SOME APPROXIMATE FIXED POINT THEOREMS WITHOUT CONTINUITY OF THE OPERATOR USING AUXILIARY FUNCTIONS

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Abstract. We introduce partial generalized convex contractions of order 4 and rank 4 using some auxiliary functions. We present some results on approximate fixed points and fixed points for such class of mappings having no continuity condition in α-complete metric spaces and μ-complete metric spaces. Also, as an application, some fixed point results in a metric space endowed with a binary relation and some approximate fixed point results in a metric space endowed with a graph have been obtained. Some examples are also provided to illustrate the main results and to show the usability of the given hypotheses.

Keywords: ε-fixed point; α-admissible mapping; partial generalized convex contraction of order 4 and rank 4; α-complete metric space

MSC 2010: 47H10, 54H25

1. Introduction

The fixed point theory is an important tool for solving various nonlinear problems. In several practical situations, the conditions in the fixed point theorems are too strong and so the existence of a fixed point is not guaranteed. In this situation, we can consider nearly fixed points or approximate fixed points. The study of approximate fixed point theorems is equally interesting to that of fixed point theorems.

In 2006, inspired and motivated by the work of Tijs et al. [15], Berinde [2] obtained some fundamental approximate fixed point theorems in metric spaces. In 2013, Dey and Saha in [6] established the existence of approximate fixed points for the Reich operator (see [13]) which in turn generalizes approximate fixed point theorems of Berinde (see [2]). Many results on approximate fixed points (see, for example, [5],

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[10], [13] and the references therein) have appeared. On the other hand, in 1982, Istrăţescu in [8] introduced the concept of convex contractions and proved that each convex contraction mapping has a unique fixed point on a complete metric space. In 2013, Miandaragh et al. [12] extended the concept of convex contractions to generalized convex contractions and generalized convex contractions of order 2. They have also proved some approximate fixed point theorems for continuous mappings satisfying such contractive conditions in complete metric spaces. The results of Miandaragh et al. (see [12]) were extended by Latif et al. in [11].

In this paper, the main results are given in three sections: In the first section, we introduce the concept of partial generalized convex contraction mappings of order 4 and rank 4 using auxiliary functions and provide sufficient conditions for the existence of approximate fixed points for such class of mappings in α-complete metric spaces and μ-complete metric spaces. In the second section, as an application, we obtain some fixed point results in metric spaces endowed with a binary relation. In the last section, we obtain some approximate fixed point results in metric spaces endowed with a graph. Some examples are also provided to illustrate the main results and to show the usability of the given hypotheses. These results extend, unify and generalize the main results of Latif et al. [11] and of Miandaragh et al. [12].

2. Preliminaries

Definition 2.1 ([13]). Let (X,d) be a metric space, $T\colon X \to X$ be a mapping and $\varepsilon > 0$ be a given real number. A point $x_0 \in X$ is said to be an ε-fixed point (approximate fixed point) of T if

$$d(x_0, Tx_0) < \varepsilon.$$

Remark 2.2. We observe that a fixed point is an ε-fixed point for all $\varepsilon > 0$. However, the converse is not true.

The set of all ε-fixed points of $T\colon X \to X$ is denoted by $F_\varepsilon(T)$, that is,

$$F_\varepsilon(T) := \{x \in X \colon d(x, Tx) < \varepsilon\}.$$

Definition 2.3. A mapping T has the approximate fixed point property (see [10]) if for all $\varepsilon > 0$ there exists an ε-fixed point of T, that is, for all $\varepsilon > 0$,

$$F_\varepsilon(T) \neq \emptyset$$

or equivalently,

$$\inf_{x \in X} d(x, Tx) = 0.$$

Definition 2.4 ([3]). A self mapping T on a metric space (X, d) is said to be asymptotically regular at a point $x \in X$ if

$$d(T^n x, T^{n+1} x) \to 0 \quad \text{as } n \to \infty,$$

where $T^n x$ denotes the nth iterate of T at x.

Definition 2.5. Let X be a nonempty set, $T\colon X \to X$ and $\alpha, \mu\colon X \times X \to \mathbb{R}^+$. We say that T is an α-admissible mapping (see [14]) or μ-subadmissible mapping if $\alpha(x, y) \geqslant 1$ or $\mu(x, y) \leqslant 1$ implies $\alpha(Tx, Ty) \geqslant 1$, $x, y \in X$ or $\mu(Tx, Ty) \leqslant 1$, $x, y \in X$, respectively.

Definition 2.6. Let (X, d) be a metric space and $\alpha, \mu\colon X \times X \to \mathbb{R}^+$ be mappings. The space X is said to be α-complete (see [7]) or μ-complete if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ or $\mu(x_n, x_{n+1}) \leqslant 1$, for all $n \in \mathbb{N}$, repectively, converges in X.

The following results will be used in the sequel.

Lemma 2.7 ([11]). *Let (X, d) be a metric space and $T\colon X \to X$ be asymptotically regular at a point $z \in X$. Then T has the approximate fixed point property.*

We recollect the following auxiliary results which will be used in the proof of our main results.

▷ If (X, d) is a complete metric space, then X is an α-complete metric space.
▷ If (X, d) is a complete metric space, then X is a μ-complete metric space.

But the converse parts of the above statements are not true.

Example 2.8. Let $X = (0, \infty)$ and the metric $d\colon X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $\alpha\colon X \times X \to \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases} \dfrac{x + y + 6}{x + y}, & x, y \in [2, 3], \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases}$$

It is easy to see that (X, d) is not a complete metric space, but (X, d) is an α-complete metric space. Indeed, if $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$, then $x_n \in [2, 3]$ for all $n \in \mathbb{N}$. Since $[2, 3]$ is a closed subset of $\mathbb{R}$, we get that $([2, 3], d)$ is a complete metric space and then there exists $x^* \in [2, 3]$ such that $x_n \to x^*$ as $n \to \infty$.

Definition 2.9 ([11])**.** Let (X,d) be a metric space, $\alpha,\mu\colon X\times X\to[0,\infty)$ and $T\colon X\to X$ be two mappings. We say that T is an α-continuous mapping or μ-continuous mapping on (X,d) if for each sequence $\{x_n\}$ in X with $x_n\to x\in X$ as $n\to\infty$ and $\alpha(x_n,x_{n+1})\geqslant 1$ or $\mu(x_n,x_{n+1})\leqslant 1$ for all $n\in\mathbb{N}\Rightarrow Tx_n\to Tx$ as $n\to\infty$.

Every continuous mapping T is an α-continuous mapping (μ-continuous mapping), where $\alpha,\mu\colon X\times X\to[0,\infty)$ is an arbitrary mapping.

Definition 2.10 ([11])**.** Let X be a nonempty set and $\alpha\colon X\times X\to\mathbb{R}^+$ be a mapping. We say that X has the property

(H) if $\forall\, x,y\in X$ there exists $z\in X$ such that $\alpha(x,z)\geqslant 1$ and $\alpha(y,z)\geqslant 1$.

Definition 2.11. Let X be a nonempty set and $\mu\colon X\times X\to\mathbb{R}^+$ be a mapping. We say that X has the property

(HH) if $\forall\, x,y\in X$ there exists $z\in X$ such that $\mu(x,z)\leqslant 1$ and $\mu(y,z)\leqslant 1$.

Definition 2.12 (see [1], [4], [9])**.** A function $h\colon\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ is a function of subclass of type I if $x\geqslant 1\Rightarrow h(1,y)\leqslant h(x,y)$ for all $y\in\mathbb{R}^+$.

Example 2.13 (see [1], [4], [9]). Define $h\colon\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ as:

(a) $h(x,y)=(y+l)^x$, $l>1$;
(b) $h(x,y)=(x+l)^y$, $l>1$;
(c) $h(x,y)=x^n y$, $n\in\mathbb{N}$;
(d) $h(x,y)=y$;
(e) $h(x,y)=(n+1)^{-1}y\sum\limits_{i=0}^{n}x^i$, $n\in\mathbb{N}$;
(f) $h(x,y)=\Big((n+1)^{-1}\sum\limits_{i=0}^{n}x^i+l\Big)^y$, $l>1$, $n\in\mathbb{N}$

for all $x,y\in\mathbb{R}^+$. Then each h is a function of subclass of type I.

Definition 2.14 (see [1], [4], [9])**.** Let $h,\mathcal{F}\colon\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$. Then we say that the pair $(\mathcal{F},h)$ is an upper class of type I if h is a function of subclass of type I and:

(i) $0\leqslant s\leqslant 1\Rightarrow\mathcal{F}(s,t)\leqslant\mathcal{F}(1,t)$,
(ii) $h(1,y)\leqslant\mathcal{F}(s,t)\Rightarrow y\leqslant st$ for all $s,t,y\in\mathbb{R}^+$.

Example 2.15. Define $h,\mathcal{F}\colon\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ as:

(a) $h(x,y)=(y+l)^x$, $l>1$ and $\mathcal{F}(s,t)=st+l$;
(b) $h(x,y)=(x+l)^y$, $l>1$ and $\mathcal{F}(s,t)=(1+l)^{st}$;
(c) $h(x,y)=x^m y$, $m\in\mathbb{N}$ and $\mathcal{F}(s,t)=st$;

(d) $h(x,y) = y$ and $\mathcal{F}(s,t) = t$;
(d) $h(x,y) = (n+1)^{-1} y \sum\limits_{i=0}^{n} x^i$, $n \in \mathbb{N}$ and $\mathcal{F}(s,t) = st$;
(e) $h(x,y) = \Big((n+1)^{-1} \sum\limits_{i=0}^{n} x^i + l\Big)^y$, $l > 1$, $n \in \mathbb{N}$ and $\mathcal{F}(s,t) = (1+l)^{st}$

for all $x, y, s, t \in \mathbb{R}^+$. Then each pair $(\mathcal{F}, h)$ is an upper class of type I.

Definition 2.16 (see [1], [4], [9]). We say that a function $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type II if $x, y \geqslant 1 \Rightarrow h(1,1,z) \leqslant h(x,y,z)$ for all $z \in \mathbb{R}^+$.

Example 2.17. Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ as:

(a) $h(x,y,z) = (z+l)^{xy}$, $l > 1$;
(b) $h(x,y,z) = (xy+l)^z$, $l > 1$;
(c) $h(x,y,z) = z$;
(d) $h(x,y,z) = x^m y^n z^p$, $m, n, p \in \mathbb{N}$;
(e) $h(x,y,z) = \frac{1}{3}(x^m + x^n y^p + y^q) z^k$, $m, n, p, q, k \in \mathbb{N}$

for all $x, y, z \in \mathbb{R}^+$. Then each h is a function of subclass of type II.

Definition 2.18 (see [1], [4], [9]). Let $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. Then we say that the pair $(\mathcal{F}, h)$ is an upper class of type II if h is a subclass of type II and:

(i) $0 \leqslant s \leqslant 1 \Rightarrow \mathcal{F}(s,t) \leqslant \mathcal{F}(1,t)$,
(ii) $h(1,1,z) \leqslant \mathcal{F}(s,t) \Rightarrow z \leqslant st$ for all $s, t, z \in \mathbb{R}^+$.

Example 2.19. Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ as:

(a) $h(x,y,z) = (z+l)^{xy}$, $l > 1$, $\mathcal{F}(s,t) = st + l$;
(b) $h(x,y,z) = (xy+l)^z$, $l > 1$, $\mathcal{F}(s,t) = (1+l)^{st}$;
(c) $h(x,y,z) = z$, $F(s,t) = st$;
(d) $h(x,y,z) = x^m y^n z^p$, $m, n, p \in \mathbb{N}$, $\mathcal{F}(s,t) = s^p t^p$;
(e) $h(x,y,z) = \frac{1}{3}(x^m + x^n y^p + y^q) z^k$, $m, n, p, q, k \in \mathbb{N}$, $\mathcal{F}(s,t) = s^k t^k$

for all $x, y, z, s, t \in \mathbb{R}^+$. Then each pair $(\mathcal{F}, h)$ is an upper class of type II.

Definition 2.20. Let $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. We say that the pair $(\mathcal{F}, h)$ is a special upper class of type I if the following conditions are satisfied:

(i) h is a subclass of type I;
(ii) $0 \leqslant s \leqslant 1 \Rightarrow \mathcal{F}(s,t) \leqslant \mathcal{F}(1,t)$ for all $t \in \mathbb{R}^+$;
(iii) $h(1,y) \leqslant \mathcal{F}(1,t) \Rightarrow y \leqslant t$ for all $y, t \in \mathbb{R}^+$.

The following are examples of the function of special upper class of type I for all $x, y, s, t \in \mathbb{R}^+$:

(1) $h(x,y) = (y^k + l)^x$, $l > 1$, $\mathcal{F}(s,t) = s^m t^k + l$;
(2) $h(x,y) = (x^n + l)^{y^k}$, $l > 1$, $\mathcal{F}(s,t) = (1+l)^{s^m t^k}$;

(3) $h(x,y) = x^n y^k$, $\mathcal{F}(s,t) = s^m t^k$;
(4) $h(x,y) = \frac{1}{2}(x+1)y$, $F(s,t) = st$;
(5) $h(x,y) = \frac{1}{3}(x+x+1)y$, $\mathcal{F}(s,t) = st$;
(6) $h(x,y) = ((x^n + x^{n-1} + \ldots + x^1 + 1)/(n+1))y^k$, $\mathcal{F}(s,t) = st^k$;
(7) $h(x,y) = ((x^n + x^{n-1} + \ldots + x^1 + 1)/(n+1) + l)^y$, $l > 1$, $\mathcal{F}(s,t) = (1+l)^{st}$.

We denote by Ψ the family of continuous and nondecreasing functions ψ, $\psi\colon [0,\infty) \to [0,\infty)$.

3. Main results

To start with, we first introduce the concept of partial generalized convex contraction mappings of order 4 and rank 4 using auxiliary functions:

Definition 3.1. Let (X,d) be a metric space. A mapping $T\colon X \to X$ is called a partial h-$\mathcal{F}$-ψ-generalized convex contraction of rank 4 if there exist a mapping $\alpha\colon X \times X \to \mathbb{R}^+$ and $a,b,c,e \in [0,1)$ with $a+b+c+e < 1$ satisfying the following condition:

$$x,y \in X, \quad h(\alpha(x,y), \psi(d(T^4x, T^4y))) \leqslant \mathcal{F}(1, \psi(ad(T^3x, T^3y) + bd(T^2x, T^2y) + cd(Tx, Ty) + ed(x,y))),$$

where the pair $(\mathcal{F}, h)$ is an upper class of type I and $\psi \in \Psi$.

Definition 3.2. Let (X,d) be a metric space. A mapping $T\colon X \to X$ is called a partial generalized convex contraction of rank 4 if there exist a mapping $\alpha\colon X \times X \to \mathbb{R}^+$ and $a,b,c,e \in [0,1)$ with $a+b+c+e < 1$ satisfying the following condition:

$$x,y \in X,\ \alpha(x,y) \geqslant 1 \Rightarrow \psi(d(T^4x, T^4y)) \leqslant \psi(ad(T^3x, T^3y) + bd(T^2x, T^2y) + cd(Tx, Ty) + ed(x,y)),$$

where $\psi \in \Psi$.

If $\alpha(x,y) \geqslant 1$ for all $x,y \in X$, then the mapping T is called a partial ψ-generalized convex contraction of rank 4 with the based mapping $\alpha\colon X \times X \to \mathbb{R}^+$.

Definition 3.3. Let (X,d) be a metric space. A mapping $T\colon X \to X$ is called a partial sub-h-$\mathcal{F}$-ψ-generalized convex contraction of rank 4 if there exist a mapping $\mu\colon X \times X \to \mathbb{R}^+$ and $a,b,c,e \in [0,1)$ with $a+b+c+e < 1$ satisfying the following condition:

$$x,y \in X,\ h(1, \psi(d(T^4x, T^4y))) \leqslant \mathcal{F}(\mu(x,y), \psi(ad(T^3x, T^3y) + bd(T^2x, T^2y) + cd(Tx, Ty) + ed(x,y))),$$

where the pair (F, h) is a special upper class of type I and $\psi \in \Psi$.

Now, we establish some approximate fixed point theorems for partial generalized convex contraction mappings in α-complete metric spaces and μ-complete metric spaces.

Theorem 3.4. *Let (X,d) be a metric space and $T\colon X\to X$ be a partial h-$\mathcal{F}$-ψ-generalized convex contraction of rank 4 with the based mapping $\alpha\colon X\times X\to\mathbb{R}^+$. Assume that T is α-admissible and there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geqslant 1$. Then T has the approximate fixed point property.*

In addition, if T is α-continuous and (X,d) is an α-complete metric space, then T has a fixed point.

Proof. Assume that $x_0\in X$ is such that $\alpha(x_0,Tx_0)\geqslant 1$. Define a sequence $\{x_n\}\subset X$ by $x_{n+1}=T^{n+1}(x_0)=Tx_n$ for $n\in\mathbb{N}\cup\{0\}$. Suppose that $x_{n_0}=x_{n_0+1}$ for some $n_0\in\mathbb{N}\cup\{0\}$. Then it is clear that x_{n_0} is a fixed point of T and hence the proof is complete. From now on, we suppose that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$.

Since T is an α-admissible mapping and $\alpha(x_0,Tx_0)\geqslant 1$, we deduce that $\alpha(x_1,Tx_1)=\alpha(Tx_0,T^2x_0)\geqslant 1$. By continuing this process, we get that $\alpha(x_n,Tx_n)=\alpha(x_n,x_{n+1})\geqslant 1$ for all $n\in\mathbb{N}\cup\{0\}$.

Suppose that $\lambda=d(T^3x_0,T^4x_0)+d(T^2x_0,T^3x_0)+d(Tx_0,T^2x_0)+d(x_0,Tx_0)=d(x_3,x_4)+d(x_2,x_3)+d(x_1,x_2)+d(x_0,x_1)$, and $\omega=a+b+c+e$. Consider

$$\begin{aligned}
&h(1,\psi(d(x_{n+4},x_{n+5})))\\
&\qquad\leqslant h(\alpha(x_n,x_{n+1}),\psi(d(x_{n+4},x_{n+5})))\\
&\qquad\leqslant \mathcal{F}(1,\psi(ad(x_{n+3},x_{n+4})+bd(x_{n+2},x_{n+3})+cd(x_{n+1},x_{n+2})+ed(x_n,x_{n+1}))).
\end{aligned}$$

This implies that

$$d(x_{n+4},x_{n+5})\leqslant ad(x_{n+3},x_{n+4})+bd(x_{n+2},x_{n+3})+cd(x_{n+1},x_{n+2})+ed(x_n,x_{n+1}).$$

So

$$d(x_4,x_5)\leqslant ad(x_3,x_4)+bd(x_2,x_3)+cd(x_1,x_2)+ed(x_0,x_1)\leqslant\omega\lambda. \tag{3.1}$$

Using the above inequality and the fact that $a,b,c,e,a+b,a+c,a+b+e\leqslant a+b+c+e=\omega$, we have

$$\begin{aligned}
d(x_5,x_6)&\leqslant ad(x_4,x_5)+bd(x_3,x_4)+cd(x_2,x_3)+ed(x_1,x_2)\\
&\leqslant ad(x_3,x_4)+ad(x_2,x_3)+ad(x_1,x_2)+ad(x_0,x_1)\\
&\qquad+bd(x_3,x_4)+cd(x_2,x_3)+ed(x_1,x_2)\\
&=(a+b)d(x_3,x_4)+(a+c)d(x_2,x_3)+(a+e)d(x_1,x_2)+ad(x_0,x_1)\\
&\leqslant\omega\lambda.
\end{aligned}\tag{3.2}$$

Using the above arguments and the fact that $a, b, c, e, a+b+c, a+b+e, a+b \leqslant a+b+c+e = \omega$, we have

$$
\begin{aligned}
d(x_6, x_7) &\leqslant ad(x_5, x_6) + bd(x_4, x_5) + cd(x_3, x_4) + ed(x_2, x_3) \\
&\leqslant ad(x_3, x_4) + ad(x_2, x_3) + ad(x_1, x_2) + ad(x_0, x_1) + bd(x_3, x_4) \\
&\quad + bd(x_2, x_3) + bd(x_1, x_2) + bd(x_0, x_1) + cd(x_3, x_4) + ed(x_2, x_3) \\
&= (a+b+c)d(x_3, x_4) + (a+b+e)d(x_2, x_3) \\
&\quad + (a+b)d(x_1, x_2) + (a+b)d(x_0, x_1) \\
&\leqslant \omega\lambda.
\end{aligned} \tag{3.3}
$$

Similarly, by using inequalities (3.1), (3.2), (3.3) and the fact that $a, b, c, e, a+b+c+e, a+b+c \leqslant a+b+c+e = \omega$, we have

$$
\begin{aligned}
d(x_7, x_8) &\leqslant ad(x_6, x_7) + bd(x_5, x_6) + cd(x_4, x_5) + ed(x_3, x_4) \\
&\leqslant ad(x_3, x_4) + ad(x_2, x_3) + ad(x_1, x_2) + ad(x_0, x_1) + bd(x_3, x_4) \\
&\quad + bd(x_2, x_3) + bd(x_1, x_2) + bd(x_0, x_1) + cd(x_3, x_4) + cd(x_2, x_3) \\
&\quad + cd(x_1, x_2) + cd(x_0, x_1) + ed(x_3, x_4) \\
&= (a+b+c+e)d(x_3, x_4) + (a+b+c)d(x_2, x_3) \\
&\quad + (a+b+c)d(x_1, x_2) + (a+b+c)d(x_0, x_1) \\
&\leqslant \omega\lambda.
\end{aligned} \tag{3.4}
$$

In the similar manner, using (3.2), (3.3), (3.4) and $a+b+c+e = \omega$, we have

$$
\begin{aligned}
d(x_8, x_9) &\leqslant ad(x_7, x_8) + bd(x_6, x_7) + cd(x_5, x_6) + ed(x_4, x_5) \\
&\leqslant a\omega\lambda + b\omega\lambda + c\omega\lambda + e\omega\lambda \leqslant \omega^2\lambda.
\end{aligned} \tag{3.5}
$$

By continuing this process, it is easy to see that

$$
d(x_n, x_{n+1}) \leqslant \omega^l\lambda, \quad n \in \mathbb{N}, \tag{3.6}
$$

where $n = 4l$ or $n = 4l+1$, $n = 4l+2$ or $n = 4l+3$ for all $l \in \mathbb{N}$. This implies that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Therefore, T is asymptotically regular at a point $x_0 \in X$.

By using Lemma 2.7, we conclude that T has the approximate fixed point property.

Next, we show that T has a fixed point provided that T is α-continuous and (X, d) is an α-complete metric space.

Firstly, we claim that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{N} \cup \{0\}$ such that $n > m$. So by (3.6), we have

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{n-1}, x_n) \\ &\leqslant \omega^l \lambda + \omega^l \lambda + \omega^l \lambda + \omega^{l+1} \lambda + \ldots \leqslant \frac{4\omega^l \lambda}{1-\omega}, \end{aligned}$$

where $m = 4l$ or $m = 4l+1$ or $m = 4l+2$ or $m = 4l+3$ for all $l \in \mathbb{N}$.

Therefore

$$d(x_m, x_n) \to 0 \quad \text{as } m \to \infty,$$

that is, $\{x_n\}$ is a Cauchy sequence in X.

Since $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ is a Cauchy sequence in X, by using α-completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since T is α-continuous, $Tx_n \to Tx^*$ as $n \to \infty$. Therefore $x^* = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} Tx_n = x^*$ and thus T has a fixed point. This completes the proof. □

Corollary 3.5. *Let (X, d) be a metric space and $T\colon X \to X$ be a partial ψ-generalized convex contraction of rank 4 with the based mapping $\alpha\colon X \times X \to \mathbb{R}^+$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Then T has the approximate fixed point property. In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point.*

Example 3.6. Let $X = (0, \infty)$ and $d\colon X \times X \to \mathbb{R}$ be defined by $d(x-y) = |x-y|$ for all $x, y \in X$. Define $T\colon X \to X$ and $\alpha\colon X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \dfrac{x+6}{7}, & x \in [1,5], \\ \dfrac{5x-6}{4}, & x \in (5,7), \\ 7x - 42, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \alpha(x,y) = \begin{cases} 1, & x, y \in [1,5], \\ 0, & \text{otherwise.} \end{cases}$$

For $\alpha(x, y) \geqslant 1$ we have $x, y \in [1, 5]$ and thus

$$\begin{aligned} d(T^4x, T^4y)) &\leqslant d\Big(T^3\frac{x+6}{7}, T^3\frac{y+6}{7}\Big) = d\Big(\frac{T^2}{7}\Big(\frac{x+6}{7}+6\Big), \frac{T^2}{7}\Big(\frac{y+6}{7}+6\Big)\Big) \\ &= d\Big(\frac{T}{7}\Big(\frac{1}{7}\Big(\frac{x+6}{7}+6\Big)+6\Big), \frac{T}{7}\Big(\frac{1}{7}\Big(\frac{y+6}{7}+6\Big)+6\Big)\Big) \\ &= d\Big(T\frac{x+342}{343}, T\frac{y+342}{343}\Big) = d\Big(\frac{1}{7}\Big(\frac{x+342}{343}+6\Big), \frac{1}{7}\Big(\frac{y+342}{343}+6\Big)\Big) \\ &= d\Big(\frac{x+2400}{2401}, \frac{y+2400}{2401}\Big) = \frac{1}{2401}|x-y| \\ &\leqslant \frac{5\cdot 6^3 + 4\cdot 36\cdot 7 + 12\cdot 49 + 343}{6^4\cdot 343}|x-y| \end{aligned}$$

$$
\begin{aligned}
&= \frac{5}{6}\Big|\frac{x+342}{343} - \frac{y+342}{343}\Big| + \frac{4}{6^2}\Big|\frac{x+48}{49} - \frac{y+48}{49}\Big| \\
&\quad + \frac{2}{6^3}\Big|\frac{x+6}{7} - \frac{y+6}{7}\Big| + \frac{1}{6^4}|x-y| \\
&= ad(T^3x, T^3y) + bd(T^2x, T^2y) + cd(Tx, Ty) + ed(x, y).
\end{aligned}
$$

Therefore T is a partial generalized convex contraction with $a = \frac{5}{6}$, $b = \frac{4}{6^2}$, $c = \frac{2}{6^3}$ and $e = \frac{1}{6^4}$.

Moreover, it is easy to see that T is α-admissible and there exists $x_0 = 1.5 \in X$ such that

$$\alpha(x_0, Tx_0) = \alpha(1.5, T(1.5)) = \alpha(1.5, 1.07) = 1.$$

One can see that all the conditions of the above theorem are true. Therefore T has a fixed point in X: $x = 6$, $x = 7$.

Theorem 3.7. *Adding property* (H) *to the hypotheses of Theorem* 3.4, *we obtain uniqueness of the fixed point of T.*

P r o o f. Let x^* and y^* be fixed points of T. By property (H), we can choose $z \in X$ such that $\alpha(x^*, z) \geqslant 1$ and $\alpha(y^*, z) \geqslant 1$. Since T is α-admissible, we get $\alpha(x^*, T^m z) \geqslant 1$ and $\alpha(y^*, T^m z) \geqslant 1$ for all $m \in N$. Put $\nu = d(x^*, T^4 z) + d(x^*, T^3 z) + d(x^*, T^2 z) + d(x^*, Tz)$, and $\omega = a + b + c + e$. Consider

$$
\begin{aligned}
&h(1, \psi(d(x^*, T^{m+4}z))) \\
&\qquad \leqslant h(\alpha(x^*, T^m z), \psi(d(x^*, T^{m+4}z))) \\
&\qquad \leqslant \mathcal{F}(1, \psi(ad(x^*, T^{m+3}z) + bd(x^*, T^{m+2}z) + cd(x^*, T^{m+1}z) + ed(x^*, T^m z))).
\end{aligned}
$$

This implies that

$$d(x^*, T^{m+4}z) \leqslant ad(x^*, T^{m+3}z) + bd(x^*, T^{m+2}z) + cd(x^*, T^{m+1}z) + ed(x^*, T^m z).$$

So

$$d(x^*, T^5 z) \leqslant ad(x^*, T^4 z) + bd(x^*, T^3 z) + cd(x^*, T^2 z) + d(x^*, Tz) \leqslant \omega\nu. \tag{3.7}$$

Using (3.7), we have

$$d(x^*, T^6 z) \leqslant ad(x^*, T^5 z) + bd(x^*, T^4 z) + cd(x^*, T^3 z) + ed(x^*, T^2 z) \leqslant \omega\nu. \tag{3.8}$$

Similarly, using (3.7), (3.8), we have

$$d(x^*, T^7 z) \leqslant ad(x^*, T^6 z) + bd(x^*, T^5 z) + cd(x^*, T^4 z) + ed(x^*, T^3 z) \leqslant \omega\nu, \tag{3.9}$$

and using (3.7), (3.8), (3.9), we have

$$d(x^*, T^8z) \leqslant ad(x^*, T^7z) + bd(x^*, T^6z) + cd(x^*, T^5z) + ed(x^*, T^4z) \leqslant \omega^2\nu. \tag{3.10}$$

By continuing this process, it is easy to see that

$$d(x^*, T^nz) \leqslant \omega^l\nu,$$

where $n = 4l$ or $n = 4l+1$ or $n = 4l+2$ or $n = 4l+3$ for all $l \in \mathbb{N}$. This implies that $d(x^*, T^nz) \to 0$ as $n \to \infty$. Similarly, we can prove that $d(y^*, T^nz) \to 0$ as $n \to \infty$. By the uniqueness of limit, we have $x^* = y^*$ and thus T has a unique fixed point. This completes the proof. □

Theorem 3.8. *Let (X, d) be a metric space and $T\colon X \to X$ be a partial sub-h-F-ψ-generalized convex contraction of rank 4 with the based mapping $\mu\colon X\times X \to \mathbb{R}^+$. Assume that T is μ-subadmissible and there exists $x_0 \in X$ such that $\mu(x_0, Tx_0) \leqslant 1$. Then T has the approximate fixed point property.*

In addition, if T is μ-continuous and (X, d) is μ-complete metric space, then T has a fixed point.

Proof. Assume that $x_0 \in X$ such that $\mu(x_0, Tx_0) \leqslant 1$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = T^{n+1}(x_0) = Tx_n$ for $n \in \mathbb{N}\cup\{0\}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}\cup\{0\}$. Then it is clear that x_{n_0} is a fixed point of T and hence the proof is complete. From now on, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$.

Since T is an μ-subadmissible mapping and $\mu(x_0, Tx_0) \leqslant 1$, we deduce that $\mu(x_1, Tx_1) = \mu(Tx_0, T^2x_0) \leqslant 1$. By continuing this process, we get that $\mu(x_n, Tx_n) = \mu(x_n, x_{n+1}) \leqslant 1$ for all $n \in \mathbb{N}\cup\{0\}$.

Let $\lambda = d(T^3x_0, T^4x_0) + d(T^2x_0, T^3x_0) + d(Tx_0, T^2x_0) + d(x_0, Tx_0) = d(x_3, x_4) + d(x_2, x_3) + d(x_1, x_2) + d(x_0, x_1)$, and $\omega = a + b + c + e$. Consider

$$\begin{aligned} h(1, \psi(d(x_{n+4}, x_{n+5}))) &\leqslant \mathcal{F}(\mu(x_n, x_{n+1}), \psi(ad(x_{n+3}, x_{n+4}) + bd(x_{n+2}, x_{n+3}) \\ &\qquad + cd(x_{n+1}, x_{n+2}) + ed(x_n, x_{n+1}))) \\ &\leqslant \mathcal{F}(1, \psi(ad(x_{n+3}, x_{n+4}) + bd(x_{n+2}, x_{n+3}) \\ &\qquad + cd(x_{n+1}, x_{n+2}) + ed(x_n, x_{n+1}))). \end{aligned}$$

This implies that

$$d(x_{n+4}, x_{n+5}) \leqslant ad(x_{n+3}, x_{n+4}) + bd(x_{n+2}, x_{n+3}) + cd(x_{n+1}, x_{n+2}) + ed(x_n, x_{n+1}).$$

So

$$d(x_4, x_5) \leqslant ad(x_3, x_4) + bd(x_2, x_3) + cd(x_1, x_2) + ed(x_0, x_1) \leqslant \omega\lambda. \tag{3.11}$$

Using the above inequality and $a, b, c, e, a+b, a+c, a+b+e \leqslant a+b+c+e = \omega$, we have

$$(3.12)\qquad \begin{aligned} d(x_5, x_6) &\leqslant ad(x_4, x_5) + bd(x_3, x_4) + cd(x_2, x_3) + ed(x_1, x_2) \\ &\leqslant ad(x_3, x_4) + ad(x_2, x_3) + ad(x_1, x_2) + ad(x_0, x_1) \\ &\quad + bd(x_3, x_4) + cd(x_2, x_3) + ed(x_1, x_2) \\ &= (a+b)d(x_3, x_4) + (a+c)d(x_2, x_3) \\ &\quad + (a+e)d(x_1, x_2) + ad(x_0, x_1) \\ &\leqslant \omega\lambda. \end{aligned}$$

Using the similar arguments as in the proof of Theorem 3.4, it is easy to see that

$$(3.13)\qquad d(x_n, x_{n+1}) \leqslant \omega^l \lambda, \quad n \in \mathbb{N},$$

where $n = 4l$ or $n = 4l+1, n = 4l+2$ or $n = 4l+3$ for all $l \in \mathbb{N}$.

This implies that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Therefore T is asymptotically regular at a point $x_0 \in X$.

By using Lemma 2.7, we conclude that T has the approximate fixed point property.

Next, we show that T has a fixed point provided that T is μ-continuous and (X, d) is an μ-complete metric space.

Firstly, we claim that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{N} \cup \{0\}$ such that $n > m$. Therefore, by using (3.13), we have

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{n-12}, x_n) \\ &\leqslant \omega^l\lambda + \omega^l\lambda + \omega^l\lambda + \omega^{l+1}\lambda + \ldots \leqslant \frac{4\omega^l\lambda}{1-\omega}, \end{aligned}$$

where $m = 4l$ or $m = 4l+1, m = 4l+2$ or $m = 4l+3$ for all $l \in \mathbb{N}$. Therefore

$$d(x_m, x_n) \to 0 \quad \text{as } m \to \infty,$$

that is, $\{x_n\}$ is a Cauchy sequence in X.

Now, as $\mu(x_n, x_{n+1}) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ is a Cauchy sequence in X, by using μ-completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. It follows from T being μ-continuous that $Tx_n \to Tx^*$ as $n \to \infty$. Therefore $x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = x^*$ and thus T has a fixed point. This completes the proof. $\square$

Definition 3.9. Let (X, d) be a metric space. A mapping $T\colon X \to X$ is called a partial h-F-ψ-generalized convex contraction of rank 4 and order 4 if there exist a mapping $\alpha\colon X \times X \to \mathbb{R}^+$and $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 \in [0, 1)$ with $a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + a_4 + b_4 < 1$, satisfying the following condition for all $x, y \in X$:

$$\begin{aligned} h(\alpha(x,y), \psi(d(T^4x, T^4y))) \leqslant \mathcal{F}(1, \psi(&a_4 d(T^3x, T^4x) + a_3 d(T^2x, T^3x) + a_2 d(Tx, T^2x) \\ &+ a_1 d(x, Tx)) + b_4 d(T^3y, T^4y) + b_3 d(T^2y, T^3y) \\ &+ b_2 d(Ty, T^2y) + b_1 d(y, Ty))), \end{aligned}$$

where the pair $(\mathcal{F}, h)$ is an upper class of type I and $\psi \in \Psi$.

Definition 3.10. Let (X, d) be a metric space. A mapping $T\colon X \to X$ is called a partial ψ-generalized convex contraction of rank 4 and order 4 with the based mapping $\alpha\colon X \times X \to \mathbb{R}^+$ if there exist $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 \in [0, 1)$ with $a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + a_4 + b_4 < 1$, satisfying the following condition for all $x, y \in X$ and $\psi \in \Psi$:

$$\begin{aligned} \alpha(x,y) \geqslant 1 \Rightarrow \psi(d(T^4x, T^4y)) \leqslant \psi(&a_4 d(T^3x, T^4x) + a_3 d(T^2x, T^3x) + a_2 d(Tx, T^2x) \\ &+ a_1 d(x, Tx) + b_4 d(T^3y, T^4y) + b_3 d(T^2y, T^3y) \\ &+ b_2 d(Ty, T^2y) + b_1 d(y, Ty)). \end{aligned}$$

Definition 3.11. Let (X, d) be a metric space. A mapping $T\colon X \to X$ is called a partial sub-h-F-ψ-generalized convex contraction of rank 4 and order 4 if there exists a mapping $\mu\colon X \times X \to \mathbb{R}^+$ and $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 \in [0, 1)$ with $a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + a_4 + b_4 < 1$, satisfying the following condition for all $x, y \in X$:

$$\begin{aligned} h(1, \psi(d(T^4x, T^4y))) \leqslant \mathcal{F}(\mu(x,y), \psi(&a_4 d(T^3x, T^4x) + a_3 d(T^2x, T^3x) \\ &+ a_2 d(Tx, T^2x) + a_1 d(x, Tx) + b_4 d(T^3y, T^4y) \\ &+ b_3 d(T^2y, T^3y) + b_2 d(Ty, T^2y) + b_1 d(y, Ty))), \end{aligned}$$

where the pair $(\mathcal{F}, h)$ is a special upper class of type I and $\psi \in \Psi$.

Theorem 3.12. *Let (X, d) be a metric space and $T\colon X \to X$ be a partial h-F-ψ-generalized convex contraction of rank 4 and order 4 with the based mapping $\alpha\colon X \times X \to \mathbb{R}^+$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Then T has the approximate fixed point property.*

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point.

P r o o f. Assume $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = T^{n+1}(x_0) = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$. Then it is clear that x_{n_0} is a fixed point of T and hence the proof is completed. From now on, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Since T is an α-admissible mapping and $\alpha(x_0, Tx_0) \geqslant 1$, we deduce that $\alpha(x_1, Tx_1) = \alpha(Tx_0, T^2x_0) \geqslant 1$. By continuing this process, we get that $\alpha(x_n, Tx_n) = \alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Assume that $s = d(T^3x_0, T^4x_0) + d(T^2x_0, T^3x_0) + d(Tx_0, T^2x_0) + d(x_0, Tx_0) = d(x_3, x_4) + d(x_2, x_3) + d(x_1, x_2) + d(x_0, x_1)$, and $\eta = 1 - b_4$ and $\varrho = a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3$. Consider

$$\begin{aligned} h(1, \psi(d(x_{n+4}, x_{n+5}))) &\leqslant h(\alpha(x_n, x_{n+1}), \psi(d(x_{n+4}, x_{n+5}))) \\ &= h(\alpha(x_n, x_{n+1}), \psi(d(T^4(T^n x_0), T^4(T^n x_1)))) \\ &\leqslant \mathcal{F}(1, \psi(a_4 d(T^{n+3}x_0, T^{n+4}x_0) + a_3 d(T^{n+2}x_0, T^{n+3}x_0) \\ &\quad + a_2 d(T^{n+1}x_0, T^{n+2}x_0) + a_1 d(T^n x_0, T^{n+1}x_0) \\ &\quad + b_4 d(T^{n+4}x_0, T^{n+5}x_0) + b_3 d(T^{n+3}x_0, T^{n+4}x_0) \\ &\quad + b_2 d(T^{n+2}x_0, T^{n+3}x_0) + b_1 d(T^{n+1}x_0, T^{n+2}x_0))). \end{aligned}$$

This implies that

$$\begin{aligned} d(x_{n+4}, x_{n+5}) \leqslant\ & a_4 d(x_{n+3}, x_{n+4}) + a_3 d(x_{n+2}, x_{n+3}) + a_2 d(x_{n+1}, x_{n+2}) \\ & + a_1 d(x_n, x_{n+1}) + b_4 d(x_{n+4}, x_{n+5}) + b_3 d(x_{n+3}, x_{n+4}) \\ & + b_2 d(x_{n+2}, x_{n+3}) + b_1 d(x_{n+1}, x_{n+2}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} d(x_4, x_5) &\leqslant a_4 d(x_3, x_4) + a_3 d(x_2, x_3) + a_2 d(x_1, x_2) + a_1 d(x_0, x_1) \\ &\quad + b_4 d(x_4, x_5) + b_3 d(x_3, x_4) + b_2 d(x_2, x_3) + b_1 d(x_1, x_2) \\ &\leqslant a_1 s + (b_3 + a_4)s + (b_2 + a_3)s + (b_1 + a_2)s + b_4 d(x_4, x_5). \end{aligned} \tag{3.14}$$

This implies that

$$d(x_4, x_5) \leqslant \frac{\varrho}{\eta} s.$$

Using similar arguments as in the proof of Theorem 3.4, it is easy to see that

$$d(x_n, x_{n+1}) \leqslant \Big(\frac{\varrho}{\eta}\Big)^l s, \quad n \in \mathbb{N}, \tag{3.15}$$

where $n = 4l$ or $n = 4l + 1$ or $n = 4l + 2$ or $n = 4l + 3$ for all $l \in \mathbb{N}$.

This implies that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Therefore T is asymptotically regular at a point $x_0 \in X$.

By using Lemma 2.7, we conclude that T has the approximate fixed point property.

Next, we show that T has a fixed point provided that T is α-continuous and (X, d) is α-complete metric space.

Firstly, we claim that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{N} \cup \{0\}$ such that $n > m$, so from (3.15),

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{n-12}, x_n) \\ &\leqslant \left(\frac{\varrho}{\eta}\right)^l \lambda + \left(\frac{\varrho}{\eta}\right)^l \lambda + \left(\frac{\varrho}{\eta}\right)^l \lambda + \left(\frac{\varrho}{\eta}\right)^{l+1} \lambda + \ldots \leqslant \frac{4(\varrho/\eta)^l \lambda}{1 - (\varrho/\eta)}, \end{aligned}$$

where $m = 4l$ or $m = 4l + 1$ or $m = 4l + 2$ or $m = 4l + 3$ for all $l \in \mathbb{N}$.

Therefore

$$d(x_m, x_n) \to 0 \quad \text{as } m \to \infty,$$

that is, $\{x_n\}$ is a Cauchy sequence in X.

As $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ is a Cauchy sequence in X, by using α-completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. It follows from T being α-continuous that $Tx_n \to Tx^*$ as $n \to \infty$. Therefore $x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = x^*$ and thus T has a fixed point. This completes the proof. □

Corollary 3.13. *Let (X, d) be a metric space and $T\colon X \to X$ be a partial ψ-generalized convex contraction of rank 4 and order 4 with the based mapping $\alpha\colon X \times X \to \mathbb{R}^+$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Then T has the approximate fixed point property. In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point.*

Proceeding in the similar way as in the results above, we obtain the following:

Theorem 3.14. *Let (X, d) be a metric space and $T\colon X \to X$ be a partial sub-h-F-generalized convex contraction of rank 4 and order 4 with the based mapping $\mu\colon X \times X \to \mathbb{R}^+$. Assume that T is μ-subadmissible and there exists $x_0 \in X$ such that $\mu(x_0, Tx_0) \leqslant 1$. Then T has the approximate fixed point property.*

In addition, if T is μ-continuous and (X, d) is μ-complete metric space, then T has a fixed point.

4. Arbitrary binary relations

In this section, we present approximate fixed point theorems on metric spaces endowed with an arbitrary binary relation. The following notions and definitions are needed.

Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. Denote $S := \mathcal{R} \cup \mathcal{R}^{-1}$. Clearly, $x, y \in X$, $x \, S \, y \Leftrightarrow x \, \mathcal{R} \, y$ or $y \, \mathcal{R} \, x$.

It is easy to see that S is a symmetric relation attached to $\mathcal{R}$.

Definition 4.1. Let T be a self mapping on a nonempty set X and $\mathcal{R}$ be a binary relation over X. We say that T is a comparative mapping if $x, y \in X$, $x \, S \, y \Rightarrow (Tx) \, S \, (Ty)$.

Definition 4.2. Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X. A mapping $T\colon X \to X$ is an $\mathcal{R}$-preserving mapping if $x, y \in X\colon x \, \mathcal{R} \, y \Rightarrow Tx \, \mathcal{R} \, Ty$.

Definition 4.3 ([11])**.** Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. The metric space X is said to be S-complete if every Cauchy sequence $\{x_n\}$ in X with $x_n \, S \, x_{n+1}$ for all $n \in \mathbb{N}$ converges in X.

Definition 4.4 ([11])**.** Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. We say that $T\colon X \to X$ is an S-continuous mapping on (X, d) if for each sequence $\{x_n\}$ in X we have $x_n \to x$ as $n \to \infty$ for some $x \in X$ and $x_n \, S \, x_{n+1}$ for all $n \in N \Rightarrow Tx_n \to Tx$ as $n \to \infty$.

Definition 4.5. Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation on X. A mapping $T\colon X \to X$ is called a partial ψ-generalized convex contraction of rank 4 with respect to S if there exist $a, b, c, e \in [0, 1)$ with $a + b + c + e < 1$ satisfying the following condition for all $x, y \in X$:

$$xSy \Rightarrow \psi(d(T^4x, T^4y)) \leqslant \psi(ad(T^3x, T^3y) + bd(T^2x, T^2y) + cd(Tx, Ty) + ed(x, y)).$$

Definition 4.6. Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation on X. A mapping $T\colon X \to X$ is called a partial ψ-generalized convex contraction of rank 4 and order 4 with respect to S if there exist $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 \in [0, 1)$ with $a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + a_4 + b_4 < 1$ satisfying the following condition for all $x, y \in X$:

$$\begin{aligned} xSy \Rightarrow \psi(d(T^4x, T^4y)) \leqslant \psi(&a_1d(T^3x, T^4x) + a_2d(T^2x, T^3x) + a_3d(Tx, T^2x) \\ &+ a_4d(x, Tx) + b_1d(T^3y, T^4y) + b_2d(T^2y, T^3y) \\ &+ b_3d(Ty, T^2y) + b_4d(y, Ty)). \end{aligned}$$

Definition 4.7. Let X be a nonempty set and $\mathcal{R}$ be a binary relation over X. We say that X has the property

($\mathrm{H_S}$) if for each $x, y \in X$ there exists $z \in X$ such that xSz and ySz.

Theorem 4.8. *Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation over X and $T\colon X \to X$ be a partial ψ-generalized convex contraction of rank 4 with respect to S. Assume that T is a comparative mapping and there exists $x_0 \in X$ such that $(x_0)\ S\ (Tx_0)$. Then T has the approximate fixed point property. In addition, if T is S-continuous and (X, d) is an S-complete metric space, then T has a fixed point, and T has a unique fixed point whenever X has the property* ($\mathrm{H_S}$).

P r o o f. Consider a mapping $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in xSy, \\ 0, & \text{otherwise.} \end{cases}$$

From the fact that there exists $x_0 \in X$ such that $(x_0)\ S\ (Tx_0)$, we get $\alpha(x_0, Tx_0) = 1$. It follows from T being a comparative mapping that T is an α-admissible mapping. Since T is a partial ψ-generalized convex contraction mapping of rank 4 with respect to S, we have for all $x, y \in X$,

$$xSy \Rightarrow \psi(d(T^4x, T^4y)) \leqslant \psi(ad(T^3x, T^3y) + bd(T^2x, T^2y) + cd(Tx, Ty) + ed(x, y)),$$

and then

$$\begin{aligned} &\alpha(x, y) \geqslant 1 \\ &\quad \Rightarrow \psi(d(T^4x, T^4y)) \leqslant \psi(ad(T^3x, T^3y) + bd(T^2x, T^2y) + cd(Tx, Ty) + ed(x, y)). \end{aligned}$$

This implies that T is a partial generalized convex contraction with based mapping α. Now, all the hypotheses of Corollary 3.5 are satisfied. So, T has an approximate fixed point. Furthermore, the S-continuity of T and the S-completeness of X yield the existence of the fixed point of T. Finally, the uniqueness of the fixed point of T follows from Theorem 3.7. This completes the proof. $\square$

Theorem 4.9. *Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation over X and $T\colon X \to X$ be a partial ψ-generalized convex contraction of rank 4 and order 4 with respect to S. Assume that T is a comparative mapping and there exists $x_0 \in X$ such that $(x_0)S(Tx_0)$. Then T has the approximate fixed point property. In addition, if T is S-continuous and (X, d) is an S-complete metric space, then T has a fixed point, and T has a unique fixed point whenever X has the property* ($\mathrm{H_S}$).

P r o o f. The proof is similar to that of Theorem 4.8. $\square$

5. Approximate fixed point analysis with graph

In this section, we give the existence of approximate fixed point theorems on a metric space endowed with a graph. Before presenting our results, we give the following notions and definitions.

Throughout this section, (X, d) is a metric space. A set $\{(x, x)\colon\ x \in X\}$ is called a diagonal of the cartesian product $X \times X$ and is denoted by $\triangle$. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e. $\triangle \subseteq E(G)$. We assume G has no parallel edges, so we can identify G by the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices.

Definition 5.1 ([11])**.** Let X be a nonempty set endowed with a graph G. We say that $T\colon\ X \to X$ preserve an edge if for $x, y \in X$,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G).$$

Definition 5.2 ([11])**.** Let (X, d) be a metric space endowed with a graph G. The metric space X is said to be $E(G)$-complete if every Cauchy sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, converges in X.

Definition 5.3 ([11])**.** Let (X, d) be a metric space endowed with a graph G and $T\colon\ X \to X$ be a mapping. We say that T is an $E(G)$-continuous mapping on (X, d) if for each sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ we have $Tx_n \to Tx$ as $n \to \infty$.

Definition 5.4. Let (X, d) be a metric space endowed with a graph G. The mapping $T\colon\ X \to X$ is called a partial ψ-generalized convex contraction mapping of rank 4 with respect to $E(G)$ if there exist $a, b, c, e \in [0, \infty)$ with $a + b + c + e < 1$ satisfying the following condition for $x, y \in X$:

$$\begin{aligned}&(x, y) \in E(G)\\ &\qquad\Rightarrow \psi(d(T^4x, T^4y)) \leqslant \psi(ad(T^3x, T^3y) + bd(T^2x, T^2y) + cd(Tx, Ty) + ed(x, y)).\end{aligned}$$

Definition 5.5. Let (X, d) be a metric space endowed with a graph G. A mapping $T\colon\ X \to X$ is called a partial ψ-generalized convex contraction of rank 4 and order 4 with respect to $E(G)$ if there exist $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 \in [0, 1)$ with $a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + a_4 + b_4 < 1$ satisfying the following condition for all $x, y \in X$:

$$\begin{aligned}(x, y) \in E(G) \Rightarrow \psi(d(T^4x, T^4y)) \leqslant \psi(&a_1d(T^3x, T^4x) + a_2d(T^2x, T^3x) + a_3d(Tx, T^2x)\\ &+ a_4d(x, Tx) + b_1d(T^2y, T^3y) + b_2d(T^2y, T^3y)\\ &+ b_3d(Ty, T^2y) + b_4d(y, Ty)).\end{aligned}$$

Definition 5.6 ([11])**.** Let X be a nonempty set endowed with a graph G. We say that X has the property

($\mathrm{H_E}$) if $\forall\, x, y \in X$ there exists $z \in X$ such that $(x, z) \in E(G)$ and $(y, z) \in E(G)$.

Theorem 5.7. *Let (X, d) be a metric space endowed with a graph G and $T\colon X \to X$ be a partial ψ-generalized convex contraction mapping of rank 4 with respect to $E(G)$. Assume that T preserves an edge and there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. Then T has the approximate fixed point property. In addition, if T is $E(G)$-continuous and (X, d) is an $E(G)$-complete metric space, then T has a fixed point, and T has a unique fixed point whenever X has the property* ($\mathrm{H_E}$)*.*

Proof. Consider a mapping $\alpha\colon X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

From the fact that there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, we get $\alpha(x_0, Tx_0) = 1$. It follows from T preserving an edge that T is an α-admissible mapping. Since T is a partial ψ-generalized convex contraction mapping of rank 4 with respect to $E(G)$, we have for all $x, y \in X$,

$$\begin{aligned}&(x, y) \in E(G) \\ &\qquad \Rightarrow \psi(d(T^4x, T^4y)) \leqslant \psi(ad(T^3x, T^3y) + bd(T^2x, T^2y) + cd(Tx, Ty) + ed(x, y))\end{aligned}$$

and then

$$\begin{aligned}&\alpha(x, y) \geqslant 1 \\ &\quad \Rightarrow \psi(d(T^4x, T^4y)) \leqslant \psi(ad(T^3x, T^3y) + bd(T^2x, T^2y) + cd(Tx, Ty) + ed(x, y)).\end{aligned}$$

This implies that T is a partial generalized convex contraction with based mapping α. Now, all the hypotheses of Corollary 3.5 are satisfied. So, T has an approximate fixed point. Furthermore, the $E(G)$-continuity of T and the $E(G)$-completeness of X yield the existence of the fixed point of T. Finally, the uniqueness of the fixed point of T follows from Theorem 3.7. This completes the proof. □

Theorem 5.8. *Let (X, d) be a metric space endowed with a graph G and $T\colon X \to X$ be a partial ψ-generalized convex contraction of rank 4 and order 4 with respect to $E(G)$. Assume that T preserves an edge and there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. Then T has the approximate fixed point property. In addition, if T is $E(G)$-continuous and (X, d) is an $E(G)$-complete metric space, then T has a fixed point, and T has a unique fixed point whenever X has the property* ($\mathrm{H_E}$)*.*

Proof. The proof is similar to that of Theorem 5.7. □

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References

[1] *A. H. Ansari, S. Shukla*: Some fixed point theorems for ordered F-$(\mathcal{F}, h)$-contraction and subcontraction in 0-f-orbitally complete partial metric spaces. J. Adv. Math. Stud. *9* (2016), 37–53. zbl MR

[2] *M. Berinde*: Approximate fixed point theorems. Stud. Univ. Babeş-Bolyai, Math. *51* (2006), 11–25. zbl MR

[3] *F. E. Browder, W. V. Petryshyn*: The solution by iteration of nonlinear functional equations in Banach spaces. Bull. Am. Math. Soc. *72* (1966), 571–575. zbl MR doi

[4] *S. Chandok, A. H. Ansari*: Some results on generalized nonlinear contractive mappings. Comm. Opt. Theory *2017* (2017), Article ID 27, 12 pages. doi

[5] *D. Dey, A. K. Laha, M. Saha*: Approximate coincidence point of two nonlinear mappings. J. Math. *2013* (2013), Article No. 962058, 4 pages. zbl MR doi

[6] *D. Dey, M. Saha*: Approximate fixed point of Reich operator. Acta Math. Univ. Comen., New Ser. *82* (2013), 119–123. zbl MR

[7] *N. Hussain, M. A. Kutbi, P. Salimi*: Fixed point theory in α-complete metric spaces with applications. Abstr. Appl. Anal. *2014* (2014), Article ID 280817, 11 pages. MR doi

[8] *V. I. Istrăţescu*: Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters. I. Ann. Mat. Pura Appl., IV. Ser. *130* (1982), 89–104. zbl MR doi

[9] *M. M. M. Jaradat, Z. Mustafa, A. H. Ansari, S. Chandok, Ć. Dolićanin*: Some approximate fixed point results and application on graph theory for partial (h, F)-generalized convex contraction mappings with special class of functions on complete metric space. J. Nonlinear Sci. Appl. *10* (2017), 1695–1708. MR doi

[10] *U. Kohlenbach, L. Leuştean*: The approximate fixed point property in product spaces. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *66* (2007), 806–818. zbl MR doi

[11] *A. Latif, W. Sintunavarat, A. Ninsri*: Approximate fixed point theorems for partial generalized convex contraction mappings in α-complete metric spaces. Taiwanese J. Math. *19* (2015), 315–333. zbl MR doi

[12] *M. A. Miandaragh, M. Postolache, S. Rezapour*: Approximate fixed points of generalized convex contractions. Fixed Point Theory Appl. *2013* (2013), Article No. 255, 8 pages. zbl MR doi

[13] *S. Reich*: Approximate selections, best approximations, fixed points, and invariant sets. J. Math. Anal. Appl. *62* (1978), 104–113. zbl MR doi

[14] *B. Samet, C. Vetro, P. Vetro*: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *75* (2012), 2154–2165. zbl MR doi

[15] *S. Tijs, A. Torre, R. Brânzei*: Approximate fixed point theorems. Libertas Math. *23* (2003), 35–39. zbl MR

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