SOME PROPERTIES OF CERTAIN SUBCLASSES OF BOUNDED MOCANU VARIATION WITH RESPECT TO 2k-SYMMETRIC CONJUGATE POINTS

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Abstract. We introduce subclasses of analytic functions of bounded radius rotation, bounded boundary rotation and bounded Mocanu variation with respect to $2k$-symmetric conjugate points and study some of its basic properties.

Keywords: $2k$-symmetric conjuqate points; bounded Mocanu variation; bounded radius rotation; bounded boundary rotation

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined on the unit disc $E = \{z \in \mathbb{C}\colon\ |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E. \tag{1.1}$$

Also, let S, K, S^* and C denote the subclasses of $\mathcal{A}$ which are univalent, close-to-convex, starlike and convex in E, respectively. Let $P_m(\gamma)$ be the class of functions $p(z)$ analytic in the unit disc E satisfying the properties $p(0) = 1$ and for $z = re^{\mathrm{i}\theta}$, $m \geqslant 2$,

$$\int_0^{2\pi} \left|\operatorname{Re} \frac{p(z) - \gamma}{1 - \gamma}\right| \mathrm{d}\theta \leqslant m\pi, \quad 0 \leqslant \gamma < 1. \tag{1.2}$$

The class $P_m(\gamma)$ for $\gamma = 0$ and $0 \leqslant \gamma < 1$ has been introduced and investigated by Pinchuk in [6], and Padmanabhan and Parvatham in [5], respectively. We note that

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$P_m(0) = P_m$ and $P_2(\gamma) = P(\gamma)$ is the class of analytic functions with positive real part greater than γ. For $m = 2$ and $\gamma = 0$ we have the class P of functions with positive real part. We can write (1.2) as

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1 + (1-2\gamma)z\mathrm{e}^{-\mathrm{i}t}}{1 - z\mathrm{e}^{-\mathrm{i}t}}\,\mathrm{d}\mu(t), \tag{1.3}$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} \mathrm{d}\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |\mathrm{d}\mu(t)| \leqslant m. \tag{1.4}$$

Also, for $p \in P_m(\gamma)$ we can write from (1.2)

$$p(z) = \Big(\frac{m}{4} + \frac{1}{2}\Big)p_1(z) - \Big(\frac{m}{4} - \frac{1}{2}\Big)p_2(z), \quad p_1, p_2 \in P_2(\gamma),\ z \in E. \tag{1.5}$$

It is known [3] that $P_m(\gamma)$ is a convex set. Also $p \in P_m(\gamma)$ is in $P_2(\gamma) = P(\gamma)$ for $|z| < r_1$, where

$$r_1 = \frac{1}{2}\big(m - \sqrt{m^2 - 4}\big). \tag{1.6}$$

The classes $V_m(\gamma)$ of functions of bounded boundary rotation of order γ and $R_m(\gamma)$ of functions of bounded radius rotation of order γ are closely related with $P_m(\gamma)$. A function $f \in \mathcal{A}$ is in $V_m(\gamma)$ if and only if $(zf'(z))'/f'(z) \in P_m(\gamma)$. Also

$$f \in R_m(\gamma) \Leftrightarrow \frac{zf'(z)}{f(z)} \in P_m(\gamma). \tag{1.7}$$

It is clear that

$$f \in V_m(\gamma) \Leftrightarrow zf'(z) \in P_m(\gamma). \tag{1.8}$$

When $m = 2, \gamma = 0$, then $V_2(0)$ coincides with the class C and $R_2(0) = S^*$. Wang et al. in [9] introduced and investigated class $S_s^{(k)}(\varphi)$, which satisfies the inequality:

$$\frac{zf'(z)}{f_k(z)} \prec \varphi(z), \quad z \in E,$$

where $\varphi(z) \in P$, $k \geqslant 2$ is a fixed positive integer and $f_k(z)$ is defined by the following equality:

$$f_k(z) = \frac{1}{k}\sum_{\upsilon=0}^{k-1} \varepsilon^{-\upsilon} f(\varepsilon^{\upsilon} z), \quad \varepsilon = \exp\frac{2\pi\mathrm{i}}{k},$$

and a function $f(z) \in E$ is in the class $C_s^{(k)}(\varphi)$ if and only if $zf'(z) \in S_s^{(k)}(\varphi)$. Also Wang and Gao (see [9]) introduced and investigated two classes $S_{sc}^{(k)}(\varphi)$ and $C_{sc}^{(k)}(\varphi)$ of functions starlike and convex with respect to $2k$-symmetric conjugate points. Noor and Mustafa in [2] introduced and investigated class $R_s^k(\gamma)$ of analytic functions which are of bounded radius rotation of order γ with respect to symmetrical points if and only if

$$\frac{2zf'(z)}{f(z)-f(-z)} \in P_k(z), \quad z \in E.$$

We now define the following.

Definition 1.1. Let $f \in \mathcal{A}$. Then f is said to be of bounded radius rotation of order γ with respect to $2k$-symmetric conjugate points if and only if

$$\frac{zf'(z)}{f_{2k}(z)} \in P_m(\gamma), \quad z \in E, \tag{1.9}$$

where $k \geqslant 1$ is a fixed positive integer and $f_{2k}(z)$ is defined as

$$f_{2k}(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}(\varepsilon^{-\upsilon} f(\varepsilon^{\upsilon} z) + \varepsilon^{\upsilon}\overline{f(\varepsilon^{\upsilon}\overline{z})}), \quad \varepsilon = \exp\frac{2\pi \mathrm{i}}{k}. \tag{1.10}$$

We shall denote the class of such functions as $R_m^{s-2k}(\gamma)$. We note that $R_2^{s-2}(\gamma)$ is the class S_s^* of univalent functions starlike with respect to symmetrical points defined by Sakaguchi (see [8]). Also we define the class $V_m^{s-2k}(\gamma)$ as follows.

Definition 1.2.

$$f \in V_m^{s-2k}(\gamma) \Leftrightarrow zf' \in R_m^{s-2k}(\gamma), \quad z \in E. \tag{1.11}$$

Motivated by the above-mentioned classes we now define the following subclasses of analytic functions.

Definition 1.3. Let $f \in \mathcal{A}$ and $f(z)f'(z)z^{-1} \neq 0$ for $z \in E$. Then f is said to be of bounded Mocanu variation of order γ with respect to $2k$-symmetric conjugate points if and only if

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f'_{2k}(z)} \in P_m(\gamma), \quad z \in E, \tag{1.12}$$

where $0 \leqslant \alpha \leqslant 1$ and $k \geqslant 1$ is a fixed positive integer and $f_{2k}(z)$ is defined by (1.10). We shall denote the class of such functions as $\mathcal{M}_m^{s-2k}(\alpha, \gamma)$.

Definition 1.4. Let $f \in \mathcal{A}$ and $f(z)f'(z)z^{-1} \neq 0$ for $z \in E$. Then f belongs to the class $\mathcal{H}^{s-2k}_{m,m_1}(\alpha, \gamma)$ if

$$\alpha \frac{zf'(z)}{g_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{g'_{2k}(z)} \in P_m(\gamma), \tag{1.13}$$

where $0 \leqslant \alpha \leqslant 1$ and $k \geqslant 1$ is a fixed positive integer and $g_{2k}(z)$ is defined as

$$g_{2k}(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}(\varepsilon^{-\upsilon}g(\varepsilon^{\upsilon}z) + \varepsilon^{\upsilon}\overline{g(\varepsilon^{\upsilon}\overline{z})}), \quad \varepsilon = \exp\frac{2\pi \mathrm{i}}{k} \tag{1.14}$$

with $g \in \mathcal{M}^{s-2k}_{m_1}(\alpha, \gamma)$.

For simplicity, we write $\mathcal{H}^{s-2k}_{m,m}(\alpha, \gamma) =: \mathcal{H}^{s-2k}_{m}(\alpha, \gamma)$.

In our investigation of the classes $R^{s-2k}_m(\gamma)$, $V^{s-2k}_m(\gamma)$, $\mathcal{M}^{s-2k}_m(\alpha, \gamma)$ and $\mathcal{H}^{s-2k}_{m,m_1}(\alpha, \gamma)$ we need the following lemmas.

Lemma 1.1 ([1]). *Let p be an analytic function in the unit disc with $P(0) = a$, where $\operatorname{Re} a > 0$. Let $P\colon E \to \mathbb{C}$ be a function such that $\operatorname{Re} P(z) > 0$ for $z \in E$. Then*

$$\operatorname{Re}[p(z) + P(z)zp'(z)] > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

Lemma 1.2 ([1]). *Let $\beta, \gamma \in \mathbb{C}$ and h be convex and univalent function in E with*

$$h(0) = 1 \quad \text{and} \quad \operatorname{Re}(\beta h(z) + \gamma) > 0, \quad z \in E.$$

If p is analytic in E with $p(0) = 1$, then subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies that

$$p(z) \prec h(z).$$

2. Basic properties of $R_m^{s-2k}(\gamma)$, $V_m^{s-2k}(\gamma)$, $\mathcal{M}_m^{s-2k}(\alpha,\gamma)$ and $\mathcal{H}_{m,m_1}^{s-2k}(\alpha,\gamma)$

Theorem 2.1. *Let $f \in \mathcal{M}_m^{s-2k}(\alpha,\gamma)$. Then the function*

$$\psi(z) = f_{2k}(z) \tag{2.1}$$

belongs to $\mathcal{M}_m^{s-2k}(\alpha,\gamma)$.

Proof. Let $f \in \mathcal{M}_m^{s-2k}(\alpha,\gamma)$. Then from Definition 1.3 we have

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f'_{2k}(z)} \in P_m(\gamma), \quad z \in E,$$

or

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{f'(z)+zf''(z)}{f'_{2k}(z)} \in P_m(\gamma), \quad z \in E. \tag{2.2}$$

Replacing z by $\varepsilon^{\upsilon} z$, $\upsilon = 0,1,2,\ldots,k-1$ in (2.2) leads to

$$\alpha\frac{\varepsilon^{\upsilon} zf'(\varepsilon^{\upsilon} z)}{f_{2k}(\varepsilon^{\upsilon} z)} + (1-\alpha)\frac{f'(\varepsilon^{\upsilon} z)+\varepsilon^{\upsilon} zf''(\varepsilon^{\upsilon} z)}{f'_{2k}(\varepsilon^{\upsilon} z)} \in P_m(\gamma). \tag{2.3}$$

We note that

$$\begin{gathered} f_{2k}(\varepsilon^{\upsilon} z) = \varepsilon^{\upsilon} f_{2k}(z), \quad f'_{2k}(\varepsilon^{\upsilon} z) = f'_{2k}(z), \\ \overline{f_{2k}(\varepsilon^{\upsilon}\overline{z})} = \varepsilon^{-\upsilon} f_{2k}(z), \quad \overline{f'_{2k}(\varepsilon^{\upsilon}\overline{z})} = f'_{2k}(z), \quad \psi_{2k}(z) = f_{2k}(z). \end{gathered} \tag{2.4}$$

Thus, in view of (2.3) and (2.4) we obtain

$$\alpha\frac{zf'(\varepsilon^{\upsilon} z)}{f_{2k}(z)} + (1-\alpha)\frac{f'(\varepsilon^{\upsilon} z)+\varepsilon^{\upsilon} zf''(\varepsilon^{\upsilon} z)}{f'_{2k}(z)} \in P_m(\gamma) \tag{2.5}$$

and

$$\alpha\frac{z\overline{f'(\varepsilon^{\upsilon}\overline{z})}}{f_{2k}(z)} + (1-\alpha)\frac{\overline{f'(\varepsilon^{\upsilon}\overline{z})}+\varepsilon^{-\upsilon} z\overline{f''(\varepsilon^{\upsilon}\overline{z})}}{f'_{2k}(z)} \in P_m(\gamma). \tag{2.6}$$

Since $P_m(\gamma)$ is a convex set, summing (2.5) and (2.6) leads to

$$\begin{aligned} &\alpha\frac{\frac{1}{2}z\big(f'(\varepsilon^{\upsilon} z)+\overline{f'(\varepsilon^{\upsilon}\overline{z})}\big)}{f_{2k}(z)} \\ &\quad + (1-\alpha)\frac{\frac{1}{2}\big(f'(\varepsilon^{\upsilon} z)+\overline{f'(\varepsilon^{\upsilon}\overline{z})}\big)+\frac{1}{2}z\big(\varepsilon^{\upsilon} f''(\varepsilon^{\upsilon} z)+\varepsilon^{-\upsilon}\overline{f''(\varepsilon^{\upsilon}\overline{z})}\big)}{f'_{2k}(z)} \in P_m(\gamma). \end{aligned} \tag{2.7}$$

Putting $\upsilon = 0, 1, 2, \dots, k-1$ in (2.7) and summing the resulting equations yields

$$\alpha \frac{\frac{1}{2} z k^{-1} \sum_{\upsilon=0}^{k-1} \left(f'(\varepsilon^{\upsilon} z) + \overline{f'(\varepsilon^{\upsilon} \overline{z})} \right)}{f_{2k}(z)} + (1-\alpha) \frac{\frac{1}{2} k^{-1} \sum_{\upsilon=0}^{k-1} \left(f'(\varepsilon^{\upsilon} z) + \overline{f'(\varepsilon^{\upsilon} \overline{z})} + z\left(\varepsilon^{\upsilon} f''(\varepsilon^{\upsilon} z) + \varepsilon^{-\upsilon} \overline{f''(\varepsilon^{\upsilon} \overline{z})} \right) \right)}{f'_{2k}(z)} \in P_m(\gamma)$$

and hence $\psi \in P_k(\gamma)$ in E. □

Putting $\alpha = 0, 1$ in Theorem 2.1 we have the following results for the classes $R_m^{s-2k}(\gamma)$ and $V_m^{s-2k}(\gamma)$.

Corollary 2.1. *Let $f \in R_m^{s-2k}(\gamma)$. Then the function $\psi(z) = f_{2k}(z)$ belongs to $R_m^{s-2k}(\gamma)$ in E.*

Corollary 2.2. *Let $f \in V_m^{s-2k}(\gamma)$. Then the function $\psi(z) = f_{2k}(z)$ belongs to $V_m^{s-2k}(\gamma)$ in E.*

In order to prove our next result we need the following lemma.

Lemma 2.1. *Let p and φ be analytic functions in E with $p(0) = 1$ and $\operatorname{Re} \varphi(z) > 0$ for $z \in E$. If*

$$p(z) + \varphi(z) z p'(z) \in P_m(\gamma),$$

then $p(z) \in P_m(\gamma)$.

P r o o f. From the definition of $P_m(\gamma)$ there exist $q_1, q_2 \in P_2(\gamma)$ such that

$$p(z) + \varphi(z) z p'(z) = m q_1(z) + (1-m) q_2(z). \tag{2.8}$$

Let p_1 and p_2 be the solutions of the Cauchy problems

$$p(z) + \varphi(z) z p'(z) = q_1(z), \quad p(0) = 1 \tag{2.9}$$

and

$$p(z) + \varphi(z) z p'(z) = q_2(z), \quad p(0) = 1, \tag{2.10}$$

respectively. In view of (2.9) and (2.10) we rewrite (2.8) as

$$p(z) + \varphi(z) z p'(z) = m(p_1(z) + \varphi(z) z p_1'(z)) + (1-m)(p_2(z) + \varphi(z) z p_2'(z)),$$

or equivalently,

$$(2.11)\quad (p(z) - mp_1(z) - (1-m)p_2(z)) + z\varphi(z)(p'(z) - mp_1'(z) - (1-m)p_2'(z)) = 0.$$

Now if we define $h(z) = p(z) - mp_1(z) - (1-m)p_2(z)$, then $h(0) = 0$ and (2.11) yields

$$(2.12)\qquad h(z) + \varphi(z)zh'(z) = 0, \quad h(0) = 0.$$

But it is clear that Cauchy problem (2.12) has the only solution $h(z) = 0$. Hence $p(z) = mp_1(z) + (1-m)p_2(z)$. For completing the proof we show that $p_1, p_2 \in P_2(\gamma)$. Form equation (2.9) we can write

$$\frac{q_1(z) - \gamma}{1-\gamma} = \frac{p_1(z) - \gamma}{1-\gamma} + \frac{\varphi(z)}{1-\gamma} zp_1'(z).$$

Since $\operatorname{Re}(q_1(z) - \gamma)/(1-\gamma) > 0$ and $\operatorname{Re}\varphi(z) > 0$, applying Lemma 1.1 we obtain $\operatorname{Re} p_1(z) > \gamma$. Similarly, we have $\operatorname{Re} p_2(z) > \gamma$ and this means that $p \in P_m(\gamma)$ and the proof is complete. □

Theorem 2.2. *Let* $0 < \alpha \leqslant 1$, $k \geqslant 1$ *and* $m \geqslant 2$. *Then*

$$\mathcal{H}^{s-2k}_{m,2}(\alpha, \gamma, g) \subseteq \mathcal{H}^{s-2k}_{m,2}(1, \gamma, g).$$

Proof. Let $f \in \mathcal{H}^{s-2k}_{m,2}(\alpha, \gamma, g)$. Then by the definition of the class $\mathcal{H}^{s-2k}_{m,2}(\alpha, \gamma, g)$ and applying Theorem 2.1 we know that $g_{2k} \in \mathcal{M}^{s-2k}_2(\alpha, \gamma)$, i.e.

$$\alpha\frac{z\varphi'(z)}{\varphi(z)} + (1-\alpha)\frac{(z\varphi')'(z)}{\varphi'(z)} \in P(\gamma),$$

where $\varphi = g_{2k}$.

Or equivalently,

$$(2.13)\qquad \alpha\frac{z\varphi'(z)}{\varphi(z)} + (1-\alpha)\frac{(z\varphi'(z))'}{\varphi'(z)} \prec h(z) := \frac{1 + (1-2\gamma)z}{1-z}.$$

Set

$$q(z) = \frac{z\varphi'(z)}{\varphi(z)},$$

then we can rewrite (2.13) as

$$(2.14)\qquad \alpha\frac{z\varphi'(z)}{\varphi(z)} + (1-\alpha)\frac{(z\varphi')'(z)}{\varphi'(z)} = q(z) + \frac{(1-\alpha)zq'(z)}{q(z)} \prec h(z).$$

Since h is convex and univalent in E with $h(0) = 1$ and $\mathrm{Re}(h(z)/(1-\alpha)) > 0$, applying Lemma 1.2, we obtain

$$q(z) \prec h(z), \quad z \in E. \tag{2.15}$$

By Setting

$$p(z) = \frac{zf'(z)}{g_{2k}(z)},$$

we get

$$\begin{aligned} zp'(z) &= z\frac{(zf'(z))'g_{2k}(z) - g'_{2k}(z)zf'(z)}{g^2_{2k}(z)} = z\frac{(zf'(z))'}{g_{2k}(z)} - \frac{zf'(z)}{g_{2k}(z)}q(z) \\ &= \frac{(zf'(z))'}{g'_{2k}(z)}q(z) - \frac{zf'(z)}{g_{2k}(z)}q(z). \end{aligned} \tag{2.16}$$

Therefore in view of $f \in \mathcal{H}^{s-2k}_{m,2}(\alpha,\gamma,g)$ and (2.16) we conclude that

$$\alpha\frac{zf'(z)}{g_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{g'_{2k}(z)} = p(z) + (1-\alpha)\frac{zp'(z)}{q(z)} \in P_m(\gamma).$$

Now from relation (2.15) it is clear that $\mathrm{Re}(q(z)/(1-\alpha)) > 0$, so applying Lemma 2.1, we get $p(z) \in P_m(\gamma)$ and the proof is complete. □

By Putting $m = 2$ and considering $g = f_{2k}$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let $0 < \alpha < 1$ and $k \geqslant 1$. Then*

$$\mathcal{M}^{s-2k}_2(\alpha,\gamma) \subseteq R^{s-2k}_2(\gamma) \subseteq K \subseteq S.$$

Theorem 2.3. *Let $0 \leqslant \alpha < 1$ and $f \in \mathcal{M}^{s-2k}_m(\alpha,\gamma)$. Then there exists a function $p \in P_m(\gamma)$ such that*

$$f_{2k}(z) = \left(\frac{1}{1-\alpha}\int_0^z u^{\alpha/(1-\alpha)}\exp\left(\frac{1}{1-\alpha}\int_0^u \frac{h(t)-1}{t}\,\mathrm{d}t\right)\mathrm{d}u\right)^{1-\alpha}, \tag{2.17}$$

where

$$h(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}\left(p(\varepsilon^\upsilon z) + \overline{p(\varepsilon^\upsilon \overline{z})}\right). \tag{2.18}$$

P r o o f. Since $f \in \mathcal{M}_m^{s-2k}(\alpha,\gamma)$, there exists a function $p \in P_m(\gamma)$ such that

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f'_{2k}(z)} = p(z). \tag{2.19}$$

Using similar arguments given in the proof of Theorem 2.1 to (2.19) we obtain

$$\alpha\frac{zf'_{2k}(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'_{2k}(z))'}{f'_{2k}(z)} = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}\bigl(p(\varepsilon^{\upsilon}z) + \overline{p(\varepsilon^{\upsilon}\overline{z})}\bigr) = h(z). \tag{2.20}$$

Let us define F as

$$\alpha\frac{zf'_{2k}(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'_{2k}(z))'}{f'_{2k}(z)} = \frac{zF'(z)}{F(z)},$$

then

$$f_{2k}(z) = \left(\frac{1}{1-\alpha}\int_0^z \frac{(F(t))^{1/(1-\alpha)}}{t}\,\mathrm{d}t\right)^{1-\alpha} \tag{2.21}$$

and the function F is analytic with $F(0)=0$ and from (2.20) we can write

$$\frac{zF'(z)}{F(z)} = h(z).$$

Now by solving the last equation and putting its response into equality (2.21) we get the result and the proof is complete. □

Theorem 2.4. *Let $0 \leqslant \alpha < 1$ and $f \in \mathcal{M}_m^{s-2k}(\alpha,\gamma)$. Then there exists a function $p \in P_m(\gamma)$ such that*

$$f'(z) = \frac{1}{(1-\alpha)^{1-\alpha}}\frac{\int_0^1 u^{\alpha/(1-\alpha)}\exp((1-\alpha)^{-1}\int_0^{uz}(h(t)-1)t^{-1}\,\mathrm{d}t)p(u)\,\mathrm{d}u}{\bigl(\int_0^1 u^{\alpha/(1-\alpha)}\exp((1-\alpha)^{-1}\int_0^{uz}(h(t)-1)t^{-1}\,\mathrm{d}t)\,\mathrm{d}u\bigr)^{\alpha}}, \tag{2.22}$$

where h is given by (2.18).

P r o o f. Suppose that $f \in \mathcal{M}_m^{s-2k}(\alpha,\gamma)$, we can get

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f'_{2k}(z)} \in P_k(\gamma),$$

so there exists a function $p \in P_k(\gamma)$ such that

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f'_{2k}(z)} = p(z).$$

Taking $F(z) = zf'(z)$ and $G(z) = f_{2k}(z)$ in the above equation yields

$$\alpha\frac{F(z)}{G(z)} + (1-\alpha)\frac{F'(z)}{G'(z)} = p(z),$$

or

$$F'(z) + \frac{\alpha}{1-\alpha}\frac{G'(z)}{G(z)}F(z) = \frac{p(z)G'(z)}{1-\alpha}. \tag{2.23}$$

Now solving Cauchy problem (2.23) and considering (2.17) we get our result and the proof is complete. □

Theorem 2.5. *Let $f, g \in \mathcal{M}_2^{s-2k}(\alpha,\gamma)$ and suppose that F is defined by*

$$F(z) = \frac{1}{\delta z^{1/\delta-1}}\int_0^z t^{1/\delta-2}(f_{2k}(t))^{\beta/(1+\beta)}(g_{2k}(t))^{1/(1+\beta)}\,\mathrm{d}t, \tag{2.24}$$

where $z \in E$, $\delta > 0$, $\beta \geqslant 0$ and $\gamma + \delta^{-1} - 1 > 0$. Then F belongs to $\mathcal{M}_2^{s-2k}(1,\gamma)$.

Proof. Since $f, g \in \mathcal{M}_2^{s-2k}(\alpha,\gamma)$, by applying Theorem 2.1 and Corollary 2.3 we obtain $f_{2k}, g_{2k} \in \mathcal{M}_2^{s-2k}(1,\gamma)$. Differentiating (2.24) logarithmically and setting $p(z) = zF'(z)/F(z)$, we have

$$p(z) + \frac{zp'(z)}{p(z)+\delta^{-1}-1} = \frac{\beta}{1+\beta}\frac{zf'_{2k}(z)}{f_{2k}(z)} + \frac{1}{1+\beta}\frac{zg'_{2k}(z)}{g_{2k}(z)}. \tag{2.25}$$

Since the functions $zf'_{2k}(z)/f_{2k}(z)$ and $zg'_{2k}(z)/g_{2k}(z)$ belong to $P_2(\gamma)$ in E, and $P_2(\gamma)$ is a convex set,

$$\frac{\beta}{1+\beta}\frac{zf'_{2k}(z)}{f_{2k}(z)} + \frac{1}{1+\beta}\frac{zg'_{2k}(z)}{g_{2k}(z)} \in P_2(\gamma).$$

We now apply Lemma 1.2 to obtain $p(z) \in P_2(\gamma)$ and the proof is complete. □

Let $L(r,f)$ denote the length of the image of the circle $|z| = r$ under f. We prove the following.

Theorem 2.6. *Let $f \in \mathcal{H}_2^{s-2k}(1,\gamma)$. Then for $0 < r < 1$,*

$$L(r,f) \leqslant \frac{4\pi(1-\gamma)}{(1-r)^{(k+2)/k}}. \tag{2.26}$$

Proof. Using Theorem 2.2 and in view of the definition of class $\mathcal{H}_2^{s-2k}(1,\gamma)$ there exists a function $g \in \mathcal{M}_2^{s-2k}(1,\gamma)$ such that

$$zf'(z) = \psi(z)h(z), \quad \psi = g_{2k} \in S^*(\gamma),\ h \in P_2(\gamma). \tag{2.27}$$

Since $\psi \in S^*(\gamma)$ and ψ is a k-fold symmetric function, there exists a k-fold symmetric function $\psi_1(z)$ such that

$$\psi(z) = z\Big(\frac{\psi_1(z)}{z}\Big)^{1-\gamma}.$$

Now for $z = re^{i\theta}$ we have

$$\begin{aligned} L(r,f) &= \int_0^{2\pi} |zf'(z)|\,d\theta \\ &= \int_0^{2\pi} \Big|z\Big(\frac{\psi_1(z)}{z}\Big)^{1-\gamma} h(z)\Big|\,d\theta = r^\gamma \int_0^{2\pi} |(\psi_1(z))^{1-\gamma} h(z)|\,d\theta, \end{aligned}$$

and so, using Hölder's inequality, we obtain

$$L(r,f) \leqslant 2\pi r^\gamma \Big(\frac{1}{2\pi}\int_0^{2\pi} |\psi_1(z)|^2\,d\theta\Big)^{1/2} \Big(\frac{1}{2\pi}\int_0^{2\pi} |h(z)|^2\,d\theta\Big)^{1/2}. \tag{2.28}$$

For $h \in P_2(\gamma)$, from the Parseval's identity it is easy to see that

$$\frac{1}{2\pi}\int_0^{2\pi} |h(z)|^2\,d\theta \leqslant \frac{1+(4(1-\gamma)^2-1)r^2}{1-r^2}. \tag{2.29}$$

Also for k-fold symmetric function ψ_1 it is known that (see [4])

$$|\psi_1(z)| \leqslant \frac{|z|}{(1-|z|^k)^{2/k}}. \tag{2.30}$$

Using (2.29) and (2.30) in (2.28), it follows that

$$L(r,f) \leqslant 2\pi r^\gamma \Big(\frac{1+(4(1-\gamma)^2-1)r^2}{1-r^2}\Big)^{1/2} \frac{r}{(1-r^k)^{2/k}} \leqslant \frac{4\pi(1-\gamma)}{(1-r)^{1+2/k}}.$$

This completes the proof. □

Theorem 2.7. *Let $f \in \mathcal{H}_2^{s-2k}(1,\gamma)$. Then for $0 < r < 1$,*

$$|a_n| \leqslant 4\pi(1-\gamma)n^{2/k}. \tag{2.31}$$

P r o o f. Since with $z = r\mathrm{e}^{i\theta}$ Cauchy Theorem gives

$$na_n = \frac{1}{2\pi r^n}\int_0^{2\pi} zf'(z)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta,$$

then

$$n|a_n| \leqslant \frac{1}{2\pi r^n}\int_0^{2\pi} |zf'(z)|\,\mathrm{d}\theta = \frac{1}{2\pi r^n}L(r,f).$$

Using Theorem 2.6 and putting $r = 1 - n^{-1}$, $n \to \infty$, we obtain the required result. $\square$

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