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EXISTENCE OF SOLUTIONS OF GENERALIZED FRACTIONAL DIFFERENTIAL EQUATION WITH NONLOCAL INITIAL CONDITION

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Abstract. This paper is devoted to studying the existence of solutions of a nonlocal initial value problem involving generalized Katugampola fractional derivative. By using fixed point theorems, the results are obtained in weighted space of continuous functions. Illustrative examples are also given.

Keywords: fractional derivative; fractional integral; existence of solution; fractional differential equation; fixed point theorem

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1. Introduction

Fractional calculus has been proved to be an adequate tool in various areas of applied mathematics, physics and engineering. The fractional differential equations are used to describe the abundant phenomena such as flow in nonlinear electric circuits, properties of viscoelastic and dielectric materials, nonlinear oscillations of earthquake, mechanics, aerodynamics, regular variations in thermodynamics, etc. Indeed, after the appearance of the work by Bagley and Torvik (see [3]–[5]), the fractional calculus ranged from theoretical aspect towards applications, also see [18], [17], [24]. In past decades, considerable attention has been given to the existence of solutions of initial and boundary value problems. For the fundamental results in existence theory see survey papers [1], [2], the monographs by Kilbas et al. [23], Podlubny [26], the papers [27], [7]–[15], [19], [22], [25], [28] and references therein.

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Recently, the authors in [7] discussed the existence and stability of solution of the initial value problem (IVP)

$$(1) \qquad \begin{cases} ({}^{\varrho}D^{\alpha,\beta}_{a+}x)(t) = f(t, x(t)), & t \in \Omega = [a, b], \\ ({}^{\varrho}I^{1-\gamma}_{a+}x)(a) = c_2, & c_2 \in \mathbb{R}, \ \gamma = \alpha + \beta(1-\alpha) \end{cases}$$

for generalized Katugampola fractional differential equation by using Schauder fixed point theorem and the equivalence between IVP (1) and the integral equation

$$(2) \qquad x(t) = \frac{c_2}{\Gamma(\gamma)}\Big(\frac{t^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\gamma-1} + \int_a^t s^{\varrho-1}\Big(\frac{t^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1}\frac{f(s, x(s))}{\Gamma(\alpha)}\,\mathrm{d}s.$$

In this paper, we study the following IVP with nonlocal initial condition:

$$(3) \qquad \begin{cases} ({}^{\varrho}D^{\alpha,\beta}_{a+}x)(t) = f(t, x(t)), & 0 < \alpha < 1, \ 0 \leqslant \beta \leqslant 1, \ t \in (a, T], \\ ({}^{\varrho}I^{1-\gamma}_{a+}x)(a+) = \sum\limits_{j=1}^{m} \eta_j x(\xi_j), & \alpha \leqslant \gamma = \alpha + \beta(1-\alpha), \ \xi_j \in (a, T], \end{cases}$$

where ${}^{\varrho}D^{\alpha,\beta}_{a+}$ is the generalized Katugampola fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$, and ${}^{\varrho}I^{1-\gamma}_{a+}$ is the Katugampola fractional integral with $\varrho > 0$, see Section 2 for definitions. Function $f\colon (a, T] \times \mathbb{R} \to \mathbb{R}$ is a given function, ξ_j are pre-fixed points satisfying $0 < a < \xi_1 \leqslant \ldots \leqslant \xi_m < T$ and η_j, $j = 1, 2, \ldots, m$ are real numbers. We study the existence of solutions of nonlocal initial value problem (NIVP) (3). First, we establish an equivalent mixed-type nonlinear Volterra integral equation

$$(4) \qquad \begin{aligned} x(t) = {} & \frac{K}{\Gamma(\alpha)}\Big(\frac{t^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s, x(s))\,\mathrm{d}s \\ & + \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\Big(\frac{t^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s, x(s))\,\mathrm{d}s, \end{aligned}$$

where

$$(5) \qquad K = \Big(\Gamma(\gamma) - \sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\gamma-1}\Big)^{-1}$$

for NIVP (3) in the weighted space of continuous functions $C_{1-\gamma,\varrho}[a, T]$ presented in the next section. We utilize the Kasnosel'skii fixed point theorem, Schauder fixed point theorem and Schaefer fixed point theorem to obtain the existence results of considered NIVP (3).

We start with some preliminaries in Section 2. We prove the equivalent integral equation in Section 3 followed by existence results in Section 4. An illustrative examples are given in the last section.

2. Preliminaries

In this section, we list some definitions and lemmas required throughout the paper. Let the Euler gamma and beta functions be defined, respectively, by

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\,\mathrm{d}x,\quad \mathbf{B}(\alpha,\beta)=\int_0^1(1-x)^{\alpha-1}x^{\beta-1}\,\mathrm{d}x,\quad \alpha>0,\ \beta>0.$$

It is well known that $\mathbf{B}(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ for $\alpha>0$, $\beta>0$, see [23]. Throughout the paper, we consider $[a,T]$, $0<a<T<\infty$ being a finite interval on $\mathbb{R}^+$ and $\varrho>0$.

Definition 2.1 ([23]). The space $X_c^p(a,T)$, $c\in\mathbb{R}$, $p\geqslant 1$ consists of those real-valued Lebesgue measurable functions g on (a,T) for which $\|g\|_{X_c^p}<\infty$, where

$$\|g\|_{X_c^p}=\Big(\int_a^b |t^c g(t)|^p\frac{\mathrm{d}t}{t}\Big)^{1/p},\quad p\geqslant 1\quad\text{and}\quad \|g\|_{X_c^\infty}=\operatorname*{ess\,sup}_{a\leqslant t\leqslant T}|t^c g(t)|.$$

In particular, when $c=1/p$, we see that $X_{1/p}^c(a,T)=L_p(a,T)$.

Definition 2.2 ([25]). We denote by $C[a,T]$ a space of continuous functions g on $(a,T]$ with the norm

$$\|g\|_C=\max_{t\in[a,T]}|g(t)|.$$

The weighted space $C_{\gamma,\varrho}[a,T]$, $0\leqslant\gamma<1$ of functions g on $(a,T]$ is defined as

$$C_{\gamma,\varrho}[a,T]=\Big\{g\colon\ (a,T]\to\mathbb{R}\colon\ \Big(\frac{t^\varrho-a^\varrho}{\varrho}\Big)^\gamma g(t)\in C[a,T]\Big\}\tag{6}$$

with the norm

$$\|g\|_{C_{\gamma,\varrho}}=\Big\|\Big(\frac{t^\varrho-a^\varrho}{\varrho}\Big)^\gamma g(t)\Big\|_C=\max_{t\in[a,T]}\Big|\Big(\frac{t^\varrho-a^\varrho}{\varrho}\Big)^\gamma g(t)\Big|,$$

and $C_{0,\varrho}[a,T]=C[a,T]$.

Definition 2.3 ([25]). Let $\delta_\varrho=(t^{\varrho-1}\mathrm{d}/\mathrm{d}t)$, $0\leqslant\gamma<1$. Denote $C^n_{\delta_\varrho,\gamma}[a,T]$ the Banach space of functions g which are continuously differentiable, with δ_ϱ, on $[a,T]$ upto order $(n-1)$ and have the derivative $\delta_\varrho^n g$ on $(a,T]$ such that $\delta_\varrho^n g\in C_{\gamma,\varrho}[a,T]$:

$$C^n_{\delta_\varrho,\gamma}[a,T]=\{\delta_\varrho^k g\in C[a,T],\ k=0,1,\dots,n-1,\ \delta_\varrho^n g\in C_{\gamma,\varrho}[a,T]\},\quad n\in\mathbb{N}$$

with the norm

$$\|g\|_{C^n_{\delta_\varrho,\gamma}}=\sum_{k=0}^{n-1}\|\delta_\varrho^k g\|_C+\|\delta_\varrho^n g\|_{C_{\gamma,\varrho}},\quad \|g\|_{C^n_{\delta_\varrho}}=\sum_{k=0}^{n}\max_{t\in\Omega}|\delta_\varrho^k g(t)|.$$

In particular, for $n=0$ we have $C^0_{\delta_\varrho,\gamma}[a,T]=C_{\gamma,\varrho}[a,T]$.

Definition 2.4 ([20]). Let $\alpha > 0$ and $f \in X_c^p(a,T)$, where X_c^p is as in Definition 2.1. The left-sided Katugampola fractional integral ${}^{\varrho}I_{a+}^{\alpha}$ of order α is defined as

$$
{}^{\varrho}I_{a+}^{\alpha} f(t) = \int_a^t s^{\varrho-1}\Big(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1}\frac{f(s)}{\Gamma(\alpha)}\,\mathrm{d}s, \quad t > a. \tag{7}
$$

Definition 2.5 ([21]). Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $n = [\alpha]+1$, where $[\alpha]$ is the integer part of α. The left-sided Katugampola fractional derivative ${}^{\varrho}D_{a+}^{\alpha}$ is defined as

$$
\begin{aligned}
{}^{\varrho}D_{a+}^{\alpha} f(t) &= \delta_{\varrho}^{n}({}^{\varrho}I_{a+}^{n-\alpha} f(s))(t) \\
&= \Big(t^{\varrho-1}\frac{\mathrm{d}}{\mathrm{d}t}\Big)^{n}\int_a^t s^{\varrho-1}\Big(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\Big)^{n-\alpha-1}\frac{f(s)}{\Gamma(n-\alpha)}\,\mathrm{d}s.
\end{aligned} \tag{8}
$$

Definition 2.6 ([25]). The left-sided generalized Katugampola fractional derivative ${}^{\varrho}D_{a+}^{\alpha,\beta}$ of order $0 < \alpha < 1$ and type $0 \leqslant \beta \leqslant 1$ is defined as

$$
({}^{\varrho}D_{a+}^{\alpha,\beta} f)(t) = ({}^{\varrho}I_{a+}^{\beta(1-\alpha)}\delta_{\varrho}\,{}^{\varrho}I_{a+}^{(1-\beta)(1-\alpha)} f)(t) \tag{9}
$$

for the functions for which the right-hand side expression exists.

Lemma 2.7. *Suppose that $\alpha > 0$, $\beta > 0$, $p \geqslant 1$ and $\varrho, c \in \mathbb{R}$ such that $\varrho \geqslant c$. Then for $f \in X_c^p(a,T)$, the semigroup property of Katugampola integral is valid. This is*

$$
{}^{\varrho}I_{a+}^{\alpha}\,{}^{\varrho}I_{a+}^{\beta} f(t) = {}^{\varrho}I_{a+}^{\alpha+\beta} f(t). \tag{10}
$$

Lemma 2.8 ([21]). *Suppose that $\alpha > 0$, $0 \leqslant \gamma < 1$ and $f \in C_{\gamma,\varrho}[a,T]$. Then for all $t \in (a,T]$,*

$$
{}^{\varrho}D_{a+}^{\alpha}\,{}^{\varrho}I_{a+}^{\alpha} f(t) = f(t).
$$

Lemma 2.9 ([21]). *Suppose that $\alpha > 0$, $0 \leqslant \gamma < 1$, $f \in C_{\gamma,\varrho}[a,T]$ and ${}^{\varrho}I_{a+}^{1-\alpha} f \in C_{\gamma,\varrho}^{1}[a,T]$. Then*

$$
{}^{\varrho}I_{a+}^{\alpha}\,{}^{\varrho}D_{a+}^{\alpha} f(t) = f(t) - \frac{{}^{\varrho}I_{a+}^{1-\alpha} f(a)}{\Gamma(\alpha)}\Big(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\alpha-1}.
$$

Lemma 2.10. *Suppose* ${}^{\varrho}I^{\alpha}_{a+}$ *and* ${}^{\varrho}D^{\alpha}_{a+}$ *are defined as in Definitions* 2.4 *and* 2.5, *respectively. Then*

$$\begin{aligned}
{}^{\varrho}I^{\alpha}_{a+}\Big(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\sigma-1} &= \frac{\Gamma(\sigma)}{\Gamma(\sigma+1)}\Big(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\alpha+\sigma-1}, \quad \alpha \geqslant 0,\ \sigma > 0,\ t > a,\\
{}^{\varrho}D^{\alpha}_{a+}\Big(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\alpha-1} &= 0, \quad 0 < \alpha < 1.
\end{aligned}$$

Remark 2.11. For $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, the generalized Katugampola fractional derivative ${}^{\varrho}D^{\alpha,\beta}_{a+}$ can be written in terms of Katugampola fractional derivative as

$$ {}^{\varrho}D^{\alpha,\beta}_{a+} = {}^{\varrho}I^{\beta(1-\alpha)}_{a+}\delta_{\varrho}\,{}^{\varrho}I^{1-\gamma}_{a+} = {}^{\varrho}I^{\beta(1-\alpha)}_{a+}\,{}^{\varrho}D^{\gamma}_{a+}, \quad \gamma = \alpha + \beta(1-\alpha).$$

Lemma 2.12 ([25])**.** *Let* $\alpha > 0$, $0 < \gamma \leqslant 1$ *and* $f \in C_{1-\gamma,\varrho}[a,b]$. *If* $\alpha > \gamma$, *then*

$$({}^{\varrho}I^{\alpha}_{a+}f)(a) = \lim_{x\to a+}({}^{\varrho}I^{\alpha}_{a+}f)(t) = 0.$$

To discuss the existence of a solution of NIVP (3), we need the following spaces:

$$C^{\alpha,\beta}_{1-\gamma,\varrho}[a,T] = \{g \in C_{1-\gamma,\varrho}[a,T]\colon\ {}^{\varrho}D^{\alpha,\beta}_{a+}g \in C_{1-\gamma,\varrho}[a,T]\}, \quad 0 < \gamma \leqslant 1 \tag{11}$$

and

$$C^{\gamma}_{1-\gamma,\varrho}[a,T] = \{g \in C_{1-\gamma,\varrho}[a,T]\colon\ {}^{\varrho}D^{\gamma}_{a+}g \in C_{1-\gamma,\varrho}[a,T]\}, \quad 0 < \gamma \leqslant 1.$$

Since ${}^{\varrho}D^{\alpha,\beta}_{a+}g = {}^{\varrho}I^{\beta(1-\alpha)}_{a+}\,{}^{\varrho}D^{\gamma}_{a+}g$, it is obvious that $C^{\gamma}_{1-\gamma,\varrho}[a,T] \subset C^{\alpha,\beta}_{1-\gamma,\varrho}[a,T]$.

Lemma 2.13. *Let* $\alpha > 0$, $\beta > 0$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $g \in C^{\gamma}_{1-\gamma,\varrho}[a,T]$, *then*

$${}^{\varrho}I^{\gamma}_{a+}\,{}^{\varrho}D^{\gamma}_{a+}g(t) = {}^{\varrho}I^{\alpha}_{a+}\,{}^{\varrho}D^{\alpha,\beta}_{a+}g(t) = {}^{\varrho}D^{\beta(1-\alpha)}_{a+}g(t).$$

3. Equivalent integral equation

To establish the equivalence between NIVP (3) with Volterra integral equation (4), we recall the following lemma.

Lemma 3.1 ([25]). *Let* $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha+\beta-\alpha\beta$. *If* $f\colon (a,T]\times\mathbb{R} \to \mathbb{R}$ *is a function such that* $f(\cdot, x(\cdot)) \in C_{1-\gamma,\varrho}[a,T]$ *for any* $x \in C_{1-\gamma,\varrho}[a,T]$, *then* $x \in C^{\gamma}_{1-\gamma,\varrho}[a,T]$ *satisfies* IVP (1) *if and only if* x *satisfies the nonlinear Volterra integral equation* (2).

Using the aforementioned equivalence, we prove a new equivalent mixed-type integral equation for NIVP (3).

Lemma 3.2. *Consider* $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *Suppose that* $f\colon (a,T] \times \mathbb{R} \to \mathbb{R}$ *is a function such that* $f(\cdot, x(\cdot)) \in C_{1-\gamma,\varrho}[a,T]$ *for any* $x(\cdot) \in C_{1-\gamma,\varrho}[a,T]$. *Function* $x(\cdot) \in C^{\gamma}_{1-\gamma,\varrho}[a,T]$ *is a solution of* NIVP (3) *if and only if* $x(\cdot)$ *is a solution of the mixed-type nonlinear Volterra integral equation* (4).

Proof. First, we prove the necessary condition. In the light of Lemma 3.1, a solution of NIVP (3) can be expressed as

$$x(t) = \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)}\Big(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\gamma-1} + \int_a^t s^{\varrho-1}\Big(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1}\frac{f(s,x(s))}{\Gamma(\alpha)}\,\mathrm{d}s. \tag{12}$$

A substitution $t = \xi_j$ in (12) yields

$$x(\xi_j) = \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)}\Big(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\gamma-1} + \int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1}\frac{f(s,x(s))}{\Gamma(\alpha)}\,\mathrm{d}s \tag{13}$$

and by multiplying both sides of (13) by η_j, we obtain

$$\eta_j x(\xi_j) = \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)}\eta_j\Big(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\gamma-1} + \eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1}\frac{f(s,x(s))}{\Gamma(\alpha)}\,\mathrm{d}s.$$

By the initial condition of NIVP (3),

$$\begin{aligned}{}^{\varrho}I^{1-\gamma}_{a+}x(a+) = \sum_{j=1}^{m}\eta_j x(\xi_j) &= \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)}\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\gamma-1}\\ &\quad+\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1}\frac{f(s,x(s))}{\Gamma(\alpha)}\,\mathrm{d}s,\end{aligned}$$

which implies

$$
\begin{aligned}
{}^{\varrho}I_{a+}^{1-\gamma}x(a+)\Big(\Gamma(\gamma) - \sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\gamma-1}\Big)
\\
= \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s,x(s))\,\mathrm{d}s
\end{aligned}
$$

i.e.

$$
{}^{\varrho}I_{a+}^{1-\gamma}x(a+) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s,x(s))\,\mathrm{d}s, \tag{14}
$$

where K is as in (5). Substituting (14) into (12), we obtain the integral equation (4).

Now we prove the sufficient condition. Applying ${}^{\varrho}I_{a+}^{1-\gamma}$ on both sides of the integral equation (4), we get

$$
\begin{aligned}
{}^{\varrho}I_{a+}^{1-\gamma}x(t) = {}^{\varrho}I_{a+}^{1-\gamma}\Big(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\gamma-1}K\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1}\frac{f(s,x(s))}{\Gamma(\alpha)}\,\mathrm{d}s
\\
+ {}^{\varrho}I_{a+}^{1-\gamma}\,{}^{\varrho}I_{a+}^{\alpha}f(t,x(t)),
\end{aligned}
$$

using Lemmas 2.7 and 2.10, we have

$$
{}^{\varrho}I_{a+}^{1-\gamma}x(t) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s,x(s))\,\mathrm{d}s + {}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}f(t,x(t)).
$$

Since $1-\gamma < 1-\beta(1-\alpha)$, Lemma 2.12 can be utilized and limit $t\to a+$ yields

$$
{}^{\varrho}I_{a+}^{1-\gamma}x(a) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s,x(s))\,\mathrm{d}s. \tag{15}
$$

A substitution $t=\xi_j$ in (4) gives us

$$
\begin{aligned}
x(\xi_j) = \frac{K}{\Gamma(\alpha)}\Big(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s,x(s))\,\mathrm{d}s
\\
+ \frac{1}{\Gamma(\alpha)}\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s,x(s))\,\mathrm{d}s.
\end{aligned}
$$

Further,

$$\begin{aligned}
(16)\quad & \sum_{j=1}^{m} \eta_j x(\xi_j) \\
&= \frac{K}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \Big(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s, x(s))\, \mathrm{d}s \sum_{j=1}^{m} \eta_j \Big(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\gamma-1} \\
&\quad + \sum_{j=1}^{m} \eta_j \frac{1}{\Gamma(\alpha)} \int_a^{\xi_j} s^{\varrho-1} \Big(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s, x(s))\, \mathrm{d}s \\
&= \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \Big(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1} \frac{f(s, x(s))}{\Gamma(\alpha)}\, \mathrm{d}s \Big(1 + K \sum_{j=1}^{m} \eta_j \Big(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\gamma-1}\Big) \\
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} K \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \Big(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1} f(s, x(s))\, \mathrm{d}s.
\end{aligned}$$

Linking (15) and (16), it follows that

$$^{\varrho}I_{a+}^{1-\gamma} x(a+) = \sum_{j=1}^{m} \eta_j x(\xi_j).$$

Applying $^{\varrho}D_{a+}^{\gamma}$ to both sides of (4), from Lemmas 2.10 and 3.1 if follows that

$$(17)\qquad ^{\varrho}D_{a+}^{\gamma} x(t) = {}^{\varrho}D_{a+}^{\beta(1-\alpha)} f(t, x(t)).$$

Since $x \in C_{1-\gamma,\varrho}^{\gamma}[a,T]$, from the definition of $C_{1-\gamma,\varrho}^{\gamma}[a,T]$ we have

$$^{\varrho}D_{a+}^{\gamma} x \in C_{1-\gamma,\varrho}[a,T] \quad \text{then} \quad {}^{\varrho}D_{a+}^{\beta(1-\alpha)} f = \delta_{\varrho}{}^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\varrho}[a,T].$$

For $f \in C_{1-\gamma,\varrho}[a,T]$, obviously $^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\varrho}[a,T]$, then $^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\varrho}^{\delta_{\varrho}}[a,T]$. This means f and $^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f$ satisfy the conditions of Lemma 2.9. Lastly, applying $^{\varrho}I_{a+}^{1-\beta(1-\alpha)}$ to both sides of (17), Lemma 2.9 helps us to obtain

$$^{\varrho}D_{a+}^{\alpha,\beta} x(t) = f(t, x(t)) - \frac{^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f(a)}{\Gamma(\beta(1-\alpha))} \Big(\frac{t^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\beta(1-\alpha)-1}.$$

By Lemma 2.12, it is easy to see that $^{\varrho}I_{a+}^{1-\beta(1-\alpha)} f(a) = 0$. Hence, it reduces to

$$^{\varrho}D_{a+}^{\alpha,\beta} x(t) = f(t, x(t)).$$

Hence, the sufficiency is proved. This completes the proof of the lemma. □

4. Existence of solutions

In this section, we state and prove the main results concerning the existence of a solution of NIVP (3). We consider the following hypotheses:

(H$_{01}$) $f\colon (a,T]\times\mathbb{R}\to\mathbb{R}$ is a function such that $f(\cdot,x(\cdot))\in C^{\beta(1-\alpha)}_{1-\gamma,\varrho}[a,T]$ for any $x\in C_{1-\gamma,\varrho}[a,T]$ and there exists a positive constant $L>0$ such that for all $x,\bar{x}\in\mathbb{R}$,
$$|f(t,x(t))-f(t,\bar{x}(t))|\leqslant L|x-\bar{x}|.$$

(H$_{02}$) The constant
$$\theta=\frac{\Gamma(\gamma)L}{\Gamma(\gamma+\alpha)}\Big(|K|\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\alpha+\gamma-1}+\Big(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\alpha}\Big)<1,$$
where K is as in (5).

Now we prove the first existence result for NIVP (3) by using Kasnosel'skii fixed point theorem.

Theorem 4.1. *Suppose that* (H$_{01}$) *and* (H$_{02}$) *are satisfied. Then* NIVP (3) *has at least one solution in* $C^{\gamma}_{1-\gamma,\varrho}[a,T]\subset C^{\alpha,\beta}_{1-\gamma,\varrho}[a,T]$.

Proof. In the light of Lemma 3.2, it is sufficient to prove the existence of a solution for mixed-type integral equation (4). Define $N\colon C_{1-\gamma,\varrho}[a,T]\to C_{1-\gamma,\varrho}[a,T]$ by

$$(18)\quad (Nx)(t)=\frac{K}{\Gamma(\alpha)}\Big(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\Big(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1}f(s,x(s))\,\mathrm{d}s$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\Big(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\Big)^{\alpha-1}f(s,x(s))\,\mathrm{d}s.$$

Clearly, the operator N is well defined.

Set $\overline{f}(s)=f(s,0)$ and
$$\varpi=\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\Big(|K|\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\alpha+\gamma-1}+\Big(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\Big)^{\alpha}\Big)\|\overline{f}\|_{C_{1-\gamma,\varrho}}.$$

Consider a ball
$$B_r=\{x\in C_{1-\gamma,\varrho}[a,T]\colon\ \|x\|_{C_{1-\gamma,\varrho}}\leqslant r\},\quad \text{where } r\geqslant\frac{\varpi}{1-\theta},\ \theta<1.$$

Now, we subdivide the operator N into two operators P and Q on B_r as follows:

$$(Px)(t) = \frac{K}{\Gamma(\alpha)}\Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{\gamma-1}\sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} f(s,x(s))\,\mathrm{d}s \tag{19}$$

and

$$(Qx)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} f(s,x(s))\,\mathrm{d}s. \tag{20}$$

The proof is given in the following several steps:

Step 1. For any $x,\bar{x} \in B_r$ we prove $Px + Q\bar{x} \in B_r$. For operator P, multiplying both sides of (19) by $((t^\varrho - a^\varrho)/\varrho)^{1-\gamma}$, we have

$$(Px)(t)\Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma} = \frac{K}{\Gamma(\alpha)}\sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} f(s,x(s))\,\mathrm{d}s,$$

then

$$\begin{aligned}
\Big|(Px)(t)\Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma}\Big| &\leqslant \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} |f(s,x(s))|\,\mathrm{d}s \\
&\leqslant \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} (|f(s,x(s)) - f(s,0)| + |f(s,0)|)\,\mathrm{d}s \\
&\leqslant \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} (L|x(s)| + |\overline{f}(s)|)\,\mathrm{d}s \\
&\leqslant \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1}\Big(\frac{s^\varrho - a^\varrho}{\varrho}\Big)^{\gamma-1} \\
&\quad \times \Big(\Big(\frac{s^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma} L|x(s)| + \Big(\frac{s^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma}|\overline{f}(s)|\Big)\,\mathrm{d}s \\
&\leqslant \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1}\Big(\frac{s^\varrho - a^\varrho}{\varrho}\Big)^{\gamma-1} \\
&\quad \times (L\|x\|_{C_{1-\gamma,\varrho}} + \|\overline{f}\|_{C_{1-\gamma,\varrho}}) \\
&= \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^m \eta_j \Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1}\int_0^1 (1-u)^{\alpha-1}u^{\gamma-1}\,\mathrm{d}u \\
&\quad \times (L\|x\|_{C_{1-\gamma,\varrho}} + \|\overline{f}\|_{C_{1-\gamma,\varrho}}) \\
&= \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^m \eta_j \Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1}\mathbf{B}(\alpha,\gamma)(L\|x\|_{C_{1-\gamma,\varrho}} + \|\overline{f}\|_{C_{1-\gamma,\varrho}}).
\end{aligned}$$

This gives

(21) $$\|Px\|_{C_{1-\gamma,\varrho}} \leqslant \frac{\Gamma(\gamma)|K|}{\Gamma(\alpha+\gamma)} \sum_{j=1}^{m} \eta_j \Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1} (L\|x\|_{C_{1-\gamma,\varrho}} + \|\overline{f}\|_{C_{1-\gamma,\varrho}}).$$

For operator Q,

$$\Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma}(Qx)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t s^{\varrho-1} \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} \Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma} f(s,x(s))\,\mathrm{d}s$$

using the same fact that we used in the case of operator P again, we obtain

$$\begin{aligned}
&\Big|(Qx)(t)\Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma}\Big| \\
&\quad\leqslant \frac{1}{\Gamma(\alpha)} \int_a^t s^{\varrho-1} \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} \Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma} |f(s,x(s))|\,\mathrm{d}s \\
&\quad\leqslant \Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_a^t s^{\varrho-1} \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} (L|x(s)| + |\overline{f}(s)|)\,\mathrm{d}s \\
&\quad\leqslant \frac{\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)} \Big(\frac{T^\varrho - a^\varrho}{\varrho}\Big)^{\alpha} (L\|x(s)\|_{C_{1-\gamma,\varrho}} + \|\overline{f}(s)\|_{C_{1-\gamma,\varrho}}).
\end{aligned}$$

This implies

(22) $$\|Qx\|_{C_{1-\gamma,\varrho}} \leqslant \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \Big(\frac{T^\varrho - a^\varrho}{\varrho}\Big)^{\alpha} (L\|x(s)\|_{C_{1-\gamma,\varrho}} + \|\overline{f}(s)\|_{C_{1-\gamma,\varrho}}).$$

Linking (21) and (22) for every $x, \bar{x} \in B_r$ we obtain

$$\|Px + Q\bar{x}\|_{C_{1-\gamma,\varrho}} \leqslant \|Px\|_{C_{1-\gamma,\varrho}} + \|Q\bar{x}\|_{C_{1-\gamma,\varrho}} \leqslant \theta r + \varpi \leqslant r,$$

which infers that $Px + Q\bar{x} \in B_r$.

Step 2. Operator P is a contraction mapping. Let $x, \bar{x} \in B_r$, for operator P we have

$$\begin{aligned}
&((Px)(t) - (P\bar{x})(t))\Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma} \\
&\quad= \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} (f(s,x(s)) - f(s,\bar{x}(s)))\,\mathrm{d}s \\
&\quad\leqslant \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} (|f(s,x(s)) - f(s,\bar{x}(s))|)\,\mathrm{d}s \\
&\quad\leqslant \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} L|x(s) - \bar{x}(s)|\,\mathrm{d}s \\
&\quad\leqslant \frac{L|K|\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1} \|x - \bar{x}\|_{C_{1-\gamma,\varrho}},
\end{aligned}$$

which is

$$\|Px - P\bar{x}\|_{C_{1-\gamma,\varrho}} \leqslant \frac{L|K|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \sum_{j=1}^{m} \eta_j \Big(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\alpha+\gamma-1} \|x - \bar{x}\|_{C_{1-\gamma,\varrho}} \leqslant \theta \|x - \bar{x}\|_{C_{1-\gamma,\varrho}}.$$

Thus, by assumption (H_{02}), operator P is a contraction mapping.

Step 3. Operator Q is compact and continuous. Since $f \in C_{1-\gamma,\varrho}[a,T]$, by the definition of $C_{1-\gamma,\varrho}[a,T]$, it is obvious that Q is continuous. By Step 1, we can write

$$\|Qx\|_{C_{1-\gamma,\varrho}} \leqslant \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \Big(\frac{T^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\alpha} (L\|x\|_{C_{1-\gamma,\varrho}} + \|\overline{f}\|_{C_{1-\gamma,\varrho}}),$$

this means Q is uniformly bounded on B_r.

To prove the compactness of Q, for any $0 < a < t_1 < t_2 \leqslant T$ we have

$$\begin{aligned}
(23)\qquad & |(Qx)(t_1) - (Qx)(t_2)| \\
&= \Big| \int_a^{t_1} s^{\varrho-1} \Big(\frac{{t_1}^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1} \frac{f(s,x(s))}{\Gamma(\alpha)}\, \mathrm{d}s \\
&\qquad - \int_a^{t_2} s^{\varrho-1} \Big(\frac{{t_2}^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1} \frac{f(s,x(s))}{\Gamma(\alpha)}\, \mathrm{d}s \Big| \\
&\leqslant \frac{\|f\|_{C_{1-\gamma,\varrho}}}{\Gamma(\alpha)} \Big| \int_a^{t_1} s^{\varrho-1} \Big(\frac{{t_1}^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1} \Big(\frac{s^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\gamma-1} \mathrm{d}s \\
&\qquad - \int_a^{t_2} s^{\varrho-1} \Big(\frac{{t_2}^{\varrho} - s^{\varrho}}{\varrho}\Big)^{\alpha-1} \Big(\frac{s^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\gamma-1} \mathrm{d}s \Big| \\
&\leqslant \frac{\|f\|_{C_{1-\gamma,\varrho}}\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \Big| \Big(\frac{{t_1}^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\alpha+\gamma-1} - \Big(\frac{{t_2}^{\varrho} - a^{\varrho}}{\varrho}\Big)^{\alpha+\gamma-1} \Big|.
\end{aligned}$$

Observe that the right-hand side of inequality (23) tends to zero as $t_2 \to t_1$ either $\alpha+\gamma < 1$ or $\alpha+\gamma \geqslant 1$. Thus, Q is equicontinuous. Hence, in the light of Arzelà-Ascoli theorem, Q is compact on B_r.

By Kasnosel'skii fixed point theorem, NIVP (3) has at least one solution $x \in C_{1-\gamma,\varrho}[a,T]$. One can easily show that this solution is actually in $C^{\gamma}_{1-\gamma,\varrho}[a,T]$ by repeating the process from the proof of Lemma 3.2. The proof is thus complete. □

We will study the next existence result by using Schauder fixed point theorem. For this, we modify hypothesis (H_{01}) to the following one:

(H_{11}) $f\colon (a,T] \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma,\varrho}[a,T]$ for any $x \in C_{1-\gamma,\varrho}[a,T]$, and for all $x \in \mathbb{R}$ there exist $L > 0$ and $M \geqslant 0$ such that

$$|f(t,x)| \leqslant L|x| + M.$$

Theorem 4.2. *Suppose that* (H_{11}) *and* (H_{02}) *hold. Then* NIVP (3) *has at least one solution in* $C^{\gamma}_{1-\gamma,\varrho}[a,T] \subset C^{\alpha,\beta}_{1-\gamma,\varrho}[a,T]$.

P r o o f. Let $B_\varepsilon = \{x \in C_{1-\gamma,\varrho}[a,T]\colon \|x\|_{C_{1-\gamma,\varrho}} \leqslant \varepsilon\}$ with $\varepsilon \geqslant \Omega/(1-\theta)$ for $\theta < 1$, where

$$\Omega = \frac{M|K|}{\Gamma(\alpha+1)}\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha} + \frac{M}{\Gamma(\alpha+1)}\Big(\frac{T^\varrho - a^\varrho}{\varrho}\Big)^{\alpha-\gamma+1}.$$

Consider the operator N on B_ε defined in (18). We prove the theorem in the following three steps:

Step 1. We check that $N(B_\varepsilon) \subset B_\varepsilon$. By assumptions ($\mathrm{H}_{11}$) and ($\mathrm{H}_{02}$), for any $x \in C_{1-\gamma,\varrho}[a,T]$ and $\|x\|_{C_{1-\gamma,\varrho}}$ we have

$$\begin{aligned}
&\Big|(Nx)(t)\Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma}\Big| \\
&\quad\leqslant \bigg(\frac{L\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1} + \frac{L\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\Big(\frac{T^\varrho - a^\varrho}{\varrho}\Big)^{\alpha}\bigg)\|x\|_{C_{1-\gamma,\varrho}} \\
&\qquad + \frac{M}{\Gamma(\alpha+1)}\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha} + \frac{M}{\Gamma(\alpha+1)}\Big(\frac{T^\varrho - a^\varrho}{\varrho}\Big)^{\alpha-\gamma+1}.
\end{aligned}$$

This is

$$\|Nx\|_{C_{1-\gamma,\varrho}} \leqslant \theta\varepsilon + \Omega \leqslant \varepsilon,$$

which yields $N(B_\varepsilon) \subset B_\varepsilon$.

Now we shall prove that N is completely continuous.

Step 2. N is continuous. Let x_n be a sequence such that $x_n \to x$ in B_ε. Then for each $t \in (a,T]$ we have

$$\begin{aligned}
&\Big|((Nx)(x_n) - (Nx)(t))\Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{\gamma-1}\Big| \\
&\quad\leqslant \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1}|f(s,x_n(s)) - f(s,x(s))|\,\mathrm{d}s \\
&\qquad + \Big(\frac{t^\varrho - a^\varrho}{\varrho}\Big)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_a^{t} s^{\varrho-1}\Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1}|f(s,x_n(s)) - f(s,x(s))|\,\mathrm{d}s \\
&\quad\leqslant \frac{|K|\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1}\|f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|_{C_{1-\gamma,\varrho}} \\
&\qquad + \Big(\frac{T^\varrho - a^\varrho}{\varrho}\Big)^{\alpha}\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\|f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|_{C_{1-\gamma,\varrho}} \\
&\quad\leqslant \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\bigg(|K|\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1} + \Big(\frac{T^\varrho - a^\varrho}{\varrho}\Big)^{\alpha}\bigg) \\
&\qquad \times \|f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|_{C_{1-\gamma,\varrho}},
\end{aligned}$$

this implies

$$\|Nx_n - Nx\|_{C_{1-\gamma,\varrho}} \leqslant \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\bigg(|K|\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1} + \Big(\frac{T^\varrho - a^\varrho}{\varrho}\Big)^{\alpha}\bigg) \times \|f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|_{C_{1-\gamma,\varrho}}.$$

Thus, N is a continuous operator.

Step 3. $N(B_\varepsilon)$ is relatively compact. Since $N(B_\varepsilon) \subset B_\varepsilon$, it follows that $N(B_\varepsilon)$ is uniformly bounded.

Furthermore, by repeating the same process as in Step 3 in Theorem 4.1, one can easily prove that N is equicontinuous on B_ε.

As $\alpha \leqslant \gamma < 1$ and noting (23), for any $0 < a < t_1 < t_2 \leqslant T$ one has

$$\begin{aligned}
&|(Nx)(t_1) - (Nx)(t_2)| \\
&\quad \leqslant \frac{\|f\|_{C_{1-\gamma,\varrho}}|K|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1} \\
&\qquad \times \bigg(\Big(\frac{{t_1}^\varrho - a^\varrho}{\varrho}\Big)^{\gamma-1} - \Big(\frac{{t_2}^\varrho - a^\varrho}{\varrho}\Big)^{\gamma-1}\bigg) + |(Qx)(t_1) - (Qx)(t_2)| \\
&\quad \leqslant \frac{\|f\|_{C_{1-\gamma,\varrho}}|K|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\sum_{j=1}^{m}\eta_j\Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1}\bigg|\frac{{t_2}^\varrho - {t_1}^\varrho}{({t_1}^\varrho - a^\varrho)({t_2}^\varrho - a^\varrho)}\bigg|^{1-\gamma} \\
&\qquad + \frac{\|f\|_{C_{1-\gamma,\varrho}}\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\bigg|\Big(\frac{{t_1}^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1} - \Big(\frac{{t_2}^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1}\bigg| \to 0
\end{aligned}$$

as $t_2 \to t_1$. Thus, Q is equicontinuous.

Hence, $N(B_\varepsilon)$ is an equicontinuous set and therefore $N(B_\varepsilon)$ is relatively compact.

As a consequence of Steps 1 to 3 together with Arzelà-Ascoli theorem, we can conclude that $N\colon B_\varepsilon \to B_\varepsilon$ is completely continuous. By applying Schauder fixed point theorem, we conclude the theorem. $\square$

Next, we will prove the following existence theorem by using Schaefer fixed point theorem. For this, we again change assumption (H_{11}) into the following one:

(H_{21}) $f\colon (a,T] \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot,x(\cdot)) \in C_{1-\gamma,\varrho}^{\beta(1-\alpha)}[a,T]$ for any $x \in C_{1-\gamma,\varrho}[a,T]$, and for all $x \in \mathbb{R}$ there exist a function $\mu(t) \in C_{1-\gamma,\varrho}[a,T]$ such that

$$|f(t,x)| \leqslant \mu(t).$$

Theorem 4.3. *Suppose that* (H_{21}) *holds. Then* NIVP (3) *has at least one solution in* $C_{1-\gamma,\varrho}^{\gamma}[a,T] \subset C_{1-\gamma,\varrho}^{\alpha,\beta}[a,T]$.

P r o o f. As in the proof of Theorem 4.2, one can repeat Steps 1 to 3 and show that $N\colon C_{1-\gamma,\varrho}[a,T]\to C_{1-\gamma,\varrho}[a,T]$ defined in (18) is a completely continuous operator. It remains to prove that $S=\{x\in C_{1-\gamma,\varrho}[a,T]\colon x=\sigma Nx$ for some $\sigma\in(0,1)\}$ is a bounded set. It follows that

$$\begin{aligned}\Big|x(t)\Big(\frac{t^\varrho-a^\varrho}{\varrho}\Big)^{1-\gamma}\Big| &\leqslant \Big\|(Nx)(t)\Big(\frac{t^\varrho-a^\varrho}{\varrho}\Big)^{1-\gamma}\Big\| \\ &\leqslant \frac{\Gamma(\gamma)\|\mu\|_{C_{1-\gamma,\varrho}}}{\Gamma(\alpha+\gamma)} \\ &\quad\times\Big(|K|\sum_{j=1}^m\eta_j\Big(\frac{\xi_j^\varrho-a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1}+\Big(\frac{T^\varrho-a^\varrho}{\varrho}\Big)^{\alpha}\Big)=C_1,\end{aligned}$$

and $\|x\|_{C_{1-\gamma,\varrho}}\leqslant C_1$, which implies the boundedness of the set S. By using Schaefer fixed point theorem, the proof can be completed. □

R e m a r k 4.4. The results in [28] for Hilfer fractional differential equations with nonlocal initial condition can be obtained for $\varrho\to 1$ from the main results of this paper.

R e m a r k 4.5. The results in [16] for Cauchy problem involving Riemann-Liouville fractional derivative with nonlocal initial condition can be obtained for $\beta=0$, $\varrho\to 1$ from the main results of this paper.

R e m a r k 4.6. The continuous and integrable solution of Cauchy-type problem (see [6]) for Hilfer-Hadamard fractional differential equations with nonlocal condition can be obtained for $\varrho\to 0^+$ from the main results of this paper.

5. Examples

We consider the following illustrative examples for the NIVP (3).

E x a m p l e 5.1. Consider the nonlocal problem

$$\begin{cases}({}^\varrho D_{a+}^{\alpha,\beta}x)(t)=f(t,x(t)), & t\in(1,2],\\ ({}^\varrho I_{a+}^{1-\gamma}x)(1+)=2x\big(\frac{5}{3}\big), & \gamma=\alpha+\beta(1-\alpha).\end{cases}\tag{24}$$

Denoting $\alpha=\frac{3}{4}$, $\beta=\frac{1}{2}$ gives $\gamma=\frac{7}{8}$. Let $\varrho=\frac{1}{2}>0$ and set

$$f(t,x)=\Big(\frac{t^\varrho-1}{\varrho}\Big)^{-1/16}+\frac{1}{4}\Big(\frac{t^\varrho-1}{\varrho}\Big)^{15/16}\sin x.$$

We can see that

$$\Big(\frac{t^{1/2}-1}{\frac12}\Big)^{1/8} f(t,x(t)) = \Big(\frac{t^{1/2}-1}{\frac12}\Big)^{1/16} + \frac14\Big(\frac{t^{1/2}-1}{\frac12}\Big)^{17/16} \sin x \in C[1,2]$$

i.e. $f(t,x) \in C_{1/8,1/2}[1,2]$.

Moreover, $|f(t,x) - f(t,\bar{x})| \leqslant \frac14|x-\bar{x}|$. Some elementary computations gives us

$$|K| = \left|\left(\Gamma(0.875) - 2\Big(\frac{(\frac53)^{1/2}-1}{\frac12}\Big)^{-1/8}\right)^{-1}\right| \approx 0.9521 < 1$$

and

$$\theta = \frac{\Gamma(0.875)}{4\Gamma(1.625)}\left(|K| \times 2\Big(\frac{(\frac53)^{1/2}-1}{\frac12}\Big)^{5/8} + \Big(\frac{2^{1/2}-1}{\frac12}\Big)^{3/4}\right) \approx 0.6763 < 1.$$

All the assumptions of Theorem 4.1 are satisfied with $|K| \approx 0.9521$ and $\theta \approx 0.6763$. Therefore, NIVP (24) has at least one solution in $C_{1/8,1/2}[1,2]$.

Example 5.2. Consider the nonlocal problem

$$\begin{cases} ({}^{\varrho}D^{\alpha,\beta}_{a+}x)(t) = f(t,x(t)), \quad t \in (1,2], \\ ({}^{\varrho}I^{1-\gamma}_{a+}x)(1+) = 3x\big(\frac87\big) + 2x\big(\frac43\big). \end{cases} \tag{25}$$

Denote $\alpha = \frac12, \beta = \frac34$ and $\varrho = \frac12 > 0$. So $\gamma = \frac78$ and $(\xi_1 = \frac87) \leqslant (\xi_2 = \frac43)$. Set $f(t,x) = \sin\big(\frac13|x|\big)$ for $t \in (1,2]$. It is easy to see that $f(t,x(t)) \in C_{1/8,1/2}[1,2]$ and $|f(t,x)| \leqslant \frac13|x|$. Moreover,

$$|K| = \left|\left(\Gamma(0.875) - \left(3\Big(\frac{(\frac87)^{1/2}-1}{\frac12}\Big)^{-1/8} + 2\Big(\frac{(\frac43)^{1/2}-1}{\frac12}\Big)^{-1/8}\right)\right)^{-1}\right| \approx 0.1973 < 1$$

and

$$\theta = \frac{\Gamma(0.875)}{3\Gamma(1.375)}\left(|K| \times 3\Big(\frac{(\frac87)^{1/2}-1}{\frac12}\Big)^{3/8} + 2\Big(\frac{(\frac43)^{1/2}-1}{\frac12}\Big)^{3/8}\right) \approx 0.71 < 1.$$

With the values of $|K|$ and θ, the NIVP (25) satisfies all the conditions of Theorem 4.2. Thus, NIVP (25) has at least one solution in $C_{1/8,1/2}[1,2]$.

Example 5.3. In the NIVP (25), we denote $\alpha = \frac12$, $\beta = \frac34$, $\varrho = \frac12$ and change $f(t,x)$ into $f(t,x(t)) = ((t^{\varrho}-1)/\varrho)^{-1/8} = \mu(t)$ for $t \in (1,2]$. It is easy to see that f so defined is in $C_{1/8,1/2}[1,2]$ and $\mu(t) \in C_{1/8,1/2}[1,2]$. Now we can apply Theorem 4.3 and easily deduce that NIVP (25) has at least one solution in $C_{1/8,1/2}[1,2]$.

6. Conclusions

In this paper we have obtained the existence results for a general class of fractional differential equations with nonlocal initial condition by using fixed point theory. Results obtained in this paper generalizes the existing results in the literature. The existence of solution is assured and some suitable illustrative examples are given in the support of our main results.

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