AN APPLICATION OF LIE GROUPOIDS TO A RIGIDITY PROBLEM OF 2-STEP NILMANIFOLDS

Hamid-Reza Fanaï, Teheran, Atefeh Hasan-Zadeh, Guilan

Received May 26, 2017. Published online August 2, 2018.
Communicated by Dagmar Medková

Abstract. We study a problem of isometric compact 2-step nilmanifolds M/Γ using some information on their geodesic flows, where M is a simply connected 2-step nilpotent Lie group with a left invariant metric and Γ is a cocompact discrete subgroup of isometries of M. Among various works concerning this problem, we consider the algebraic aspect of it. In fact, isometry groups of simply connected Riemannian manifolds can be characterized in a purely algebraic way, namely by normalizers. So, suitable factorization of normalizers and expression of a vector bundle as an associated fiber bundle to a principal bundle, lead us to a general framework, namely groupoids. In this way, drawing upon advanced ingredients of Lie groupoids, normal subgroupoid systems and other notions, not only an answer in some sense to our rigidity problem has been given, but also the dependence between normalizers, automorphisms and especially almost inner automorphisms, has been clarified.

Keywords: nilpotent Lie group; isometric nilmanifolds; normalizer; Lie algebroid; normal subgroupoid system; inner automorphism

MSC 2010: 53C24, 22A22, 22F05

1. Introduction

Rigidity problems of compact nilmanifolds via some information about their geodesic flows have been studied well for compact surfaces with nonpositive Gaussian curvature (see [5], [6], [21]), and locally symmetric spaces (see [2], [3]). In this paper, we consider the geodesic conjugacy problem for compact 2-step nilmanifolds, specifically:

Problem 1.1 (Rigidity Problem, [8]). Are two compact 2-step nilmanifolds M/Γ and M'/Γ' isometric or not if they have conjugated geodesic flows?

The research has been supported by the Research Council of Sharif University of Technology, and Fouman Faculty of Engineering, College of Engineering, University of Tehran.

DOI: 10.21136/MB.2018.0041-17

In the special case when the geodesic conjugacy between their sphere bundles is the restriction of a symplectic diffeomorphism, this problem has been solved in [12] and its extension to Poisson structure has been proved in [9].

However, in the general case, most important rigidity theorems are Mostow's rigidity theorem about locally symmetric spaces of noncompact type (see [10], [13], and [14]) for hyperbolic spaces, also its generalization, i.e. Gromow's rigidity theorem in locally irreducible setting (see [1], [4], and [19]) for hyperbolic setting. All of these results show that the study of isometric manifolds reduces in some sense to the study of isomorphisms of isometric groups. The spirit of all these results is the same: given a sufficiently large class of isometries, the existence of an isomorphism between these classes implies that two manifolds are isometric and the isometry is induced by an automorphism (in reality a diffeomorphism). This generic approach can be found in new works, such as [18].

Inspired by it and after explanation of the rigidity problem in Section 2, we choose an algebraic approach to study our problem. In fact, the isometry group of a simply connected Riemannian manifold can be characterized in a purely algebraic way, namely by normalizers. But any suitable factorization of normalizers can be useful only for finite or finitely generated groups while we work on isometry groups and automorphisms of Lie groups. Furthermore, in expression of a vector bundle as an associated fiber bundle to a principal bundle, the notion of a groupoid automatically appears. These lead us to this general setting, which is studied in Section 2.2. Passing from the case of actions on groups to actions on vector bundles, the notions of automorphisms and isometries admit straightforward generalizations, which however, have rarely been spelled out in the literature. Our methodology has been presented in Section 3, step by step.

Using powerful notions of Lie groupoid theory, we will be able to give an answer in some sense to our rigidity problem in Problem 4.1, which can be applied to the analytic results for 2-step nilpotent Lie algebras such as Corollary 4.2. Also, in Theorem 4.1 and Corollary 4.1, we gain more results, which highlight the dependence of normalizers on automorphisms induced by them, for Lie groups and their Lie algebras.

2. Preliminaries

2.1. Explanation of the problem. Let $(M/\Gamma, g)$ and $(M'/\Gamma', g')$ be two compact 2-step Riemannian nilmanifolds. (M, M' are simply connected 2-step nilpotent Lie groups and Γ, Γ' are discrete subgroups of M, M', respectively, with Riemannian metrics whose lifts to M, M', respectively, are left invariant.) Let $\mathcal{M}$, $\mathcal{M}'$ be Lie algebras of M, M', respectively. Lifting a homeomorphism $F\colon\ T(M/\Gamma)\setminus\{0\}\to$

$T(M'/\Gamma') \setminus \{0\}$ which intertwines the geodesic flows to the map $\widetilde{F}\colon M \times (\mathcal{M} \setminus \{0\}) \to M' \times (\mathcal{M}' \setminus \{0\})$, as explained in [8], we have:

Theorem 2.1 ([8]). *Suppose $(M/\Gamma, g)$ and $(M'/\Gamma', g')$ are compact 2-step Riemannian nilmanifolds, and $F_*\colon \Gamma \to \Gamma'$ is an isomorphism which induces a marking between their length spectra. Then there exists a Γ-almost inner automorphism Φ of M and an isometric automorphism $\Psi : (M, g) \to (M', g')$ with $\Psi(\Phi(\Gamma)) = \Gamma'$ such that $F_* = \Psi_{|\Phi(\Gamma)} \circ \Phi_{|\Gamma}$.*

Note 2.1. $\varphi \in \Phi(\Gamma)$ is a Γ-almost inner automorphism of M if for every element $\gamma \in \Gamma$ there exists an element $a \in M$, possibly depending on γ, such that $\varphi(\gamma) = a_\gamma^{-1} \gamma a_\gamma$. The set of Γ-almost inner automorphisms is denoted by $AIA(\Gamma)$ and similarly, $AID(\Gamma)$ is its infinitesimal counterpart, namely Γ-almost inner derivations.

Let Ψ be as in Theorem 2.1. Then it induces an isometry, also denoted by Ψ, from $(M/\Phi(\Gamma), g)$ to $(M'/\Gamma', g')$, thus we may replace $(M'/\Gamma', g')$ by $(M/\Phi(\Gamma), g)$. Moreover, we replace $\Psi_*^{-1} \circ F\colon T(M/\Gamma) \setminus \{0\} \to T(M/\Phi(\Gamma)) \setminus \{0\}$ by F and $\Psi_*^{-1} \circ \widetilde{F}\colon TM \setminus \{0\} \to TM \setminus \{0\}$ by $\widetilde{F}$. The new F is a geodesic conjugacy from $(M/\Gamma, g)$ to $(M/\Phi(\Gamma), g)$ and the new $\widetilde{F}\colon TM \setminus \{0\} \to TM \setminus \{0\}$ is a lift of F which satisfies $\widetilde{F} \circ dL_\gamma = dL_{\Phi(\gamma)} \circ \widetilde{F}$ for all $\gamma \in \Gamma$. So, it intertwines the geodesic flow of (M, g). Then for Γ as a properly discontinuous group of isometries of M we have the characterization of the isometry group of M/Γ in a purely algebraic way.

Theorem 2.2 ([20]). *Let Γ be a properly discontinuous group of isometries of a simply connected Riemannian manifold M. Then the group $I(M/\Gamma)$ of isometries of M/Γ is isomorphic to $N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normalizer of Γ in $I(M)$.*

Now, $(M/\Phi(\Gamma), g)$ and $(M'/\Gamma', g')$ as isometric manifolds have isomorphic isometry groups (by the map which takes the isometry α of $(M/\Phi(\Gamma), g)$ to isometry $\beta\alpha\beta^{-1}$ of $(M'/\Gamma', g')$, $\beta \in I(M'/\Gamma')$), and as in the proof of Theorem 2.2 (see [20]), without details, one time for M' and Γ' and another time for M and $\Phi(\Gamma)$, respectively, we have

$$\frac{N(\Gamma')}{\Gamma'} \simeq \frac{N(\Phi(\Gamma))}{\Phi(\Gamma)}. \tag{2.1}$$

Note 2.2. If Γ-almost inner automorphisms equal to inner automorphisms, then $M/\Phi(\Gamma)$ would be isometric to M/Γ, hence

$$\frac{N(\Phi(\Gamma))}{\Phi(\Gamma)} \simeq \frac{N(\Gamma)}{\Gamma}. \tag{2.2}$$

This isometry is induced from the left translations of (M, g). Therefore, it is reasonable to seek some assumptions which give us equality (2.2).

Corollary 2.1. *The problem of checking, whether two 2-step nilmanifolds with conjugated geodesic flows are isometric or not, reduces to the examination of automorphisms induced only by a part of their isometry groups, i.e. their normalizers.*

2.2. Lie groupoids. In this section, we recall some facts from groupoid theory. In fact, any suitable factorization of normalizers, that we might need as stated in Corollary 2.1, can be applied only for finite or finitely generated groups, while we use groups such as $I(M)$ and $AIA(\Gamma)$, which are not necessarily finitely generated; this will be solved by Lie groupoids.

Basic notions of Lie groupoids and Lie algebroids which will be used through this paper and which we will not describe in details can be found in Chapters 1, 2 and 3 of [17]. However, for the convenience of the reader we give some main concepts of it. There are relations between some concepts of Lie groupoids and the notions of Lie groups mentioned up to now:

By Proposition 1.3.5 of [17], there is the mutual correspondence between a principal bundle $P(M, G, \pi)$ and a locally trivial Lie groupoid Ω, namely Lie groupoid $\Omega \rightrightarrows M$, whose anchor map is a surjective submersion. Proposition 1.3.9 of [17] says that the associated inner Lie group bundle $(\Omega_m \times \Omega_m^m)/\Omega_m^m$ defined by the inner automorphism action of Ω_m^m on itself, carries to the inner subgroupoid $I\Omega$ of locally trivial Lie groupoid Ω. Finally, left invariant metrics on nilmanifolds, adjoint and exponential formulas which are necessary for us, covered to a generalized element of Lie groupoid $G \rightrightarrows M$ named as bisection, a smooth map $\sigma\colon M \to G$ which is right-inverse to $s\colon G \to M$ and is such that $t \circ \sigma\colon M \to M$, is a diffeomorphism.

Let $\Omega \rightrightarrows M$ be locally trivial, and $\Omega * \widehat{M} \to \widehat{M}$ be an action of Ω on a surjective submersion $q : \widehat{M} \to M$. Then $(\widehat{M}, q, M)$ is a fiber bundle by taking a section atlas $\{\sigma_i\colon U_i \to \Omega_m\}$, $m \in M$ and with it defining charts $\psi_i\colon U_i \times \widehat{M}_m \to \widehat{M}_{U_i}, (x, a) \mapsto \sigma_i(x)a$. Now, define $\Omega_m \times \widehat{M}_m \to \widehat{M}$ by $(\xi, a) \mapsto \xi a$. In terms of the charts ψ_i for $\widehat{M}$ and $(x, g) \mapsto \sigma_i(x)g$ for Ω_m, this is $U_i \times \Omega_m^m \times \widehat{M}_m \to U_i \times \widehat{M}_m$, $(x, g, a) \mapsto (x, ga)$, and is then a surjective submersion. Hence $(\Omega_m \times \widehat{M}_m)/\Omega_m^m \to \widehat{M}$, $\lfloor \xi, a \rfloor \to \xi a$ is a diffeomorphism and is equivariant with respect to the isomorphism $(\Omega_m \times \Omega_m)/\Omega_m^m \to \Omega$, $\langle \eta, \xi \rangle \mapsto \eta\xi^{-1}$ from gauge groupoid of the vertex bundle at m. These structures can be applied to inner automorphism action $\Omega * I\Omega \to I\Omega$, $(\xi, \varsigma) \mapsto I_\xi(\varsigma) = \xi\varsigma\xi^{-1}$. In this way, for Ω corresponding to a principal bundle $P(M, G)$, inner subgroupoid $I\Omega$ is equivariantly isomorphic to the inner group bundle $(P \times G)/G$.

Definition 2.1 ([17]). Let $\Omega * \widehat{M} \to \widehat{M}$ be an action of a locally trivial Lie groupoid $\Omega \rightrightarrows M$ on a surjective submersion $(\widehat{M}, p, M)$. Then $\lambda \in \Gamma\widehat{M}$ is Ω-deformable if for all $x, y \in M$ there exists $\xi \in \Omega_x^y$ such that $\xi\lambda(x) = \lambda(y)$. If

$\lambda \in \Gamma\widehat{M}$ is Ω-deformable, then the stabilizer subgroupoid of Ω at λ is $\Omega\{\lambda\} = \{\xi \in \Omega\colon\ \xi\lambda(s(\xi)) = \lambda(t(\xi))\}$.

N o t e 2.3. $\Omega\{\lambda\}$ is closed in Ω since $\widehat{M}$ is Hausdorff. Also, a section λ is Ω-deformable if and only if its values lie in a single orbit. In this case the stabilizer subgroupoid is transitive. For example, Baer groupoid, as a refinement of the conjugacy relation on subgroups of a group, is a special case of stabilizer subgroupoids with $\xi = gH$ and $x = y$.

Now, define $f\colon\ \Omega \to \widehat{M} \times_q \widehat{M}$ by $\xi \mapsto (\lambda(t(\xi)), \xi\lambda(s(\xi)))$. Then $\Omega\{\lambda\} = f^{-1}(\Delta_{\widehat{M}})$. From Note 2.3 and since f is transversal to $\Delta_{\widehat{M}}$ in $\widehat{M} \times_q \widehat{M}$, $\Omega\{\lambda\}$ is a closed embedded reduction of Ω, namely a wide Lie subgroupoid of Ω such that it is locally trivial by itself. Also, by the paragraph before Definition 2.1, let the fiber bundle $\widehat{M} = (\Omega_m \times (\Omega_m^m/\Upsilon_m^m))/\Omega_m^m$, where Υ is a reduction of Ω for which the vertex groups are embedded subgroups of the vertex groups of Ω. Now, let $\Omega * \widehat{M} \to \widehat{M}$ be the associated action $\xi\lfloor\eta, g\Upsilon_m^m\rfloor = \lfloor\xi\eta, g\Upsilon_m^m\rfloor$. Then λ with $\lambda(x) = \lfloor\zeta, \Upsilon_m^m\rfloor$ for $x \in M$ and $\zeta \in \Upsilon_m^x$ is a well-defined smooth Ω-deformable section of $\widehat{M}$, and $\Omega\{\lambda\} = \Upsilon$.

N o t e 2.4. The above mutual structure gives a classification of those closed embedded reductions of a locally trivial Lie groupoid which have a specified vertex group at a chosen point of the base.

3. Proposed methodology

In this section, we apply the ingredients from Lie groupoid theory to our rigidity problem and some questions around it, step by step.

Step 3.1.

Definition 3.1 ([17]). A normal subgroupoid system in $G \rightrightarrows M$ is a triple $\aleph = (N, R, \theta)$, where N is a closed, embedded, wide Lie subgroupoid of G, R is a closed, embedded, wide Lie subgroupoid of the pair groupoid $M \times M$, and θ is an action of R on the map $s\colon\ G \cdot\!\!\cdot\!\!\cdot N \to M$, of the set of right cosets of N as g ranges through G to M, such that the following conditions hold:

(N1) Consider $(n, m) \in R$ and $Ng \in G \cdot\!\!\cdot\!\!\cdot N$ with $s(Ng) = m$. Then writing $\theta(n, m)(Ng) = Nh$, we have $(t(h), t(g)) \in R$.

(N2) For any $(n, m) \in R$ we have $\theta(n, m)(N1_m) = N1_n$.

(N3) Consider $(n, m) \in R$ and $Ng \in G \cdot\!\!\cdot\!\!\cdot N$ with $s(Ng) = m$, and $h \in G$ with $s(h) = t(g)$. Then if $\theta(n, m)(Ng) = Ng_1$ and $\theta(t(g_1), t(g))(Nh) = Nh_1$, we have $\theta(n, m)(Nhg) = Nh_1g_1$.

By applying (N1) of Definition 3.1 to $\tau \in N$, as $\theta(n,m)(Ng) = N\tau = N$, we see that $(t(\tau), s(\tau)) \in R$, so by properties (N2) and (N3), if $Ng \in G \cdot\cdot\cdot N$ with $s(Ng) = s(\tau)$, then (by applying (N3) to Ng and $N\tau^{-1}$), $\theta(t(\tau), s(\tau))(Ng) = Ng\tau^{-1}$. In particular, if τ is an element of inner subgroupoid IN, $g \in G$ and $g\tau g^{-1}$ is defined, then $g\tau g^{-1} \in N$ (since $\theta(t(\tau), s(\tau))(Ng\tau g^{-1}) = Ng\tau g^{-1}\tau^{-1} = N\tau^{-1}$). Also, if $\aleph$ is uniform, namely the anchor of G, restricted to $N \to R$, is a surjective submersion, then $\aleph$ is entirely determined by N, since $\theta(t(\tau), s(\tau))(Ng) = Ng\tau^{-1}$ for $\tau \in N$ with $s(g) = s(\tau)$; this identifies θ. Also, N identifies R as the image of its anchor and both $\aleph$ and $G/\aleph$ are entirely determined by N.

Step 3.2. There is a bijective correspondence between normal subgroupoid systems of G and fibrations $F\colon G \to G'$, $f\colon M \to M'$. Also, from Theorem 2.4.16 of [17], $\aleph$ is uniform if and only if the fibration (F, f) is uniform (i.e. both f and $F^{\Downarrow}\colon G \to f^{\Downarrow}G'$ are surjective submersions).

Step 3.3. Now, by backtrack search in (N3), we can say that the normalizer $N_G(U)$ of a subgroup U of a finite generating group G induces automorphisms of U (by conjugating), and the kernel of this action is the centralizer $C_G(U)$. Thus, we can consider the quotient $N_G(U)/C_G(U)$ as a subgroup of $\mathrm{Aut}(U)$. Then for finitely generated subgroup $U \leqslant G$ we have following Lemma 3.1, which has been proved in [16] by another approach.

Lemma 3.1 ([16])**.** *For $\alpha \in \mathrm{Aut}(U)$, $U = \langle u_1, \ldots, u_m \rangle$ we have that α is induced by $N_G(U)$ if and only if there exists*

(1) $g_1 \in G$ *such that* $u_1^\alpha = u_1^{g_1}$,

(2) $g_2 \in C_G(u_1^{g_1})$ *such that* $u_2^\alpha = (u_2^{g_1})^{g_2}$,

$\vdots$

(m) $g_m \in C_G(u_1^{g_1}, u_2^{g_1 g_2}, \ldots, u_{m-1}^{g_1 g_2 \cdots g_{m-1}})$ *such that* $u_m^\alpha = (u_m^{g_1 g_2 \cdots g_{m-1}})^{g_m}$.

In this case the element $x = g_1 g_2 \ldots g_m$ is an element that induces α. Any other element inducing the same automorphism will differ from x only by an element of $C_G(U)$.

Step 3.4. Now we can use the uniqueness of identifying the centralizers as the stabilizer of conjugacy action as in Definition 2.1 and the concepts mentioned after Note 2.3, and automorphisms induced by this conjugation. Then the similarity between them and normal subgroupoid systems justifies Proposition 2.1.6 of [17] and in a special case of fibrations results in the following classification, which has been obtained in [17] for another setting.

Theorem 3.1 ([17]). *Let $F\colon G \to G'$, $f\colon M \to M'$ be a fibration of Lie groupoids with normal subgroupoid system $\aleph = (N, R(f), \theta)$. Suppose that $\Phi\colon G \to H$ is any morphism of Lie groupoids over a smooth map $\varphi\colon M \to P$ such that:*

(i) *for all $n \in N$, $\Phi(n)$ is an identity of H;*

(ii) *$\varphi \times \varphi$ maps $R(f) = \{(n,m) \in M \times M\colon f(n) = f(m)\}$ into the diagonal of $P \times P$;*

(iii) *if $\bar{\Phi}$ is an induced map $G \cdot\!\!\cdot^{\,\cdot} N \to H$, then $\bar{\Phi}(\theta(n,m)(Ng)) = \bar{\Phi}(Ng)$ for all $(n,m) \in R(f)$ and $Ng \in G \cdot\!\!\cdot^{\,\cdot} N$ with $s(g) = m$.*

Then there is a unique morphism of Lie groupoids $\Psi\colon G' \to H$ over $\psi\colon M' \to P$ such that $\Psi \circ F = \Phi$ and $\psi \circ f = \varphi$. In particular, if (Φ, φ) is a fibration and $\aleph$ is its normal subgroupoid system, then (Ψ, ψ) is an isomorphism of Lie groupoids.

4. Applications of proposed methodology

Now, we come back to 2-step nilmanifolds mentioned in Section 2.2.

From Corollary 2 on page 33 of [22], discrete subgroup Γ of nilpotent Lie group M is finitely generated and Lemma 3.1 is applicable to it, especially for properly discontinuous subgroup $\Gamma = \langle \gamma_1, \ldots, \gamma_{n\rangle}$ of $G = I(M)$, $N_G(\Gamma) = \langle C_G(\Gamma), g_\alpha \rangle$; $\alpha(\langle \gamma_1, \ldots, \gamma_n \rangle) = g_\alpha(\langle \gamma_1, \ldots, \gamma_n \rangle) g_\alpha^{-1}$, where g_α is a composition of particular isometries as in Lemma 3.1.

Also, from Theorem 2.3 of [13], $AIA(M)$ is a connected, simply connected nilpotent Lie group of Step 3.1 (i.e. abelian). Then, from Corollary 1 on page 34 of [22], any automorphism of Γ extends to a unique automorphism of M.

Theorem 4.1. *Extended element $\alpha \in \mathrm{Aut}(\Gamma)$ induced by $N_G(\Gamma)$, $G = I(M)$ as in Lemma 3.1, to $\widetilde{\alpha} \in \mathrm{Aut}(AIA(\Gamma))$ is induced by conjugation in $I(M)$.*

Proof. The first approach: Let the principal bundle $P(M, G, \pi) = \Omega_m(M, \Omega_m^m, \pi)$, $\Omega = \mathrm{Aut}(AIA(\Gamma))$ and let $\widetilde{\alpha}(\alpha, \mathrm{id})$ be an automorphism. Thus $\pi \circ \widetilde{\alpha} = \alpha \circ \pi$ and $\widetilde{\alpha}(ug) = \widetilde{\alpha}(u)g$ for all $u \in \Omega_m$, $g \in \Omega_m^m$, which are sufficient for our nilpotent Lie groups because of left invariant metrics. For $m \in M$ choose $u \in \pi^{-1}(m)$ and write $\sigma(m) = \langle \widetilde{\alpha}(u), u \rangle$. This σ is a bisection of $(\Omega_m \times \Omega_m)/\Omega_m^m$ and I_σ is $\langle v, u \rangle \mapsto \langle \widetilde{\alpha}(v), \widetilde{\alpha}(u) \rangle$, which from Section 2.2 is the automorphism corresponding to $\widetilde{\alpha}\colon \Omega_m \to \Omega_m$.

Conversely, consider the locally trivial Lie groupoid Ω on M, and an inner automorphism $I_\sigma\colon \Omega \to \Omega$ over $t \circ \sigma\colon M \to M$. At each $m \in M$ there is the vertex principal bundle $\Omega_m(M, \Omega_m^m)$. Fix $m_0 \in M$ and write $m_1 = t(\sigma(m_0))$. Then I_σ restricts to $\Omega_{m_0} \to \Omega_{m_1}$. However, we can identify Ω_{m_0} and Ω_{m_1} by right translation by the single element $\sigma(m_0)^{-1}$, without reference to the rest of σ. If we do this, then the restriction of I_σ is identified with the restriction of L_σ.

Automorphisms of principal bundles of the form $\widetilde{\alpha}(\alpha, \mathrm{id})$ thus are either inner automorphisms or left translations, namely gauge transformations. Those for which $f = \mathrm{id}$ correspond to those bisections of $(\Omega_m \times \Omega_m)/\Omega_m^m$ which take values in the inner group bundle $(\Omega_m \times \Omega_m^m)/\Omega_m^m$ which carries to inner subgroupoid $I\Omega$ of Ω as in Section 2.2.

The second approach: Inspired by Section 4 of [16] in relation to equidistributing automorphism and point stabilizer notions and the structure of the proof of Lemma 3.1, our theorem will also be proved in the following way.

By notations in the previous section, there is a transitive action of $\Omega = \widetilde{\alpha}^{-1}(\Delta_{\widehat{M}})$, $\widehat{M} = AIA(\Gamma)$ with a cyclic point stabilizer, namely inner subgroupoid system and since $\Omega/I\Omega$ is isomorphic to the image of (t, s) in $M \times M$, it is uniform. Then the corresponding uniform fibration, as in Step 3.4, maps a point stabilizer to a point stabilizer and the other properties of fibrations of Theorem 3.1 have been satisfied by it. Therefore, it is induced by conjugation in $I(M)$. Finally, from Note 2.4, this selection is sufficient for us, too. □

Corollary 4.1. *Positive answer to the rigidity problem (problem mentioned in the Introduction) results in a bijective correspondence between the cosets* $\{g_\alpha\}_\alpha$ *and* $\{g'_\beta\}_\beta$ *of Theorem* 4.1, *where* $\{g'_\beta\}_\beta$ *are cosets of* $C_G(AIA(\Gamma))$ *in* $N_G(AIA(\Gamma))$ *as in Theorem* 4.1.

Proof. At first, we study the structure of the proof of Theorem 2.2. Let Γ be a properly discontinuous group of $I(M)$. For $\varphi \in N(\Gamma)$ we have $\varphi\Gamma p = \Gamma\varphi p$ for all $p \in M$. Then there is a unique function $\varphi'\colon M/\Gamma \to M/\Gamma$ such that $\varphi' \circ h = h \circ \varphi$, where $h\colon M \to M/\Gamma$ is the natural projection.

Local cross sections show that φ' is smooth and is a diffeomorphism with inversion $(\varphi^{-1})'$. Since h is a local isometry, φ' is an isometry. We can show that the map $\varphi \to \varphi'$ is a homomorphism onto $I(M)$ and its kernel equals to Γ.

Now, for isomorphisms $l\colon I(M/\Phi(\Gamma)) \to N(\Phi(\Gamma))/\Phi(\Gamma)$ and $\kappa\colon I(M/\Gamma) \to N(\Gamma)/\Gamma$ we can say:

(1) l^{-1} affects the element which comes from one conjugacy.

(2) The similar relation is true for κ.

(3) The function $\tau\colon I(M/\Phi(\Gamma)) \to I(M/\Gamma)$ assigns to every element of $I(M/\Phi(\Gamma))$ its conjugacy by an element of $I(M/\Gamma)$.

(4) $\{g_\alpha\}_\alpha$ (or $\{g'_\beta\}_\beta$) are cosets of $C_G(\Gamma)$ (or $C_G(AIA(\Gamma))$) in $N_G(\Gamma)$ (or $N_G(AIA(\Gamma))$), and therefore $g_\alpha = C_G(\Gamma)\cdot a$ (or $g'_\alpha = C_G(\Phi(\Gamma))\cdot a'$), where $a \in N_G(\Gamma)$ (or $a' \in N_G(AIA(\Gamma))$).

(5) l, τ and κ assigns centralizers to the corresponding centralizers.

(6) Finally, $C_G(\Phi(\gamma)) = C_G(a_\gamma \gamma a_\gamma^{-1}) = a_\gamma C_G(\gamma) a_\gamma^{-1}$ for all $\gamma \in \Gamma$ and for all $\Phi \in AIA(\Gamma)$ and therefore $C_G(AIA(\Gamma)) = AIA(C_G(\Gamma))$.

Now, from these 6 steps and from Note 2.2 we obtain the desired bijective correspondence by passing of the following diagram:

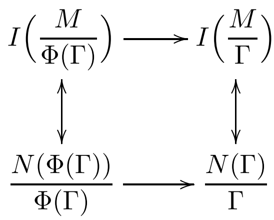

Figure 4.1.

P r o b l e m 4.1. Are the two compact 2-step nilmanifolds M/Γ and M'/Γ' isometric or not if they have conjugated geodesic flows?

A n s w e r . We try to give a partial answer. In fact, from Sections 2 and 3 which are depicted in Figure 4.1, we want to determine whether two quotients $N(\Gamma)/\Gamma$ and $N(\Phi(\Gamma))/\Phi(\Gamma)$ are equal or not if two corresponding nilmanifolds have conjugated geodesic flows. For this we use the Lie groupoid framework explained before. For $\xi\colon T_xM \to T_yM$, $s(\xi)$ is x and $t(\xi)$ is y; the object inclusion map is $x \to 1_x = \mathrm{id}_{T_xM}$, and the partial multiplication is the composition of maps. The inverse of $\xi \in \Upsilon(TM)$ is its inverse as a map. The smooth structure on $\Upsilon(TM)$ is induced from that of TM as follows:

Let $\{\psi_i\colon U_i \times T_mM \to T_{U_i}\}$ be an atlas for TM. For each i and j, define $\overline{\psi}_i^j\colon U_j \times GL(T_mM) \times U_i \to \Upsilon(T)_{U_i}^{U_j}$, $(y, A, x) \mapsto \psi_{j,y} \circ A \circ \psi_{i,x}^{-1}$. Now, from the charts $\psi_i'\colon U_i \times \mathrm{Hom}^1(T_mM, T_mM) \to \mathrm{Hom}^1(TM, TM)_{U_i}$ defined by $\psi_i'(x, f)(\varphi) = \varphi \circ \psi_{i,x}^{-1}$, where $\varphi \in \mathrm{Hom}^1(TM, TM)_x$, and in a similar manner for $n > 1$, we have the smoothness of the action

$$\Upsilon(TM) * \mathrm{Hom}^n(TM; TM) \to \mathrm{Hom}^n(TM; TM); \quad \xi\varphi = \varphi \circ (\xi^{-1})^n. \tag{4.1}$$

Now for Riemannian structure $\langle\cdot,\cdot\rangle$ in TM, regarded as a section of $\mathrm{Hom}^2(TM; M \times \mathbb{R})$, $\langle\cdot,\cdot\rangle$ is $\Upsilon(TM)$-deformable with respect to action (4.1), since any two vector spaces of the same dimension with any positive-definite inner products are isometric. Thus, we have that the stabilizer subgroupoid of $\langle\cdot,\cdot\rangle$ is locally trivial Lie groupoid on M. In fact, a section of this subgroupoid is the same as a moving frame of TM and the local triviality from it is equivalent to the existence of moving frames in TM. In particular, by applying this structure to Lie groupoid $\mathrm{Inn}(\Gamma)$ and then to $AIA(\Gamma)$, as we have explained above, the quotients in relation as stabilizers of these groupoids are reductions of $\mathrm{Aut}(M)$. Therefore the equality in relation (2.1), and therefore a positive answer to the rigidity problem, is guaranteed if the two reductions are equal up to the conjugacy relation induced by $I\Omega$ from Section 2.2. The structure of the proof is described in Figure 4.2.

$$\left(\frac{M}{\Gamma}, g\right),\ \left(\frac{M'}{\Gamma'}, g'\right) \text{ are compact Riemannian 2-step nilmanifolds with conjugated geodesic flows.}$$

$$\Downarrow$$

$$\left(\frac{M}{\Phi(\Gamma)}, g\right),\ \left(\frac{M'}{\Gamma'}, g'\right) \text{ are compact Riemannian 2-step nilmanifolds with conjugated geodesic flows.}$$

$$\Downarrow$$

$$\left(\frac{M}{\Phi(\Gamma)}, g\right),\ \left(\frac{M'}{\Gamma'}, g'\right) \text{ are isometric manifolds.}$$

$$\Downarrow$$

$$I\left(\frac{M}{\Phi(\Gamma)}\right) \cong I\left(\frac{M'}{\Gamma'}\right) \Longleftrightarrow \frac{N(\Phi(\Gamma))}{\Gamma} \cong \frac{N(\Gamma')}{\Gamma'}.$$

$$\Downarrow$$

$$\text{Reduced main problem: } \frac{N(\Phi(\Gamma))}{\Gamma} \cong \frac{N(\Gamma)}{\Gamma}.$$

$$\Uparrow$$

$$\text{The reductions } \frac{N(\Phi(\Gamma))}{\Gamma} \text{ and } \frac{N(\Gamma)}{\Gamma} \text{ are equal.}$$

Figure 4.2.

Corollary 4.2 ([11]). *Let $\mathcal{M}$ be a 2-step nilpotent Lie algebra with an inner product $\langle\cdot,\cdot\rangle$ and φ be an almost inner derivation of continuous type on $\mathcal{M}$, i.e. $\varphi(x) = [\sigma(x), x]$ with σ continuous on $\mathcal{M} \setminus \{0\}$. Let $z \in Z(\mathcal{M})$ and $y \in \ker(J(z))$. Then $\langle\varphi(x), z\rangle = \langle[\sigma(y), x], z\rangle$ for all $x \in \mathcal{M}$, where $J(z)\colon \mathcal{V} \to \mathcal{V}$; $\mathcal{V} = \mathcal{M} \setminus Z(\mathcal{M})$, $z \in Z(\mathcal{M})$ is a skew symmetric linear transformation defined by $J(z)x = (ad(x))^* z$ for $x \in \mathcal{V}$.*

In particular, if the center of $\mathcal{M}$ properly contains the derived algebra, then every almost inner derivation of continuous type on $\mathcal{M}$ is inner.

P r o o f. In fact, the equality $\langle[\sigma(x), x], z\rangle = \langle[\sigma(y), x], z\rangle$; $\langle[y, x], z\rangle = 0$ must be proved.

But the continuity of σ implies that the above equality holds for $y + tx$, $t \in \mathbb{R}$ or as Lie groupoid notions, for the second reference point in $e^{t\sigma(x)} = \pi^{-1}(m)$ of a principal bundle $P(M, G, \pi) = \Omega_m(M, \Omega_m^m, \pi)$ corresponding to locally trivial Lie groupoid $\Omega = AIA(\Gamma)$. But from Section 2.2, we reach to a groupoid morphism $R_{\Phi^{-1}(m)}(\mathrm{id}_M, I_{\Phi(m)})$ for $\Phi \in AIA(\Gamma)$ (equivalently for $\varphi \in AID(\Gamma)$) which corresponds to the identity automorphism of the corresponding gauge groupoid.

In this way, using the above answer, we achieve the equality of two corresponding reductions up to conjugacy relation mentioned there. This shows the independence of σ on the choice of the points contained in $\ker(J(z))$ as in the above equality or in Lie algebroid letters, two elements $\sigma(x)$ and $\sigma(y)$ in their corresponding gauge algebroids are an identity derivation. □

5. Conclusions

In this paper, a rigidity problem of two compact 2-step nilmanifolds M/Γ and M'/Γ' with conjugated flows was studied. A novel methodology was presented to answer this problem by developing a reformulation based on Lie groupoids and Lie algebroids. For this purpose, we applied algebraic notions of the factorization of the isometry groups of the simply connected Riemannian manifolds whose corresponding normalizers can be factorized by ingredients of Lie groupoids. The machinery of Lie groupoid theory was applied to provide a systematic method to answer in some sense our rigidity problem.

The main advantage of the proposed methodology is that it can be applied to general rigidity problems of compact 2-step nilmanifolds with analytic conditions for their almost inner automorphisms. Also, it provides the exact relation between the normalizers and the automorphisms induced by them, for Lie groups and their Lie algebras. Also, positive answer to the rigidity problem results in a bijective correspondence between the cosets of centralizers in normalizers.

References

[1] *R. Baer*: Primary Abelian groups and their automorphisms. Am. J. Math. *59* (1937), 99–117. zbl MR doi

[2] *G. Besson, G. Courtois, S. Gallot*: Entropy and rigidity of locally symmetric spaces of strictly negative curvature. Geom. Func. Anal. *5* (1995), 731–799. (In French.) zbl MR doi

[3] *G. Besson, G. Courtois, S. Gallot*: Minimal entropy and Mostow's rigidity theorems. Ergodic Theory Dyn. Syst. *16* (1996), 623–649. zbl MR doi

[4] *A. Châtelet*: Les groupes abéliens finis et les modules de points entiers. Bibliothèque universitaire. Travaux et mémoires de l'Université de Lille. Nouv. série II, 3. Gauthier-Villars, Lille, 1925. (In French.) zbl

[5] *C. Croke*: Rigidity for surfaces of non-positive curvature. Comment. Math. Helvet. *65* (1990), 150–169. zbl MR doi

[6] *C. Croke, P. Eberlein, B. Kleiner*: Conjugacy and rigidity for nonpositively curved manifolds of higher rank. Topology *35* (1996), 273–286. zbl MR doi

[7] *J. J. Duistermaat, J. A. C. Kolk*: Lie Groups. Universitext. Springer, Berlin, 2000. zbl MR doi

[8] *P. Eberlein*: Geometry of two-step nilpotent groups with a left invariant metric. Ann. Sci. Éc. Norm. Supér. (4) *27* (1994), 611–660. zbl MR doi

[9] *H.-R. Fanaï, A. Hasan-Zadeh*: A symplectic rigidity problem for 2-step nilmanifolds. Houston J. Math. *43* (2017), 363–374. zbl MR

[10] *D. Gabai*: On the geometric and topological rigidity of the hyperbolic 3-manifolds. Bull. Am. Math. Soc., New Ser. *31* (1994), 228–232. zbl MR doi

[11] *C. S. Gordon, Y. Mao*: Geodesic conjugacies of two-step nilmanifolds. Mich. Math. J. *45* (1998), 451–481. zbl MR doi

[12] *C. S. Gordon, Y. Mao, D. Schueth*: Symplectic rigidity of geodesic flows on two-step nilmanifolds. Ann. Sci. Éc. Norm. Supér. (4) *30* (1998), 417–427. zbl MR doi

[13] *C. S. Gordon, E. N. Wilson*: Isospectral deformations of compact solvmanifolds. J. Differ. Geom. *19* (1984), 241–256. zbl MR doi

[14] *J. T. Hallett, K. A. Hirsch*: Torsion-free groups having finite automorphism groups. J. Algebra *2* (1965), 287–298. zbl MR doi

[15] *J. T. Hallett, K. A. Hirsch*: Die Konstruktion von Gruppen mit vorgeschriebenen Automorphismengruppen. J. Reine Angew. Math. *239–240* (1969), 32–46. zbl MR doi

[16] *A. Hulpke*: Normalizer calculation using automorphisms. Computational Group Theory and the Theory of Groups. AMS special session On Computational Group Theory, Davidson, USA, 2007 (L.-C. Kappe et al., eds.). Contemporary Mathematics 470. AMS, Providence, 2008, pp. 105–114. zbl MR

[17] *K. C. H. Mackenzie*: General Theory of Lie Groupoids and Lie Algebroids. London Mathematical Society Lecture Note Series 213. Cambridge University Press, Cambridge, 2005. zbl MR doi

[18] *K. Mann*: Homomorphisms between diffeomorphism groups. Avaible at `https://arxiv.org/abs/1206.1196v1` (2012).

[19] *G. D. Mostow*: Strong Rigidity of Locally Symmetric Spaces. Annals of Mathematics Studies. No. 78. Princeton University Press, Princeton, 1973. zbl MR doi

[20] *B. O'Neill*: Semi Riemannian Geometry. With Applications to Relativity. Pure and Applied Mathematics 103. Academic Press, London, 1983. zbl MR

[21] *J. P. Otal*: The spectrum marked by lengths of surfaces with negative curvature. Ann. Math. (2) *131* (1990), 151–162. (In French.) zbl MR doi

[22] *M. S. Raghunathan*: Discrete Subgroups of Lie Groups. Ergebnisse der Mathematik und ihrer Grenzgebiete. Band 68. Springer, Berlin, 1972. zbl MR

Authors' addresses: *Hamid-Reza Fanaï*, Department of Mathematical Sciences, Sharif University of Technology, P.O.Box 11155-9415, Tehran, Iran, e-mail: `fanai@sharif.ir`; *Atefeh Hasan-Zadeh* (corresponding author), Fouman Faculty of Engineering, College of Engineering, University of Tehran, P.O.Box 43581-39115, Guilan, Iran, e-mail: `hasanzadeh.a@ut.ac.ir`.