DERIVATIVE OF THE DONSKER DELTA FUNCTIONALS

Herry Pribawanto Suryawan, Yogyakarta

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Abstract. We prove that derivatives of any finite order of Donsker's delta functionals are well-defined elements in the space of Hida distributions. We also show the convergence to the derivative of Donsker's delta functionals of two different approximations. Finally, we present an existence result of finite product and infinite series of the derivative of the Donsker delta functionals.

Keywords: Donsker delta functional; white noise analysis; distributional derivative

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1. Introduction

In the area of Gaussian analysis, the composition of Dirac delta function δ_c at point c with a stochastic process $X=(X_t)_{t\geqslant 0}$, i.e., $\delta_c(X_t)$, is a well-studied object. Let $C_0(\mathbb{R}^n)$ be the Banach space of continuous functions $w\colon [0,T]\to\mathbb{R}^n$ such that $w(0)=0$ endowed with the supremum norm, and $\mathbb{P}$ be the Wiener measure on $C_0(\mathbb{R}^n)$. The n-dimensional Brownian motion $(B_t)_{t\geqslant 0}$ on the Wiener space is given by the projection map $B_t(w)=w(t)$, $w\in C_0(\mathbb{R}^n)$. The Donsker delta functional of the Brownian motion is a formal composition $\delta(B_T-c)=\delta(w(T)-c)$ where $\delta(\cdot - c):=\delta_c$, which can be viewed as a formal density of the n-dimensional Brownian bridge measure $\mathbb{P}^{T,c}_{0,0}:=\mathbb{P}(\cdot|w(T)=c)$ with respect to the Wiener measure $\mathbb{P}$. The Brownian bridge measure is singular to the Wiener measure. However, it is still useful to consider this formal density on the Wiener space, analogously as Dirac delta functions are useful in the calculus on the Euclidean space. The Donsker delta functional was first introduced, for the case of Brownian motion, in the framework of white noise analysis in [12]. Later on, it has been also studied by using Malliavin calculus techniques in [20]. Donsker's delta functional of a stochastic process has found many applications, for example in the context of local times of stochastic

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processes, polymer physics, Feynman path integral, and mathematical finance. For some studies on the Donsker delta functionals and their applications we refer to [1], [2], [3], [5], [7], [9], [14], [15], [19], [21] and references therein. The notion of the derivative of Donsker's delta functional has been mentioned briefly in [13]. Another motivation for the study of the derivative of Donsker's delta functional comes also from the work on the derivative of self-intersection local times of stochastic process introduced in [17]. The present work can be construed as a step towards a more comprehensive study on the white noise approach to the derivative of Donsker's delta functionals and their applications.

The organization of the present paper is as follows. In Section 2 we briefly review the necessary background of Gaussian analysis. As main results in Section 3 we prove that the derivative of any order $k \in \mathbb{N}$ of the Donsker delta functional of a regular random variable exists as a Hida distribution. We obtain also an explicit expression for the Wiener-Itô chaos decomposition of the derivative of Donsker's delta functional. Convergence results to the derivative of Donsker's delta functional are also presented. Section 4 contains results on the existence of finite product and infinite series of the derivative of Donsker's delta functional.

2. Elements of Gaussian analysis

This section contains pertinent notions and results from Gaussian analysis used throughout this paper. For more details we refer the interested reader to [8], [11], [16] and the literature quoted there. We start with a real separable Hilbert space $\mathcal{H}$ equipped with inner product $(\cdot,\cdot)$ and corresponding norm $|\cdot|$. Let $\mathcal{N}$ be a nuclear Fréchet space which is densely and continuously embedded into $\mathcal{H}$, and $\mathcal{N}'$ be its topological dual space. By identifying $\mathcal{H}$ with its dual space via Riesz isomorphism we get the Gelfand triple $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$. The dual pairing between $\mathcal{N}'$ and $\mathcal{N}$ is denoted by $\langle\cdot,\cdot\rangle$, and considered as an extension of the inner product in $\mathcal{H}$. Let us denote by $\mathcal{C}$ the σ-algebra on $\mathcal{N}'$ generated by the cylinder sets. Then, the canonical Gaussian measure μ on the measurable space $(\mathcal{N}',\mathcal{C})$ is established by using the Minlos theorem via the relation $\int_{\mathcal{N}'} \mathrm{e}^{\mathrm{i}\langle\omega,\eta\rangle}\,\mathrm{d}\mu(\omega) = \mathrm{e}^{-|\eta|^2/2}$, $\eta \in \mathcal{N}$.

The space of square-integrable complex-valued functions over the probability space $(\mathcal{N}',\mathcal{C},\mu)$ will be denoted by $L^2(\mu)$. The notation $\mathbb{E}_\mu$ means the expectation with respect to the probability measure μ. An important element of $L^2(\mu)$ is the so-called Wick exponential $:\mathrm{e}^{\langle\cdot,\eta\rangle}: := \mathrm{e}^{\langle\cdot,\eta\rangle - |\eta|^2/2}$, $\eta \in \mathcal{N}$. For $f \in \mathcal{H}$ we define the random variable $X_f := \langle\cdot,f\rangle$, which is a centered Gaussian with variance $\mathbb{E}_\mu(X_f^2) = |f|^2$. We call X_f a regular random variable on Gaussian space $(\mathcal{N}',\mathcal{C},\mu)$. From the Segal isomorphism between $L^2(\mu)$ and the complex Fock space $\Gamma(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ over the complexification $\mathcal{H}_{\mathbb{C}}$ over $\mathcal{H}$, one has, for every $f \in L^2(\mu)$, the so-called Wiener-Itô chaos decomposition $f(\omega) = \sum_{n=0}^{\infty} \langle :\omega^{\otimes n}:, f_{(n)} \rangle$, $f_{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$, where $:\omega^{\otimes n}:$ is the Wick tensor of order n of $\omega \in \mathcal{N}'$. Here $\widehat{\otimes}$ is the notation of symmetric tensor product. The function $f_{(n)}$ is called the nth kernel of f.

For our purpose we need a Gelfand triple around the Hilbert space $L^2(\mu)$, namely the Hida triple $(\mathcal{N}) \subset L^2(\mu) \subset (\mathcal{N})'$. Here $(\mathcal{N})$ is called the space of Hida test functions and can be constructed by taking the intersection of a family of Hilbert subspaces of $L^2(\mu)$. It is equipped with the projective limit topology and has the structure of nuclear countably Hilbert space. Moreover, the topology on $(\mathcal{N})$ is uniquely determined by the topology on $\mathcal{N}$. The space of Hida distributions $(\mathcal{N})'$ is defined as the topological dual space of $(\mathcal{N})$. The Wiener-Itô chaos decomposition can be extended to elements of $(\mathcal{N})'$, that is, for any $\Phi \in (\mathcal{N})'$ it holds that

$$\Phi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, \Phi_{(n)} \rangle, \quad \Phi_{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}. \tag{2.1}$$

Example 2.1. By choosing the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ of Lebesgue square-integrable functions, nuclear space $\mathcal{N} = \mathcal{S}(\mathbb{R})$ of Schwartz test functions, and $\mathcal{N}' = \mathcal{S}'(\mathbb{R})$, the space of Schwartz distributions, one gets the concrete Gelfand triple $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$. The probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{C}, \mu)$ is known as the white noise space. A standard Brownian motion on the white noise space is given by $B_t := \langle \cdot, \mathrm{ind}_{1_{[0,t]}} \rangle$, where ind_{1_A} denotes the indicator function of $A \subset \mathbb{R}$. Within this setting, white noise at each time t is a well-defined mathematical object given by $W_t := \mathrm{d}/\mathrm{d}t B_t = \langle \cdot, \delta_t \rangle$, where the convergence takes place in $(\mathcal{N})'$ with respect to the inductive limit topology. Here δ_t denotes the Dirac delta function at t. Further, all derivatives of any order of Brownian motion are Hida distributions with the represention $B_t^{(k)} = (-1)^{k-1} \langle \cdot, \delta_t^{(k-1)} \rangle$, where $\delta_t^{(n)}$ is the notation for the nth distributional derivative of the Dirac delta function at t.

The S-transform of $\Phi \in (\mathcal{N})'$ is the mapping from $\mathcal{N}$ into $\mathbb{C}$ given by $S\Phi(\eta) := \langle\langle \Phi, :\mathrm{e}^{\langle \cdot, \eta \rangle}: \rangle\rangle$, $\eta \in \mathcal{N}$. Here $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the bilinear dual pairing on $(\mathcal{N})' \times (\mathcal{N})$ which extends the inner product on $L^2(\mu)$. Since the Wick exponentials $\{:\mathrm{e}^{\langle \cdot, \eta \rangle}:, \eta \in \mathcal{N}\}$ form a total set in $(\mathcal{N})$, we have that elements of $(\mathcal{N})'$ are characterized by their S-transform. Moreover, the S-transform of $\Phi \in (\mathcal{N})'$ also extends to $\mathcal{N}_{\mathbb{C}}$. If Φ is given via the chaos decomposition (2.1), then its S-transform can be calculated by $(S\Phi)(\eta) = \sum_{n=0}^{\infty} \langle \Phi_{(n)}, \eta^{\otimes n} \rangle$. We define also the generalized expectation of $\Phi \in (\mathcal{N})'$ as $\mathbb{E}_\mu(\Phi) := (S\Phi)(0) = \Phi_{(0)}$.

We will use the following three important results on the characterization of Hida distributions via S-transform. We only state the theorems, while for details and proofs we refer the reader to [11].

Theorem 2.2. *A mapping $F\colon \mathcal{N} \to \mathbb{C}$ is the S-transform of an element in $(\mathcal{N})'$ if and only if it is a U-functional, that is,*

(1) *for all $\xi, \eta \in \mathcal{N}$, the mapping $\lambda \mapsto F(\eta + \lambda\xi)$ from $\mathbb{R}$ into $\mathbb{C}$ has an entire extension to $\lambda \in \mathbb{C}$, and*
(2) *for some continuous quadratic form Q on $\mathcal{N}$ there exist constants $C, K > 0$ such that for all $\eta \in \mathcal{N}$ and $z \in \mathbb{C}$ it holds that $|F(z\eta)| \leqslant C\mathrm{e}^{K|z|^2|Q(\eta)|}$.*

Theorem 2.3. *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, and $\lambda \mapsto \Phi_\lambda$ a mapping from Ω to $(\mathcal{N})'$. If $F_\lambda := S\Phi_\lambda$ fulfils the two conditions:*

(1) *for every $\eta \in \mathcal{N}$ the mapping $\lambda \mapsto F_\lambda(\eta)$ is measurable,*
(2) *there exists a continuous norm $|\cdot|_*$ on $\mathcal{N}$ such that for all $\lambda \in \Omega$, $\eta \in \mathcal{N}$, $z \in \mathbb{C}$, F_λ satisfies the bound $|F_\lambda(z\eta)| \leqslant C(\lambda)\mathrm{e}^{K(\lambda)|z|^2|\eta|_*^2}$ with $C(\lambda) \in L^1(\nu)$ and $K(\lambda) \in L^\infty(\nu)$,*

then Φ_λ is Bochner integrable with respect to some Hilbertian norm which topologizes $(\mathcal{N})'$. Thus, in particular, $\int_\Omega \Phi_\lambda \,\mathrm{d}\nu(\lambda) \in (\mathcal{N})'$, and the S-transform and Bochner integral commute.

Theorem 2.4. *Let $\Phi_n \in (\mathcal{N})'$, $n \in \mathbb{N}$. If $F_n := S\Phi_n$ satisfies the two conditions:*

(1) *$(F_n(\eta))_{n\in\mathbb{N}}$ is a Cauchy sequence for all $\eta \in \mathcal{N}$,*
(2) *there exists a continuous norm $|\cdot|_*$ on $\mathcal{N}$ and $C, K > 0$ such that for all $\eta \in \mathcal{N}$, $z \in \mathbb{C}$ and for almost all $n \in N$ it holds that $|F_n(z\eta)| \leqslant C\mathrm{e}^{K|z|^2|\eta|_*^2}$,*

then $(\Phi_n)_{n\in\mathbb{N}}$ converges in the strong topology to a unique $\Phi \in (\mathcal{N})'$.

3. Main results

Now we prove our main results on the derivative of Donsker's delta functional using tools from Gaussian analysis. Instead of using the Wiener-Itô chaos decomposition approach as in [13], we established the existence of the derivative of Donsker's delta functional via Bochner integration in $(\mathcal{N})'$. As a starting point we use a formal Fourier-transform representation of the derivatives of Dirac delta function $\delta^{(k)}(x) := \frac{1}{2}\pi^{-1}\int_{\mathbb{R}} \mathrm{i}^k\lambda^k\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda$, where $k \in \mathbb{N} \cup \{0\}$ with the convention $\delta^{(0)} := \delta$. In the rest of the paper, by p_ε and H_k we denote the heat kernel $p_\varepsilon(x) := \mathrm{e}^{-x^2/2\varepsilon}/\sqrt{2\pi\varepsilon}$, $\varepsilon > 0$

and the Hermite polynomial of order k defined by the Rodrigues formula $H_k(x) = (-1)^k \mathrm{e}^{x^2}(\mathrm{d}^k/\mathrm{d}x^k)\mathrm{e}^{-x^2}$, respectively.

Theorem 3.1. *Let f be a nonzero element of $\mathcal{H}$. For any $c \in \mathbb{R}$ the Bochner integral*

$$\delta_c^{(k)}(X_f) := \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{i}^k\lambda^k \mathrm{e}^{\mathrm{i}\lambda(\langle\cdot,f\rangle - c)}\,\mathrm{d}\lambda, \quad k \in \mathbb{N}\cup\{0\}$$

exists as a Hida distribution with the S-transform given by

$$S(\delta_c^{(k)}(X_f))(\eta) = p_{|f|^2}(\langle f,\eta\rangle - c)\Big(\frac{-1}{\sqrt{2|f|^2}}\Big)^k H_k\Big(\frac{\langle f,\eta\rangle - c}{\sqrt{2|f|^2}}\Big), \quad \eta \in \mathcal{N}.$$

The generalized expectation of $\delta_c^{(k)}(X_f)$ is given by

$$\mathbb{E}_\mu(\delta_c^{(k)}(X_f)) = p_{|f|^2}(c)\Big(\frac{1}{\sqrt{2|f|^2}}\Big)^k H_k\Big(\frac{c}{\sqrt{2|f|^2}}\Big).$$

Proof. Let us define $F_\lambda(\eta) := S(\mathrm{i}^k\lambda^k\mathrm{e}^{\mathrm{i}\lambda(\langle\cdot,f\rangle-c)})(\eta)$, $\eta\in\mathcal{N}$. Then,

$$\begin{aligned}
F_\lambda(\eta) &= \langle\!\langle \mathrm{i}^k\lambda^k\mathrm{e}^{\mathrm{i}\lambda(\langle\cdot,f\rangle-c)}, {:}\mathrm{e}^{\langle\cdot,\eta\rangle}{:}\rangle\!\rangle = \int_{\mathcal{N}'} \mathrm{i}^k\lambda^k\mathrm{e}^{\mathrm{i}\lambda(\langle\omega,f\rangle-c)}\mathrm{e}^{\langle\omega,\eta\rangle-|\eta|^2/2}\,\mathrm{d}\mu(\omega)\\
&= \mathrm{i}^k\lambda^k\mathrm{e}^{-|\eta|^2/2}\mathrm{e}^{-\mathrm{i}c\lambda}\int_{\mathcal{N}'}\mathrm{e}^{\langle\omega,\mathrm{i}\lambda f+\eta\rangle}\,\mathrm{d}\mu(\omega) = \mathrm{i}^k\lambda^k\mathrm{e}^{-|\eta|^2/2}\mathrm{e}^{-\mathrm{i}c\lambda}\mathrm{e}^{|\mathrm{i}\lambda f+\eta|^2/2}\\
&= \mathrm{i}^k\lambda^k\mathrm{e}^{-|\lambda|^2|f|^2/2}\mathrm{e}^{\mathrm{i}\lambda(\langle f,\eta\rangle-c)}.
\end{aligned}$$

The mapping $\lambda\mapsto F_\lambda(\eta)$ is continuous for all $\eta\in\mathcal{N}$, and hence it is measurable with respect to the Lebesgue measure $\mathrm{d}\lambda$. Furthermore, for $z\in\mathbb{C}$ and $\eta\in\mathcal{N}$ we have

$$\begin{aligned}
|F_\lambda(z\eta)| &= |\lambda^k|\mathrm{e}^{-|\lambda|^2|f|^2/2}|\mathrm{e}^{\mathrm{i}\lambda(\langle f,\eta\rangle)}| \leqslant |\lambda^k|\mathrm{e}^{-|\lambda|^2|f|^2/2}\mathrm{e}^{|z||\langle f,\lambda\eta\rangle|}\\
&\leqslant |\lambda^k|\mathrm{e}^{-|\lambda|^2|f|^2/4}\mathrm{e}^{(|z|^2|\langle f,\lambda\eta\rangle|^2)/(|\lambda|^2|f|^2)} \leqslant |\lambda^k|\mathrm{e}^{-|\lambda|^2|f|^2/4}\mathrm{e}^{|z|^2|\eta|^2}.
\end{aligned}$$

The first factor $C(\lambda) := |\lambda^k|\mathrm{e}^{-|\lambda|^2|f|^2/4}$ is an integrable function of λ, while the second factor $K(\lambda) := \mathrm{e}^{|z|^2|\eta|^2}$ is constant function of λ. Hence, according to Theorem 2.3 $\delta^{(k)}(\langle\cdot,f\rangle - c)$ is well-defined as a Bochner integral in $(\mathcal{N})'$. To obtain its S-transform we integrate $\frac{1}{2}\pi^{-1}F_\lambda$ over $\mathbb{R}$ as follows:

$$\begin{aligned}
S(\delta_c^{(k)}(X_f))(\eta) &= \frac{1}{2\pi}\int_{\mathbb{R}} S(\mathrm{i}^k\lambda^k\mathrm{e}^{\mathrm{i}\lambda(\langle\cdot,f\rangle-c)})(\eta)\,\mathrm{d}\lambda\\
&= \frac{1}{2\pi}\mathrm{i}^k\int_{\mathbb{R}}\lambda^k\mathrm{e}^{-|\lambda|^2|f|^2/2}\mathrm{e}^{\mathrm{i}\lambda(\langle f,\eta\rangle-c)}\,\mathrm{d}\lambda\\
&= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(\langle f,\eta\rangle-c)^2/(2|f|^2)}\frac{(-1)^k k!}{|f|^{2k+1}}(\langle f,\eta\rangle-c)^k\sum_{l=0}^{\lfloor\frac{k}{2}\rfloor}\frac{1}{(k-2l)!\,l!}\Big(-\frac{|f|^2}{2(\langle f,\eta\rangle-c)^2}\Big)^l.
\end{aligned}$$

In the last equality we have used a particular definite integral of the product of an exponential function and a polynomial, see e.g. formula 3.462 (2) in [6]. By recalling the explicit formula for Hermite polynomial

$$H_k(x) = k! \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^l}{(k-2l)!\, l!}(2x)^{k-2l},$$

we obtain

$$S(\delta_c^{(k)}(X_f))(\eta) = \frac{1}{\sqrt{2\pi|f|^2}} \mathrm{e}^{-(\langle f,\eta\rangle - c)^2/(2|f|^2)} \frac{(-1)^k 2^{-k/2}}{|f|^k} H_k\left(\frac{\langle f,\eta\rangle - c}{\sqrt{2|f|^2}}\right).$$

By using the expression for the heat kernel we arrive at the desired expression. To compute the generalized expectation of $\delta^{(k)}(\langle\cdot, f\rangle - c)$ we simply substitute the value $\eta = 0$ in the expression of the S-transform. This finishes the proof. □

In the following theorem we derive the Wiener-Itô chaos decomposition of the derivative of Donsker's delta functional.

Theorem 3.2. *For $f \in \mathcal{H} \setminus \{0\}$, $c \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$ it holds that*

$$\delta_c^{(k)}(X_f) = p_{|f|^2}(c) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\sqrt{2|f|^2}}\right)^{n+k} H_{n+k}\left(\frac{c}{\sqrt{2|f|^2}}\right) \langle :\cdot^{\otimes n}:, f^{\otimes n}\rangle$$

in $(\mathcal{N})'$.

P r o o f. We begin with the Bochner integral representation as in Theorem 3.1 and make use of the generating function of the Hermite polynomial to infer that

$$\begin{aligned}
\delta_c^{(k)}(X_f) &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{i}^k \lambda^k \mathrm{e}^{\mathrm{i}\lambda(\langle\omega, f\rangle - c)}\, \mathrm{d}\lambda \\
&= \frac{1}{2\pi} \mathrm{i}^k \sum_{n=0}^{\infty} \frac{\mathrm{i}^n |f|^n}{n! 2^{n/2}} H_n\left(\frac{\langle\cdot, f\rangle}{\sqrt{2|f|^2}}\right) \int_{\mathbb{R}} \lambda^{n+k} \mathrm{e}^{-|f|^2\lambda^2/2 - \mathrm{i}c\lambda}\, \mathrm{d}\lambda \\
&= \left(\frac{1}{2\pi} \mathrm{i}^k \sum_{n=0}^{\infty} \frac{\mathrm{i}^n |f|^n}{n!\, 2^{n/2}} H_n\left(\frac{\langle\cdot, f\rangle}{\sqrt{2|f|^2}}\right)\right) \mathrm{e}^{-c^2/(2|f|^2)} \frac{\sqrt{2\pi}}{|f|^{n+k+1}} (-\mathrm{i})^{n+k} 2^{-(n+k)/2} \\
&\quad \times (n+k)! \sum_{l=0}^{\lfloor \frac{n+k}{2} \rfloor} \frac{(-1)^l}{(n+k-2l)!\, l!} \left(\sqrt{2}\frac{c}{|f|}\right)^{n+k-2l} \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-c^2/(2|f|^2)} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{2^{-(n+k)/2}}{|f|^{n+k+1}} H_{n+k}\left(\frac{c}{\sqrt{2|f|^2}}\right) \left(\frac{|f|^n}{2^{n/2}} H_n\left(\frac{\langle\cdot, f\rangle}{\sqrt{2|f|^2}}\right)\right).
\end{aligned}$$

In the computation above we use again formula 3.462 (2) in [6]. Moreover, by using the identity $\langle :\cdot^{\otimes n}:, f^{\otimes n}\rangle = |f|^n 2^{-n/2} H_n(\langle\cdot, f\rangle/\sqrt{2|f|^2})$, see e.g. [8], [13], we arrive at the stated expression. In addition, the nth kernel of $\delta^{(k)}(\langle\cdot, f\rangle - c)$ is given by

$$\delta^{(k)}_{c,(n)}(X_f) := p_{|f|^2}(c)\frac{1}{n!}\left(\frac{1}{\sqrt{2|f|^2}}\right)^{n+k} H_{n+k}\left(\frac{c}{\sqrt{2|f|^2}}\right) f^{\otimes n}.$$

One can check easily that the zeroth kernel is exactly the generalized expectation obtained in Theorem 3.1. □

We give some remarks.

(1) By taking $k = 0$ in the above result we recover the classical result on Donsker's delta functional and its S-transform as in, for example, [8], [14]. The generalized expectation of Donsker's delta functional is the heat kernel. The complex-scaled Donsker's delta functional has been also of interest, since it formally gives the Schrödinger kernel, see e.g. [14].

(2) Consider the expression for the S-transform of $\delta^{(k)}(\langle\cdot, f\rangle - c)$. It is analytic in the parameter $c \in \mathbb{R}$. Therefore, we can extend it to complex parameter $c \in \mathbb{C}$ and the resulting expression is still a U-functional. Hence, Theorem 2.2 enables us to define $\delta^{(k)}(\langle\cdot, f\rangle - c)$ for $c \in \mathbb{C}$.

(3) The same argument holds if we extend it to $f \in \mathcal{H}_{\mathbb{C}}$. In order to avoid problems with complex square root we cut the complex plane along the negative axis. So we have to exclude $f \in \mathcal{H}_{\mathbb{C}}$ with negative value of $(f, f)_{\mathcal{H}_{\mathbb{C}}}$. So, again using Theorem 2.2, it is possible to define $\delta^{(k)}(\langle\cdot, f\rangle - c)$ for complex parameters c and f.

Definition 3.3. Let f be a nonzero element of $\mathcal{H}_{\mathbb{C}}$, $\arg(f, f) \neq \pi$, $c \in \mathbb{C}$ and $k \in \mathbb{N} \cup \{0\}$. The generalized function $\delta^{(k)}_c(X_f)$ defined via its S-transform

$$S(\delta^{(k)}_c(X_f))(\eta) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(\langle f,\eta\rangle - c)^2/(2|f|^2)} \frac{\mathrm{i}^{2k} 2^{-k/2}}{(f, f)^{(k+1)/2}} H_k\left(\frac{\langle f, \eta\rangle - c}{\sqrt{2(f, f)}}\right), \quad \eta \in \mathcal{N}_{\mathbb{C}}$$

is called the derivative of order k of Donsker's delta functional.

We have constructed the derivative of any finite order of Donsker's delta functional as a Bochner integral in $(\mathcal{N})'$. In the following theorem we show that it can be approximated by a sequence of square integrable functions. Precisely speaking, the sequence consists of Bochner integrals in $L^2(\mu)$. We need the following well-known Bochner integrability criterion whose proof can be found in, e.g., [10].

Proposition 3.4. *Let X be a Banach space with norm $\|\cdot\|$ and $(A, \mathcal{A}, \varrho)$ be a measure space. If $f\colon A \to X$ is separably-valued and weakly measurable, and $\int_A \|f(a)\| \,\mathrm{d}\varrho(a) < \infty$, then f is Bochner integrable.*

Theorem 3.5. *For $f \in \mathcal{H} \setminus \{0\}$, $c \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$ the following convergence holds in $(\mathcal{N})'$:*

$$\delta_c^{(k)}(X_f) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} \mathrm{i}^k \lambda^k \mathrm{e}^{\mathrm{i}\lambda(\langle\cdot,f\rangle - c)}\,\mathrm{d}\lambda.$$

Proof. For $n \in \mathbb{N}$ let us define $\Psi_n := \frac{1}{2}\pi^{-1} \int_{-n}^{n} \mathrm{i}^k \lambda^k \mathrm{e}^{\mathrm{i}\lambda(\langle\cdot,f\rangle - c)}\,\mathrm{d}\lambda$ and $G\colon [-n,n] \to L^2(\mu)$ by $G(\lambda) := \mathrm{i}^k\lambda^k \mathrm{e}^{\mathrm{i}\lambda(\langle\cdot,f\rangle - c)}$. Note that $[-n,n] \ni \lambda \mapsto (g, G(\lambda))_{L^2(\mu)}$ is a continuous mapping for all $g \in L^2(\mu)$ and therefore measurable. It is clear that G is separably-valued. Moreover, for each $\lambda \in [-n,n]$ we have $\|G(\lambda)\|^2_{L^2(\mu)} = \int_{\mathcal{N}'} |\mathrm{i}^k\lambda^k \mathrm{e}^{\mathrm{i}\lambda(\langle\cdot,f\rangle - c)}|^2\,\mathrm{d}\mu(\omega) = |\lambda|^{2k}$. This implies

$$\int_{-n}^{n} \|G(\lambda)\|_{L^2(\mu)}\,\mathrm{d}\lambda = \frac{2n^{k+1}}{k+1} < \infty.$$

By using Proposition 3.4 we know that for any $n \in \mathbb{N}$, Ψ_n exists as a Bochner integral in $L^2(\mu)$. Next, we prove that the sequence $(\Psi_n)_{n\in\mathbb{N}}$ converges in $(\mathcal{N})'$. We have the S-transform of Ψ_n at $\eta \in \mathcal{N}$:

$$\begin{aligned} S\Psi_n(\eta) &= \frac{1}{2\pi} \int_{-n}^{n} S(\mathrm{i}^k\lambda^k \mathrm{e}^{\mathrm{i}\lambda(\langle\cdot,f\rangle - c)})(\eta)\,\mathrm{d}\lambda \\ &= \frac{\mathrm{i}^k}{2\pi} \int_{\mathbb{R}} \operatorname{ind}_{1_{[-n,n]}}(\lambda)\lambda^k \mathrm{e}^{-|\lambda|^2|f|^2/2} \mathrm{e}^{\mathrm{i}\lambda(\langle f,\eta\rangle - c)}\,\mathrm{d}\lambda. \end{aligned}$$

Observe that for each $\lambda \in \mathbb{R}$ the integrand in the last integral converges pointwise to $\lambda^k \mathrm{e}^{-|\lambda|^2|f|^2/2}\mathrm{e}^{\mathrm{i}\lambda(\langle f,\eta\rangle - c)}$ as $n \to \infty$. Moreover, it is bounded by the function $|\lambda^k|\mathrm{e}^{-|\lambda|^2|f|^2/2}$ which is integrable over $\mathbb{R}$ since it is a product of a polynomial with a rapidly decreasing function. By using the Lebesgue dominated convergence theorem we obtain that $(\Psi_n)_{n\in\mathbb{N}}$ converges to $\frac{1}{2}\mathrm{i}^k\pi^{-1}\int_{\mathbb{R}} \lambda^k \mathrm{e}^{-|\lambda|^2|f|^2/2}\mathrm{e}^{\mathrm{i}\lambda(\langle f,\eta\rangle - c)}\,\mathrm{d}\lambda$ as $n \to \infty$. This means that $(\Psi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence for every $\eta \in \mathcal{N}$. For $\eta \in \mathbb{N}$, $z \in \mathbb{C}$, and $n \in \mathbb{N}$ we have

$$|S\Psi_n(z\eta)| \leqslant \frac{1}{2\pi}\int_{-n}^{n} |\lambda^k|\mathrm{e}^{-|\lambda|^2|f|^2/2}\,\mathrm{d}\lambda \leqslant \frac{1}{2\pi}\mathrm{e}^{|z|^2|\eta|^2}\int_{\mathbb{R}} |\lambda^k|\mathrm{e}^{-|\lambda|^2|f|^2/4}\,\mathrm{d}\lambda \leqslant C\mathrm{e}^{|z|^2|\eta|^2},$$

where $C = 2^{(k+1)/2}k!\,D_{-k-1}(0)\pi^{-1}|f|^{-(k+1)}$ and D_ν is the notation of the parabolic cylinder function. The constant C is obtained by an application of formula 3.462 (1) in [6]. This shows the boundedness condition, and the assertion of the theorem now follows from Theorem 2.4. □

The Dirac delta function also appears as limit in the distribution sense of a sequence of approximale identities. Most notably, it holds that $\delta(x) = \lim\limits_{\varepsilon\to 0} p_\varepsilon(x) =$

$\lim_{\varepsilon\to 0} \mathrm{e}^{-x^2/2\varepsilon}/\sqrt{2\pi\varepsilon}$. It is easy to see that the kth derivative of the heat kernel is given by $p_\varepsilon^{(k)}(x) := (-1)^k(\sqrt{2\varepsilon})^{-k}H_k(x/\sqrt{2\varepsilon})p_\varepsilon(x)$, $k \in \mathbb{N}\cup\{0\}$. This motivates the following.

Theorem 3.6. *For $f \in \mathcal{H}\setminus\{0\}$, $c \in \mathbb{R}$ and $k \in \mathbb{N}\cup\{0\}$ it holds that*

$$\delta_c^{(k)}(X_f) = \lim_{\varepsilon\to 0} \frac{(-1)^k}{(\sqrt{2\varepsilon})^k} H_k\Big(\frac{\langle\cdot,f\rangle - c}{\sqrt{2\varepsilon}}\Big)p_\varepsilon(\langle\cdot,f\rangle - c)$$

in $(\mathcal{N})'$.

P r o o f. First, we show that for each $\varepsilon > 0$,

$$\delta_{c,\varepsilon}^{(k)}(X_f) := \frac{(-1)^k}{(\sqrt{2\varepsilon})^k} H_k\Big(\frac{\langle\cdot,f\rangle - c}{\sqrt{2\varepsilon}}\Big)p_\varepsilon(\langle\cdot,f\rangle - c)$$

is a square-integrable function. To this end we compute its $L^2(\mu)$-norm:

$$\begin{aligned}
\|\delta_{c,\varepsilon}^{(k)}(X_f)\|_{L^2(\mu)}^2 &= \int_{\mathcal{N}'} \Big|\frac{(-1)^k}{(\sqrt{2\varepsilon})^k} H_k\Big(\frac{\langle\cdot,f\rangle - c}{\sqrt{2\varepsilon}}\Big)\frac{1}{\sqrt{2\pi\varepsilon}}\mathrm{e}^{-(\langle\cdot,f\rangle-c)^2/(2\varepsilon)}\Big|^2 \,\mathrm{d}\mu(\omega)\\
&= \frac{1}{(\sqrt{2\varepsilon})^{2k}}\frac{1}{2\pi\varepsilon}\frac{1}{\sqrt{2\pi|f|^2}}\int_{\mathbb{R}} H_k^2\Big(\frac{x-c}{\sqrt{2\varepsilon}}\Big)\mathrm{e}^{-(x-c)^2/\varepsilon}\mathrm{e}^{-x^2/(2|f|^2)}\,\mathrm{d}x\\
&= \frac{1}{(\sqrt{2\varepsilon})^{2k-1}}\frac{1}{2\pi\varepsilon}\frac{1}{\sqrt{2\pi|f|^2}}\mathrm{e}^{-c^2/(2|f|^2)}\\
&\quad\times \int_{\mathbb{R}} H_k^2(x)\mathrm{e}^{-(2+\varepsilon/|f|^2)x^2 - xc\sqrt{2\varepsilon}/|f|^2}\,\mathrm{d}x.
\end{aligned}$$

Based on an asymptotic consideration it is clear that the last integral is finite, and hence, we can conclude the square-integrability of $\delta_{c,\varepsilon}^{(k)}(X_f)$. Nevertheless, with a little effort we can compute the value of the integral explicitly. By using the formula

$$H_k^2(x) = 2^k(k!)^2\sum_{s=0}^{k}\frac{H_{2s}(x)}{2^s(s!)^2(k-s)!}$$

(see [4]) and formula 7.374 (8) in [6] we get

$$\begin{aligned}
&\int_{\mathbb{R}} H_k^2(x)\mathrm{e}^{-(2+\varepsilon/|f|^2)x^2 - xc\sqrt{2\varepsilon}/|f|^2}\,\mathrm{d}x\\
&\qquad= \sum_{s=0}^{k}\frac{2^k(k!)^2}{2^s(s!)^2(k-s)!}\int_{\mathbb{R}} H_{2s}(x)\mathrm{e}^{-(2+\varepsilon/|f|^2)x^2 - xc\sqrt{2\varepsilon}/|f|^2}\,\mathrm{d}x\\
&\qquad= \sqrt{\pi}\,2^k k!\,\mathrm{e}^{c^2\varepsilon/(2|f|^2(\varepsilon+2|f|^2))}\sum_{s=0}^{k}\frac{1}{2^s(s!)^2(k-s)!}\Big(\frac{|f|^2+\varepsilon}{2|f|^2+\varepsilon}\Big)^s\\
&\qquad\quad\times H_{2s}\Big(c\sqrt{\frac{\varepsilon}{(2|f|^2+\varepsilon)(2|f|^2+2\varepsilon)}}\Big).
\end{aligned}$$

Hence,

$$\|\delta_{c,\varepsilon}^{(k)}(X_f)\|_{L^2(\mu)}^2 = \varepsilon^{-(k+1/2)} \frac{k!}{2\pi|f|} \mathrm{e}^{-c^2/(2|f|^2+\varepsilon)} \sum_{s=0}^{k} \frac{1}{2^s (s!)^2 (k-s)!} \Big(\frac{|f|^2+\varepsilon}{2|f|^2+\varepsilon}\Big)^s \times H_{2s}\bigg(c\sqrt{\frac{\varepsilon}{(2|f|^2+\varepsilon)(2|f|^2+2\varepsilon)}}\bigg) < \infty.$$

We have proved that for each $\varepsilon > 0$, $\delta_\varepsilon^{(k)}(\langle\cdot, f\rangle - c) \in L^2(\mu)$. Next, we show the convergence in $(\mathcal{N})'$ as $\varepsilon \to 0$. By the Cameron-Martin theorem we have

$$\begin{aligned}
&S\delta_{c,\varepsilon}^{(k)}(X_f)(\eta)\\
&\quad= \int_{\mathcal{N}'} \delta_{c,\varepsilon}^{(k)}(\langle \omega+\eta, f\rangle)\,\mathrm{d}\mu(\omega)\\
&\quad= \frac{(-1)^k}{(\sqrt{2\varepsilon})^k}\frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathcal{N}'} H_k\Big(\frac{\langle\omega,f\rangle + \langle f,\eta\rangle - c}{\sqrt{2\varepsilon}}\Big) \mathrm{e}^{-(\langle\omega,f\rangle+\langle f,\eta\rangle-c)^2/(2\varepsilon)}\,\mathrm{d}\mu(\omega)\\
&\quad= \frac{(-1)^k}{(\sqrt{2\varepsilon})^k}\frac{1}{\sqrt{2\pi\varepsilon}}\frac{1}{\sqrt{2\pi|f|^2}} \int_{\mathbb{R}} H_k\Big(\frac{x+\langle f,\eta\rangle - c}{\sqrt{2\varepsilon}}\Big) \mathrm{e}^{-(x+\langle f,\eta\rangle-c)^2/(2\varepsilon)} \mathrm{e}^{-x^2/(2|f|^2)}\,\mathrm{d}x\\
&\quad= \frac{(-1)^k 2^{-k/2}}{\sqrt{2\pi|f|^2}} \Big(\frac{1}{|f|^2+\varepsilon}\Big)^{k/2} \mathrm{e}^{-(\langle f,\eta\rangle-c)^2/(2(|f|^2+\varepsilon))} H_k\Big(\frac{\langle f,\eta\rangle - c}{\sqrt{2(|f|^2+\varepsilon)}}\Big).
\end{aligned}$$

As $\varepsilon \to 0$ we have that $S\delta_\varepsilon^{(k)}(\langle\cdot, f\rangle - c)(\eta)$ converges to

$$\frac{(-1)^k 2^{-k/2}}{\sqrt{2\pi|f|^2}} \frac{1}{|f|^k} \mathrm{e}^{-(\langle f,\eta\rangle-c)^2/(2|f|^2)} H_k\Big(\frac{\langle f,\eta\rangle - c}{\sqrt{2|f|^2}}\Big) = p_{|f|^2}^{(k)}(\langle f,\eta\rangle - c).$$

By applying the bound $|H_n(x)| < \alpha(n!)^{1/2} 2^{n/2} \mathrm{e}^{x^2/2}$ for the Hermite polynomials, we have for any $\eta \in \mathcal{N}$, $z \in \mathbb{C}$, and $\varepsilon > 0$,

$$\begin{aligned}
|S\delta_{c,\varepsilon}^{(k)}(X_f)(z\eta)| &= \frac{2^{-k/2}}{\sqrt{2\pi|f|^2}} \Big(\frac{1}{|f|^2+\varepsilon}\Big)^{k/2}\\
&\qquad \times |\mathrm{e}^{-(\langle f,z\eta\rangle-c)^2/(2(|f|^2+\varepsilon))}| \bigg|H_k\Big(\frac{\langle f,z\eta\rangle - c}{\sqrt{2(|f|^2+\varepsilon)}}\Big)\bigg|\\
&\leqslant \frac{2^{-k/2}}{\sqrt{2\pi|f|^2}} \Big(\frac{1}{|f|^2+\varepsilon}\Big)^{k/2}\\
&\qquad \times |\mathrm{e}^{-(\langle f,z\eta\rangle-c)^2/(2(|f|^2+\varepsilon))}| \alpha(k!)^{1/2} 2^{k/2} |\mathrm{e}^{(\langle f,z\eta\rangle-c)^2/(4(|f|^2+\varepsilon))}|\\
&\leqslant \frac{\alpha(k!)^{1/2}}{\sqrt{2\pi}} \frac{1}{|f|^{k+1}} |\mathrm{e}^{-(\langle f,z\eta\rangle-c)^2/(4(|f|^2+\varepsilon))}|
\end{aligned}$$

$$\leqslant \frac{\alpha(k!)^{1/2}}{\sqrt{2\pi}} \frac{1}{|f|^{k+1}} \mathrm{e}^{c^2/(2|f|^2)} \mathrm{e}^{|\langle f, z\eta\rangle|^2/(2|f|^2)}$$

$$\leqslant \frac{\alpha(k!)^{1/2}}{\sqrt{2\pi}} \frac{1}{|f|^{k+1}} \mathrm{e}^{c^2/(2|f|^2)} \mathrm{e}^{|z|^2|\eta|^2/2}.$$

Hence, we have shown the boundedness condition needed for the application of Theorem 2.4. □

Note that the limit object in $(\mathcal{N})'$ obtained in Theorem 3.6 must coincide with the Bochner integral in $(\mathcal{N})'$ in Theorem 3.1. This follows immediately from Theorem 2.2 and the fact that they have identical S-transforms.

4. Product and series of the derivative of Donsker's delta functional

In this section we will prove that finite products and infinite series of the derivatives of Donsker's delta functional are Hida distributions. We use the following notations:

$$\vec{f} := (f_1, \ldots, f_n), \quad \vec{c} := (c_1, \ldots, c_n), \quad \text{and} \quad \vec{\lambda} := (\lambda_1, \ldots, \lambda f_n).$$

Theorem 4.1. *Let $f_j \in \mathcal{H}$ be linearly independent vectors, $M = (f_k, f_l)_{k,l=1,\ldots,n}$ be the corresponding Gram matrix, and $c_j \in \mathbb{R}$. Then,*

$$\Phi := \delta^{(k),n}(\langle \cdot, \vec{f}\rangle - \vec{c}) := \prod_{j=1}^{n} \delta_{c_j}^{(k)}(X_{f_j})$$

is a Hida distribution with S-transform is given by

$$S\Phi(\eta) = \frac{(-1)^{nk}}{\sqrt{(2\pi)^n 2^{nk} (\det(M))^{k+1}}} \mathrm{e}^{-(\langle \vec{f},\eta\rangle - \vec{c})^T M^{-1} (\langle \vec{f},\eta\rangle - \vec{c})/2} H_k^n\left(\frac{\langle \vec{f}, \eta\rangle - \vec{c}}{\sqrt{2|\vec{f}|^2}}\right),$$

where

$$H_k^n\left(\frac{\langle \vec{f}, \eta\rangle - \vec{c}}{\sqrt{2|\vec{f}|^2}}\right) := \prod_{j=1}^{n} H_k\left(\frac{\langle f_j, \eta\rangle - c_j}{\sqrt{2|f_j|^2}}\right) \quad \text{and} \quad \eta \in \mathcal{N}.$$

The generalized expectation can be written as

$$\mathbb{E}_\mu(\Phi) = \frac{1}{\sqrt{(2\pi)^n \det(M)}} \mathrm{e}^{-\vec{c}^T M^{-1} \vec{c}/2} \left(\frac{1}{\sqrt{2^n \det(M)}}\right)^k H_k^n\left(\frac{\vec{c}}{\sqrt{2|\vec{f}|^2}}\right).$$

P r o o f. Let $\eta \in \mathcal{N}$, $\vec{\lambda}^k := \prod_{j=1}^{n} \lambda_j^k$, $\mathrm{e}^{-\mathrm{i}\vec{\lambda}\vec{c}} := \mathrm{e}^{-\mathrm{i}\sum_{j=1}^{n}\lambda_j c_j}$, and $\mathrm{e}^{\langle \omega, \mathrm{i}\vec{\lambda}\vec{f}\eta\rangle} := \mathrm{e}^{\sum_{j=1}^{n}\langle \omega, \mathrm{i}\lambda_j f_j + \eta\rangle}$. By Fubini's theorem, we obtain

$$\begin{aligned}
(S\Phi)(\eta) &= S\Big(\prod_{j=1}^{n} \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{i}^k \lambda_j^k \mathrm{e}^{\mathrm{i}\lambda_j(\langle\cdot, f_j\rangle - c_j)}\,\mathrm{d}\lambda\Big)(\eta)\\
&= \frac{\mathrm{i}^{nk}}{(2\pi)^n}\Big\langle\!\!\Big\langle \prod_{j=1}^{n}\int_{\mathbb{R}} \lambda_j^k \mathrm{e}^{\mathrm{i}\lambda_j(\langle\cdot, f_j\rangle - c_j)}\,\mathrm{d}\lambda, {:}\mathrm{e}^{\langle\cdot,\eta\rangle}{:}\Big\rangle\!\!\Big\rangle\\
&= \frac{\mathrm{i}^{nk}}{(2\pi)^n}\mathrm{e}^{-|\eta|^2/2}\int_{\mathbb{R}^n} \vec{\lambda}^k \mathrm{e}^{-\mathrm{i}\vec{\lambda}\vec{c}}\int_{\mathcal{N}'} \mathrm{e}^{\langle\omega, \mathrm{i}\vec{\lambda}\vec{f}+\eta\rangle}\,\mathrm{d}\mu(\omega)\,\mathrm{d}^n\vec{\lambda}\\
&= \frac{\mathrm{i}^{nk}}{(2\pi)^n}\mathrm{e}^{-|\eta|^2/2}\int_{\mathbb{R}^n} \vec{\lambda}^k \mathrm{e}^{-\mathrm{i}\vec{\lambda}\vec{c}}\mathrm{e}^{|\mathrm{i}\vec{\lambda}\vec{f}+\eta|^2}\,\mathrm{d}^n\vec{\lambda}\\
&= \frac{\mathrm{i}^{nk}}{(2\pi)^n}\int_{\mathbb{R}^n} \vec{\lambda}^k \mathrm{e}^{-\mathrm{i}\vec{\lambda}\vec{c}}\mathrm{e}^{\langle\mathrm{i}\vec{\lambda}\vec{f}, \mathrm{i}\vec{\lambda}\vec{f}\rangle/2 + \langle\mathrm{i}\vec{\lambda}\vec{f}, \eta\rangle}\,\mathrm{d}^n\vec{\lambda}.
\end{aligned}$$

Next, we consider

$$\langle \mathrm{i}\vec{\lambda}\vec{f}, \mathrm{i}\vec{\lambda}\vec{f}\rangle = -\sum_{k,l=1}^{n}\lambda_k\lambda_l(f_k, f_l) = -\vec{\lambda}^{\top}M\vec{\lambda},$$

where $M = (f_k, f_l)_{k,l=1,\ldots,n}$. This is a Gram matrix of linearly independent vectors in $\mathcal{H}$, and thus positive definite. Hence,

$$\begin{aligned}
(S\Phi)(\eta) &= \frac{\mathrm{i}^{nk}}{(2\pi)^n}\int_{\mathbb{R}^n} \vec{\lambda}^k \mathrm{e}^{\vec{\lambda}^{\top}M\vec{\lambda}/2 + \mathrm{i}\vec{\lambda}(\langle\vec{f},\eta\rangle - \vec{c})}\,\mathrm{d}^n\vec{\lambda}\\
&= \frac{\mathrm{i}^{nk}}{(2\pi)^n}\prod_{j=1}^{n}\sqrt{2\pi}\mathrm{e}^{-(\langle f_j,\eta\rangle - c_j)^2/(2|f_j|^2)}\frac{\mathrm{i}^k 2^{-k/2}}{|f_j|^{k+1}}H_k\Big(\frac{\langle f_j,\eta\rangle - c_j}{\sqrt{2|f_j|^2}}\Big)\\
&= \frac{(-1)^{nk}}{\sqrt{(2\pi)^n 2^{nk}(\det(M))^{k+1}}}\mathrm{e}^{-(\langle\vec{f},\eta\rangle - \vec{c})^T M^{-1}(\langle\vec{f},\eta\rangle - \vec{c})/2}H_k^n\Big(\frac{\langle\vec{f},\eta\rangle - \vec{c}}{\sqrt{2|\vec{f}|^2}}\Big).
\end{aligned}$$

By using the same argument as in the proof of Theorem 3.6 we obtain

$$\begin{aligned}
|S\Phi(z\eta)| &= \frac{1}{\sqrt{(2\pi)^n 2^{nk}(\det(M))^{k+1}}}\prod_{j=1}^{n}\big|\mathrm{e}^{-(\langle f_j,\eta\rangle - c_j)^2/(2|f_j|^2)}\big|\Big|H_k\Big(\frac{\langle f_j,\eta\rangle - c_j}{\sqrt{2|f_j|^2}}\Big)\Big|\\
&\leqslant \sqrt{\Big(\frac{\alpha^2 k!}{2\pi}\Big)^n\frac{1}{\det(M)^{k+1}}}\prod_{j=1}^{n}|\mathrm{e}^{-(\langle f_j,\eta\rangle - c_j)^2/(4|f_j|^2)}|\\
&\leqslant \sqrt{\Big(\frac{\alpha^2 k!}{2\pi}\Big)^n\frac{1}{\det(M)^{k+1}}}\mathrm{e}^{\vec{c}^{\top}M^{-1}\vec{c}/2}\mathrm{e}^{n|z|^2|\eta|^2/2}.
\end{aligned}$$

Therefore, the S-transform of Φ is analytically continued and is uniformly bounded of second order. From Theorem 2.2 we conclude that $\Phi \in (\mathcal{N})'$. □

Theorem 4.2. *Let f_j, $j = 1,\dots,n$ be linearly independent vectors in $\mathcal{H}$ and $M = (f_k, f_l)_{k,l=1,\dots,n}$ the corresponding Gram matrix. If $F\colon \mathbb{R}^n \to \mathbb{C}$ is in $L^p(\mathbb{R}^n, \mathrm{e}^{-\vec{x}^\top M^{-1}\vec{x}/4}\,\mathrm{d}^n\vec{x})$ for some $p > 1$, then $\Psi := \int_{\mathbb{R}^n} F(\vec{x})\delta^{(k),n}(\langle\cdot,\vec{f}\rangle - \vec{x})\mathrm{d}^n\vec{x}$ is a Bochner integral in $(\mathcal{N})'$.*

Proof. From the proof of Theorem 4.1 we have for $\eta \in \mathcal{N}$

$$
\begin{aligned}
&|S\delta^{(k),n}(\langle\cdot,\vec{f}\rangle - \vec{x})(\eta)| \\
&\quad\leqslant \sqrt{\Big(\frac{\alpha^2 k!}{2\pi}\Big)^n \frac{1}{\det(M)^{k+1}}}\mathrm{e}^{-\vec{x}^\top M^{-1}\vec{x}/4}\prod_{j=1}^{n}\mathrm{e}^{|\langle f_j, z\eta\rangle|^2/(4|f_j|^2)}\mathrm{e}^{\varepsilon x_j^2/|f_j|^2 + |\langle f_j, z\eta\rangle|^2/(4\varepsilon|f_j|^2)} \\
&\quad= \sqrt{\Big(\frac{\alpha^2 k!}{2\pi}\Big)^n \frac{1}{\det(M)^{k+1}}}\mathrm{e}^{-(1/4-\varepsilon)\vec{x}^\top M^{-1}\vec{x}}\mathrm{e}^{n(1+|z|^2|\eta|^2/\varepsilon)/4}.
\end{aligned}
$$

Now we choose $q > 1$ with $p^{-1} + q^{-1} = 1$ and $\varepsilon > 0$ such that $q\varepsilon < \frac{1}{4}$. Then, by Hölder's inequality

$$
\begin{aligned}
|S\Psi(z\eta)| &\leqslant \sqrt{\Big(\frac{\alpha^2 k!}{2\pi}\Big)^n \frac{1}{\det(M)^{k+1}}}\mathrm{e}^{n(1+|z|^2|\eta|^2/\varepsilon)/4}\int_{\mathbb{R}^n}|F(\vec{x}|\mathrm{e}^{-(1/4-\varepsilon)\vec{x}^\top M^{-1}\vec{x}}\,\mathrm{d}^n\vec{x} \\
&\leqslant \sqrt{\Big(\frac{\alpha^2 k!}{2\pi}\Big)^n \frac{1}{\det(M)^{k+1}}}\mathrm{e}^{n(1+|z|^2|\eta|^2/\varepsilon)/4} \\
&\quad\times\Big(\int_{\mathbb{R}^n}|F(\vec{x})|^p\mathrm{e}^{-\vec{x}^\top M^{-1}\vec{x}/4}\,\mathrm{d}^n\vec{x}\Big)^{1/p}\Big(\int_{\mathbb{R}^n}\mathrm{e}^{-(1/4-q\varepsilon)\vec{x}^\top M^{-1}\vec{x}}\,\mathrm{d}^n\vec{x}\Big)^{1/q}.
\end{aligned}
$$

The last integrals are finite by our assumptions. Hence Theorem 2.3 applies and the proof is finished. □

Now we prove a result on a series of the derivative of Donsker's delta functional.

Theorem 4.3. *Let $f \in \mathcal{H}\setminus\{0\}$, $c \in \mathbb{R}$, and $k \in \mathbb{N}\cup\{0\}$. The infinite series $\varphi := \sum\limits_{n=-\infty}^{\infty}\delta_c^{(k)}(X_f + n)$ is a Hida distribution with S-transform*

$$
S\varphi(\eta) = p_{|f|^2}(\langle\cdot,f\rangle - c)\Big(\frac{-1}{\sqrt{2|f|^2}}\Big)^k\sum_{n=-\infty}^{\infty}\mathrm{e}^{(2n\langle f,\eta\rangle - n^2)/(2|f|^2)}H_k\Big(\frac{\langle f,\eta\rangle - c + n}{\sqrt{2|f|^2}}\Big).
$$

P r o o f. Let us define for $N \in \mathbb{N}$ the finite sum $\varphi_N := \sum_{n=-N}^{N} \delta^{(k)}(\langle \cdot, f\rangle - c + n)$. It is obvious that $\varphi_N \in (\mathcal{N})'$ and its S-transform is given by

$$S\varphi_N(\eta) = \frac{1}{\sqrt{2\pi}} \frac{(-1)^k 2^{-k/2}}{|f|^{k+1}} \sum_{n=-N}^{N} \mathrm{e}^{-(\langle f,\eta\rangle - c+n)^2/(2|f|^2)} H_k\Big(\frac{\langle f,\eta\rangle - c + n}{\sqrt{2|f|^2}}\Big).$$

Our goal is to show that φ_N converges in $(\mathcal{N})'$ as $N \to \infty$. We find a uniform bound in N as follows. Let $\eta \in \mathcal{N}$ and $z \in \mathbb{C}$. Then,

$$\begin{aligned}
|S\varphi_N(z\eta)| &\leqslant \frac{\alpha(k!)^{1/2}}{\sqrt{2\pi}|f|^{k+1}} \sum_{n=-N}^{N} \big|\mathrm{e}^{-(\langle f,z\eta\rangle - c+n)^2/(4|f|^2)}\big| \\
&\leqslant \frac{\alpha(k!)^{1/2}}{\sqrt{2\pi}|f|^{k+1}} \mathrm{e}^{|\langle f,z\eta\rangle - c|^2/|f|^2} \sum_{n=-N}^{N} \mathrm{e}^{-n^2/(8|f|^2)} \\
&\leqslant \frac{\alpha(k!)^{1/2}}{\sqrt{2\pi}|f|^{k+1}} \mathrm{e}^{\beta c^2/|f|^2} \mathrm{e}^{\gamma|z|^2|\eta|^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-n^2/(8|f|^2)}
\end{aligned}$$

for some constants $\beta, \gamma > 0$. The last series can be written in terms of the Jacobi theta function $\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi \mathrm{i}\tau n^2 + 2\pi \mathrm{i} n z}$, which converges for all $z \in \mathbb{C}$ and τ in the upper half plane, see e.g. [18]. Hence,

$$|S\varphi_N(z\eta)| \leqslant \frac{\alpha(k!)^{1/2}}{\sqrt{2\pi}|f|^{k+1}} \mathrm{e}^{\beta c^2/|f|^2} \mathrm{e}^{\gamma|z|^2|\eta|^2} \vartheta\Big(0, \frac{\mathrm{i}}{8\pi|f|^2}\Big).$$

Now, Theorem 2.4 delivers the convergence of φ_N to φ in $(\mathcal{N})'$ as $N \to \infty$. Furthermore,

$$\begin{aligned}
S\varphi(\eta) &= \frac{1}{\sqrt{2\pi}} \frac{(-1)^k 2^{-k/2}}{|f|^{k+1}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-(\langle f,\eta\rangle - c+n)^2/(2|f|^2)} H_k\Big(\frac{\langle f,\eta\rangle - c + n}{\sqrt{2|f|^2}}\Big) \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(\langle f,\eta\rangle - c)^2/(2|f|^2)} \frac{(-1)^k 2^{-k/2}}{|f|^{k+1}} \\
&\quad \times \sum_{n=-\infty}^{\infty} \mathrm{e}^{(2n\langle f,\eta\rangle - n^2)/(2|f|^2)} H_k\Big(\frac{\langle f,\eta\rangle - c + n}{\sqrt{2|f|^2}}\Big) \\
&= p_{|f|^2}(\langle \cdot, f\rangle - c)\Big(\frac{-1}{\sqrt{2|f|^2}}\Big)^k \sum_{n=-\infty}^{\infty} \mathrm{e}^{(2n\langle f,\eta\rangle - n^2)/(2|f|^2)} H_k\Big(\frac{\langle f,\eta\rangle - c + n}{\sqrt{2|f|^2}}\Big).
\end{aligned}$$

□

In the case $k = 0$ we obtain a nice expression

$$S\varphi(\eta) = p_{|f|^2}(\langle \cdot, f\rangle - c)\vartheta\Big(-\frac{\mathrm{i}\langle f, \eta\rangle}{2\pi|f|^2}, \frac{\mathrm{i}}{2\pi|f|^2}\Big).$$

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Author's address: *Herry Pribawanto Suryawan*, Department of Mathematics, Sanata Dharma University, Yogyakarta, Indonesia, e-mail: herrypribs@usd.ac.id.