intervals I and J in $\mathbb{R}^m$ are said to be *non-overlapping* if $I^\circ\cap J^\circ=\emptyset$, where I° denotes the *interior* of I. The family of all closed non-degenerate subintervals in $\mathbb{R}^m$ is denoted by $\mathcal{I}$ and the family of all closed non-degenerate subintervals in $E\subset\mathbb{R}^m$ is denoted by $\mathcal{I}_E$.

An interval function $F\colon\mathcal{I}_E\to X$ is said to be an *additive interval function* if for each two non-overlapping intervals $I,J\in\mathcal{I}_E$ such that $I\cup J\in\mathcal{I}_E$ we have

$$F(I\cup J)=F(I)+F(J).$$

A pair (t,I) of a point $t\in E$ and an interval $I\in\mathcal{I}_E$ is called an $\mathcal{M}$*-tagged interval* in E, t is the tag of I. A finite collection $\{(t_i,I_i)\colon\ i=1,\ldots,p\}$ of $\mathcal{M}$-tagged intervals in E is called an $\mathcal{M}$*-partition* in E if $\{I_i\colon\ i=1,\ldots,p\}$ is a collection of pairwise non-overlapping intervals in $\mathcal{I}_E$. Given $Z\subset E$, a positive function $\delta\colon\ Z\to(0,\infty)$ is called a *gauge* on Z. We say that an $\mathcal{M}$-partition $\pi=\{(t_i,I_i)\colon\ i=1,\ldots,p\}$ in E is

▹ $\mathcal{M}$-partition of E if $\bigcup\limits_{i=1}^{p} I_i = E$,

▹ Z-tagged if $\{t_1, \ldots, t_p\} \subset Z$,

▹ δ-fine if for each $i = 1, \ldots, p$ we have $I_i \subset B_m(t_i, \delta(t_i))$.

We now fix an interval $W \in \mathcal{I}$ and let $f\colon W \to X$ be a function. The function f is said to be *McShane integrable* on W if there is a vector $x_f \in X$ such that for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\Bigl\| \sum_{(t,I)\in\pi} f(t)|I| - x_f \Bigr\| < \varepsilon.$$

In this case, vector x_f is said to be the *McShane integral* of f on W and we set $x_f = (M)\int_W f \, \mathrm{d}\lambda$. Function f is said to be *McShane integrable* over a subset $A \subset W$ if the function $f \cdot \mathbb{1}_A\colon W \to X$ is McShane integrable on W, where $\mathbb{1}_A$ is the characteristic function of the set A. The McShane integral of f over A will be denoted by $(M)\int_A f \, \mathrm{d}\lambda$. If $f\colon W \to X$ is McShane integrable on W, then by Theorem 4.1.6 in [13], function f is McShane integrable on each $E \in \mathcal{L}(W)$.

Function $f\colon W \to X$ is said to be *variationally McShane integrable* (or *strongly McShane integrable*) on W if there exists an additive interval function $F\colon \mathcal{I}_W \to X$ such that for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\sum_{(t,I)\in\pi} \| f(t)|I| - F(I) \| < \varepsilon.$$

Function F is said to be the primitive of f. Clearly, if f is variationally McShane integrable with the primitive F, then f is McShane integrable, and by Proposition 3.6.16 in [13] we also have

$$F(I) = (M)\int_I f \, \mathrm{d}\lambda \quad \text{for every } I \in \mathcal{I}_W.$$

For more information about the McShane integral we refer to [17], [2], [4], [5]–[7], [11], [10], [14] and [13].

Given an additive interval function $F\colon \mathcal{I}_W \to X$, a subset $Z \subset W$ and a gauge δ on Z, we define

$$V_{\mathcal{M}}F(Z, \delta) = \sup\Biggl\{ \sum_{(t,I)\in\pi} \|F(I)\|\colon \ \pi \text{ is a } Z\text{-tagged } \delta\text{-fine } \mathcal{M}\text{-partition in } W \Biggr\}.$$

Then we set

$$V_{\mathcal{M}}F(Z) = \inf\{V_{\mathcal{M}}F(Z, \delta)\colon \ \delta \text{ is a gauge on } Z\}.$$

The set function $V_{\mathcal{M}}F(\cdot)$ is said to be the *McShane variational measure generated by* F. It is a Borel metric outer measure on W, see [1] or [15]. The McShane variational measure has been used extensively for studying the primitives (indefinite integrals) of real functions. See e.g. paper [1] by Di Piazza, [12] by Pfeffer for relations to integration and the fundamental general work [16] by Thomson.

An additive interval function $F\colon \mathcal{I}_G \to X$ is said to be *strongly absolutely continuous* (sAC) on $E \subset G$ if for each $\varepsilon > 0$ there exists $\eta > 0$ such that for each finite collection $\{I_1, \ldots, I_p\}$ of pairwise non-overlapping subintervals in $\mathcal{I}_E$ we have

$$\sum_{i=1}^{p} |I_i| < \eta \Rightarrow \sum_{i=1}^{p} \|F(I_i)\| < \varepsilon.$$

Assume that a point $t \in G$ is given. We set

$$\mathcal{I}_G(t) = \{I \in \mathcal{I}_G\colon\ t \in I,\ I \text{ is a cube}\}.$$

We say that F is *cubic differentiable* at t if there exists a vector $F'_{\mathrm{c}}(t) \in X$ such that

$$\lim_{\substack{I \in \mathcal{I}_G(t)\\ |I| \to 0}} \frac{F(I)}{|I|} = F'_{\mathrm{c}}(t);$$

$F'_{\mathrm{c}}(t)$ is said to be the *cubic derivative* of F at t.

A sequence (I_k) of pairwise non-overlapping intervals in $\mathcal{I}_G$ is said to be a *division* of G if

$$\bigcup_{k=1}^{\infty} I_k = G.$$

We denote by $\mathscr{D}_G$ the family of all divisions of the set G. By Lemma 2.43 in [3], the family $\mathscr{D}_G$ is not empty.

An additive interval function $F\colon \mathcal{I}_G \to X$ is said to be a *strong-Hake-function* if for each division (I_k) of G we have:

- ▷ the series $\sum\limits_{\{k\colon |I \cap I_k| > 0\}} \|F(I \cap I_k)\|$ converges in $\mathbb{R}$ for each $I \in \mathcal{I}$,
- ▷ $F(I) = \sum\limits_{\{k\colon |I \cap I_k| > 0\}} F(I \cap I_k)$ for all $I \in \mathcal{I}_G$.

We say that the additive interval function $F\colon \mathcal{I}_G \to X$ has the *strong-$\mathcal{M}$-negligible variation* over a subset $Z \subset \mathbb{R}^m$ if for each $\varepsilon > 0$ there exists a gauge δ_ε on Z such that for each Z-tagged δ_ε-fine $\mathcal{M}$-partition π in $\mathbb{R}^m$ we have:

- ▷ the series $\sum\limits_{\{k\colon |I \cap I_k| > 0\}} F(I \cap I_k)$ is unconditionally convergent in X for each $(t, I) \in \pi$,

▷ $\sum_{(t,I)\in\pi}\Big\|\Big(\sum_{\{k:|I\cap I_k|>0\}} F(I\cap I_k)\Big)\Big\| < \varepsilon$

whenever (I_k) is a division of G. We say that F has *strong-$\mathcal{M}$-negligible variation outside* of G if F has the strong-$\mathcal{M}$-negligible variation over $G^{\rm c} = \mathbb{R}^m \setminus G$.

We say that a function $f\colon G \to X$ is *Hake-variationally McShane integrable* on G with the primitive $F\colon \mathcal{I}_G \to X$ if we have:

▷ for each $\varepsilon > 0$ there exists a gauge δ on G such that for each δ-fine $\mathcal{M}$-partition π in G we have

$$\sum_{(t,I)\in\pi} \|f(t)|I| - F(I))\| < \varepsilon,$$

▷ F is the strong-Hake-function,

▷ F has the strong-$\mathcal{M}$-negligible variation outside of G.

In this case, we define the Hake-variational McShane integral of f over I as

$$(HvM)\int_I f\,\mathrm{d}\lambda = F(I) \quad \forall\, I \in \mathcal{I}_G.$$

2. The main results

Since G is a bounded subset of $\mathbb{R}^m$, there exists $I_0 \in \mathcal{I}$ such that $G \subset I_0$. Given a function $f : G \to X$, we define a function $f_0\colon I_0 \to X$ as

$$f_0(t) = \begin{cases} f(t) & \text{if } t \in G,\\ 0 & \text{if } t \in I_0 \setminus G.\end{cases}$$

Theorem 2.1. *Assume that a division (C_k) of G, a function $f\colon G \to X$ and an additive interval function $F\colon \mathcal{I}_G \to X$ are given. Define*

$$f_k = f|_{C_k} \quad \text{and} \quad F_k = F|_{\mathcal{I}_{C_k}} \quad \text{for each } k \in \mathbb{N}.$$

Then the following statements are equivalent:

(i) *f is Hake-variationally McShane integrable on G with the primitive F,*

(ii) *f_0 is variationally McShane integrable on I_0 with the primitive F_0 such that $F_0(I) = F(I)$ for all $I \in \mathcal{I}_G$,*

(iii) *for each $k \in \mathbb{N}$, function f_k is variationally McShane integrable on C_k with the primitive F_k, F is a strong-Hake function and has the strong-$\mathcal{M}$-negligible variation outside of G.*

P r o o f. (i) $\Rightarrow$ (iii): Assume that f is Hake-variationally-McShane integrable on G with the primitive F. Then given $\varepsilon > 0$ there exists a gauge δ on G such that for each δ-fine $\mathcal{M}$-partition π in G we have

$$\sum_{(t,I)\in\pi} \|f(t)|I| - F(I)\| < \varepsilon. \tag{2.1}$$

We can choose $\delta(t)$ such that $B_m(t,\delta(t)) \subset G$ for all $t \in G$.

Since F is a strong-Hake function and has the strong-$\mathcal{M}$-negligible variation outside of G, it is enough to prove that each f_k is variationally McShane integrable on C_k with the primitive F_k. Let π_k be a δ_k-fine $\mathcal{M}$-partition of C_k, where $\delta_k = \delta|_{C_k}$. Then π_k is a δ-fine $\mathcal{M}$-partition in G and therefore

$$\sum_{(t,I)\in\pi_k} \|f_k(t)|I| - F_k(I)\| = \sum_{(t,I)\in\pi_k} \|f(t)|I| - F(I)\| < \varepsilon.$$

This means that f_k is variationally McShane integrable on C_k with the primitive F_k.

(iii) $\Rightarrow$ (ii): Assume that (iii) holds and an arbitrary $\varepsilon > 0$ is given. Then, since each function f_k is variationally McShane integrable on C_k with the primitive F_k, by Lemma 3.6.15 in [13] there exists a gauge δ_k on C_k such that for each δ_k-fine $\mathcal{M}$-partition π_k in C_k we have

$$\sum_{(t,I)\in\pi_k} \|f_k(t)|I| - F_k(I)\| \leqslant \frac{\varepsilon}{2^k}. \tag{2.2}$$

Note that for $t \in G$ we have the following possible cases:

▷ There exists $i_0 \in \mathbb{N}$ such that $t \in (C_{i_0})^\circ$;

▷ There exists $j_0 \in \mathbb{N}$ such that $t \in C_{j_0} \setminus (C_{j_0})^\circ$. In this case, there exists a finite set $\mathcal{N}_t = \{j \in \mathbb{N}\colon\ t \in C_j \setminus (C_j)^\circ\}$ such that $t \in \bigcap\limits_{j\in\mathcal{N}_t} C_j$ and $t \notin C_k$ for all $k \in \mathbb{N} \setminus \mathcal{N}_t$.
Hence, $t \in \Big(\bigcup\limits_{j\in\mathcal{N}_t} C_j\Big)^\circ$, where $\Big(\bigcup\limits_{j\in\mathcal{N}_t} C_j\Big)^\circ$ is the interior of $\bigcup\limits_{j\in\mathcal{N}_t} C_j$.
We can choose each δ_k so that $B_m(t,\delta_k(t)) \subset C_k$ if $t \in (C_k)^\circ$, and

$$B_m(t,\delta_k(t)) \subset \bigcup_{j\in\mathcal{N}_t} C_j \quad \text{if } t \in C_k \setminus (C_k)^\circ.$$

Since F has the strong-$\mathcal{M}$-negligible variation outside of G, there exists a gauge δ_v on $G^{\rm c}$ such that for each $G^{\rm c}$-tagged δ_v-fine $\mathcal{M}$-partition π_v in $\mathbb{R}^m$ we have

$$\sum_{(t,I)\in\pi_v} \bigg\|\Big(\sum_{\{k\colon |I\cap C_k|>0\}} F(I\cap C_k)\Big)\bigg\| < \varepsilon. \tag{2.3}$$

By hypothesis, we have also that F is a strong-Hake-function. Therefore we can define an additive interval function $F_0\colon \mathcal{I}_{I_0} \to X$ as

$$F_0(I) = \sum_{\{k\colon |I\cap C_k|>0\}} F(I\cap C_k) \quad \forall\, I \in \mathcal{I}_{I_0}.$$

Clearly, $F_0(I) = F(I)$ for all $I \in \mathcal{I}_G$. We will show that f_0 is variationally McShane integrable on I_0 with the primitive F_0. To see this, we first define a gauge $\delta_0\colon I_0 \to (0,\infty)$ as follows. For any $t \in G$ we choose $\delta_0(t) = \delta_{i_0}(t)$ if $t \in (C_{i_0})^\circ$, and $\delta_0(t) = \min\{\delta_j(t)\colon j \in \mathcal{N}_t\}$ otherwise. If $t \in I_0 \setminus G$, then we choose $\delta_0(t) = \delta_v(t)$. Let π be an arbitrary δ_0-fine $\mathcal{M}$-partition of I_0. Then $\pi = \pi_a \cup \pi_b \cup \pi_c$, where

$$\pi_a = \{(t,I) \in \pi\colon\ (\exists\, i_0 \in \mathbb{N})\,[t \in (C_{i_0})^\circ]\}$$

and

$$\pi_b = \{(t,I) \in \pi\colon\ (\exists\, j_0 \in \mathbb{N})[t \in C_{j_0} \setminus (C_{j_0})^\circ]\},$$
$$\pi_c = \{(t,I) \in \pi\colon\ t \in I_0 \setminus G\}.$$

Therefore

$$\begin{aligned}(2.4)\quad &\sum_{(t,I)\in\pi} \|f_0(t)|I| - F_0(I)\| \\ &= \sum_{(t,I)\in\pi_a} \|f(t)|I| - F(I)\| + \sum_{(t,I)\in\pi_b} \|f(t)|I| - F(I)\| + \sum_{(t,I)\in\pi_c} \|F_0(I)\|.\end{aligned}$$

Note that if we define

$$\pi_a^k = \{(t,I)\colon\ (t,I) \in \pi_a,\ t \in (C_k)^\circ\}$$

and

$$\pi_b^k = \{(t, I\cap C_k)\colon\ (t,I) \in \pi_b,\ t \in C_k \setminus (C_k)^\circ,\ |I\cap C_k| > 0\},$$

then π_a^k and π_b^k are δ_k-fine $\mathcal{M}$-partitions in C_k. Therefore by (2.2) it follows that

$$\begin{aligned}(2.5)\quad \sum_{(t,I)\in\pi_a} \|f(t)|I| - F(I)\| &= \sum_k \Bigg(\sum_{\substack{t\in(C_k)^\circ\\ (t,I)\in\pi_a}} \|f_k(t)|I| - F_k(I)\| \Bigg) \\ &= \sum_k \Bigg(\sum_{(t,I)\in\pi_a^k} \|f_k(t)|I| - F_k(I)\| \Bigg) \leqslant \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon\end{aligned}$$

and

$$\begin{aligned}
(2.6)\qquad & \sum_{(t,I)\in\pi_b} \|f(t)|I| - F(I)\| \\
&= \sum_{(t,I)\in\pi_b} \Big\| \Big(\sum_{\substack{|I\cap C_j|>0\\ j\in\mathcal{N}_t}} (f(t)|I\cap C_j| - F(I\cap C_j)) \Big) \Big\| \\
&= \sum_{(t,I)\in\pi_b} \Big\| \Big(\sum_{\substack{|I\cap C_j|>0\\ j\in\mathcal{N}_t}} (f_j(t)|I\cap C_j| - F_j(I\cap C_j)) \Big) \Big\| \\
&\leqslant \sum_{(t,I)\in\pi_b} \Big(\sum_{\substack{|I\cap C_j|>0\\ j\in\mathcal{N}_t}} \|f_j(t)|I\cap C_j| - F_j(I\cap C_j)\| \Big) \\
&\leqslant \sum_k \Big(\sum_{(t,I)\in\pi_b^k} \|f_k(t)|I\cap C_k| - F_k(I\cap C_k)\| \Big) \leqslant \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.
\end{aligned}$$

By (2.3), the equality

$$\sum_{(t,I)\in\pi_c} \|F_0(I)\| = \sum_{(t,I)\in\pi_c} \Big\| \Big(\sum_{\{k\colon |I\cap C_k|>0\}} F(I\cap C_k) \Big) \Big\|$$

together with the fact that π_c is a G^{c}-tagged δ_v-fine $\mathcal{M}$-partition in $\mathbb{R}^m$ yields

$$\sum_{(t,I)\in\pi_c} \|F_0(I)\| < \varepsilon.$$

Hence, by (2.4), (2.5) and (2.6) we obtain

$$\sum_{(t,I)\in\pi} \|f_0(t)|I| - F_0(I)\| < 3\varepsilon,$$

and since π is an arbitrary δ_0-fine $\mathcal{M}$-partition of I_0, it follows that f_0 is variationally McShane integrable on I_0 with the primitive F_0.

(ii) $\Rightarrow$ (i): Assume that f_0 is variationally McShane integrable on I_0 with the primitive F_0 such that $F_0(I) = F(I)$ for all $I \in \mathcal{I}_G$. By Lemma 3.6.15 in [13], given $\varepsilon > 0$ there exists a gauge δ_0 on I_0 such that for each δ_0-fine $\mathcal{M}$-partition π in I_0 we have

$$(2.7)\qquad \sum_{(t,I)\in\pi} \|f_0(t)|I| - F_0(I)\| < \varepsilon.$$

We can choose δ_0 so that $B_m(t,\delta_0(t)) \subset G$ for all $t \in G$. Hence, if we define $\delta = \delta_0|_G$, then for each δ-fine $\mathcal{M}$-partition π in G we have

$$\sum_{(t,I)\in\pi} \|f(t)|I| - F(I)\| < \varepsilon.$$

Thus, it remains to show that F is a strong-Hake function and has strong-$\mathcal{M}$-negligible variation outside of G. Let (I_k) be an arbitrary division of G.

We first show that F is a strong-Hake function. Since f_0 is variationally McShane integrable on I_0, $\|f_0\|$ is McShane integrable on I_0 . Hence, by Theorem 4.1.11 and Theorem 7.5.4 in [13] we obtain

$$(2.8)\qquad (M)\int_G \|f_0\|\,\mathrm{d}\lambda = \sum_k (M)\int_{I_k} \|f_0\|\,\mathrm{d}\lambda = \sum_k V_{\mathcal{M}}F_0(I_k),$$

and since

$$V_{\mathcal{M}}F_0(I_k) \geqslant V_{\mathcal{M}}F_0(I\cap I_k) \geqslant V_{\mathcal{M}}F(I\cap I_k) \geqslant \|F(I\cap I_k)\| \quad \text{for each } I\in\mathcal{I},$$

it follows that the series $\sum\limits_{\{k\colon |I\cap I_k|>0\}} \|F(I\cap I_k)\|$ converges in $\mathbb{R}$. By hypothesis, for each $I\in\mathcal{I}_G$ we have also

$$\begin{aligned} F(I) = F_0(I) = (M)\int_I f_0\,\mathrm{d}\lambda &= (M)\int_{\bigcup\limits_k (I\cap I_k)} f_0\,\mathrm{d}\lambda = \sum_k (M)\int_{I\cap I_k} f_0\,\mathrm{d}\lambda \\ &= \sum_k F_0(I\cap I_k) = \sum_{\{k\colon |I\cap I_k|>0\}} F(I\cap I_k). \end{aligned}$$

Thus, F is a strong-Hake-function.

It remains to prove that F has the strong-$\mathcal{M}$-negligible variation outside of G. To see this, we first define a gauge $\delta_v\colon G^{\mathrm{c}} \to (0,\infty)$ as follows: $\delta_v(t) = \delta_0(t)$ if $t\in Z = I_0\setminus G$, while for $t\notin I_0$ we choose $\delta_v(t)$ so that $B_m(t,\delta_v(t))\cap I_0 = \emptyset$. Assume that π_v is a G^{c}-tagged δ_v-fine $\mathcal{M}$-partition in $\mathbb{R}^m$. Hence,

$$\pi_Z = \{(t, I\cap I_0)\colon (t,I)\in\pi_v,\ t\in Z,\ |I\cap I_0|>0\}$$

is a δ_0-fine $\mathcal{M}$-partition in I_0. Then by (2.7), it follows that

$$\varepsilon \geqslant \sum_{(t,J)\in\pi_Z} \|f_0(t)|J| - F_0(J)\| = \sum_{(t,J)\in\pi_Z} \|F_0(J)\|$$

$$
\begin{aligned}
&= \sum_{(t,J)\in\pi_Z} \Bigl\|(M)\int_J f_0\,\mathrm{d}\lambda\Bigr\| = \sum_{(t,J)\in\pi_Z} \Bigl\|(M)\int_{J\cap G} f_0\,\mathrm{d}\lambda\Bigr\| \\
&= \sum_{(t,J)\in\pi_Z} \Bigl\|\sum_k (M)\int_{J\cap I_k} f_0\,\mathrm{d}\lambda\Bigr\| = \sum_{(t,J)\in\pi_Z} \Bigl\|\sum_{\{k\colon |J\cap I_k|>0\}} F_0(J\cap I_k)\Bigr\| \\
&= \sum_{(t,J)\in\pi_Z} \Bigl\|\sum_{\{k\colon |J\cap I_k|>0\}} F(J\cap I_k)\Bigr\| = \sum_{(t,I)\in\pi_v} \Bigl\|\sum_{\{k\colon |I\cap I_k|>0\}} F(I\cap I_k)\Bigr\|.
\end{aligned}
$$

This means that F has the strong-$\mathcal{M}$-negligible variation outside of G, and this ends the proof. $\square$

The following theorem shows a full descriptive characterization of Hake-variation-McShane integral.

Theorem 2.2. *Assume that a division (C_k) of G, a function $f\colon G\to X$ and an additive interval function $F\colon \mathcal{I}_G\to X$ are given. Then the following statements are equivalent:*

(i) *f is Hake-variationally McShane integrable with the primitive F,*

(ii) *$F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t)=f(t)$ at almost all $t\in G$, F is sAC on each C_k, F is a strong-Hake function and has the strong-$\mathcal{M}$-negligible variation outside of G.*

Proof. (i) $\Rightarrow$ (ii): Assume that f is Hake-variationally McShane integrable with the primitive F. Then by Theorem 2.1, for $I_0\in\mathcal{I}$ with $G\subset I_0$, the function f_0 is variationally McShane integrable on I_0 with the primitive F_0 such that $F(I)=F_0(I)$ for all $I\in\mathcal{I}_G$. Hence, by Theorem 1.4 in [9], F_0 is sAC on I_0, $(F_0)'_{\mathrm{c}}(t)$ exists and $(F_0)'_{\mathrm{c}}(t)=f_0(t)$ at almost all $t\in I_0$.

Since $\mathcal{I}_G\subset\mathcal{I}_{I_0}$, F is sAC on G and therefore F is sAC on each C_k.

Fix an arbitrary $k\in\mathbb{N}$ and $t\in(C_k)^\circ$ such that $(F_0)'_{\mathrm{c}}(t)=f_0(t)$. It follows that

$$
\lim_{\substack{I\in\mathcal{I}_{I_0}(t)\\ |I|\to 0}} \frac{F_0(I)}{|I|} = \lim_{\substack{I\in\mathcal{I}_G(t)\\ |I|\to 0}} \frac{F(I)}{|I|}.
$$

Thus, $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t)=f(t)$. Since k and t are arbitrary, the last result holds at almost all $t\in\bigcup_k(C_k)^\circ$, and since

$$
\Bigl|G\setminus\bigcup_k(C_k)^\circ\Bigr| = 0,
$$

it follows that $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t)=f(t)$ at almost all $t\in G$.

By the definition of the Hake-variationally McShane integrability, we have also that F is a strong-Hake function and has the strong-$\mathcal{M}$-negligible variation outside of G.

(ii) $\Rightarrow$ (i): Assume that (ii) holds and define

$$f_k = f|_{C_k} \quad \text{and} \quad F_k = F|_{\mathcal{I}_{C_k}} \quad \text{for each } k \in \mathbb{N}.$$

Then each F_k is sAC on C_k, $(F_k)'_{\rm c}(t)$ exists and $(F_k)'_{\rm c}(t) = f_k(t)$ at almost all $t \in C_k$. Therefore by Theorem 1.4 in [9], each f_k is variational McShane integrable on C_k with the primitive F_k. Therefore by Theorem 2.1, f is Hake-variationally McShane integrable with the primitive F, and this ends the proof. □

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