INVERSE TOPOLOGY IN MV-ALGEBRAS

Fereshteh Forouzesh, Farhad Sajadian, Mahta Bedrood, Bam

Received October 15, 2017. Published online September 25, 2018.
Communicated by Sándor Radeleczki

Abstract. We introduce the inverse topology on the set of all minimal prime ideals of an MV-algebra A and show that the set of all minimal prime ideals of A, namely $\operatorname{Min}(A)$, with the inverse topology is a compact space, Hausdorff, T_0-space and T_1-space.

Furthermore, we prove that the spectral topology on $\operatorname{Min}(A)$ is a zero-dimensional Hausdorff topology and show that the spectral topology on $\operatorname{Min}(A)$ is finer than the inverse topology on $\operatorname{Min}(A)$. Finally, by open sets of the inverse topology, we define and study a congruence relation of an MV-algebra.

Keywords: minimal prime; spectral topology; inverse topology; congruence

MSC 2010: 06D35, 06F30

1. Introduction and preliminaries

MV-algebras were introduced by Chang to provide algebraic semantics for Łukasiewicz infinite-valued propositional logics (see [3]). Eslami introduced the prime spectrum of a BL-algebra (see [5]).

Belluce et al. introduced the prime spectrum of an MV-algebra and studied in [1] a topological space on $\operatorname{Spec}(A)$. They defined the topological space for MV-algebras as follows:

Let A be an MV-algebra. The set of all prime ideals of A is denoted by $\operatorname{Spec}(A)$. $\operatorname{Spec}(A)$ can be endowed with a spectral topology. Thus, if I is an ideal of A, then $u_A(I) = \{P \in \operatorname{Spec}(A) \colon I \nsubseteq P\}$ is an open set in $\operatorname{Spec}(A)$, while $v_A(I) = \{P \in \operatorname{Spec}(A) \colon I \subseteq P\}$ is closed. Also, let $a \in A$. The open sets $u_A(a) = \{P \in \operatorname{Spec}(A) \colon a \notin P\}$ constitute a basis for the open sets of $\operatorname{Spec}(A)$. Topological space $\operatorname{Spec}(A)$ is called the prime spectrum of A.

Also, Forouzesh et al. introduced the spectral topology and quasi-spectral topology of proper prime A-ideals in MV-modules and proved some properties of them (see [6]).

DOI: 10.21136/MB.2018.0117-17

In addition, Bhattacharjee et al. studied the minimal prime spectra of commutative rings with identity. They had been able to identify several interesting types of extensions of rings. Also, they introduced inverse topology on the minimal prime spectra in reduced rings (see [2]). We take this idea from this paper.

In the present paper, we define the inverse topology on the set of all minimal prime ideals of an MV-algebra A and prove some important results. In fact, let $\mathrm{Min}(A)$ be the set of all minimal prime ideals of A. Since $\mathrm{Min}(A) \subseteq \mathrm{Spec}(A)$, we consider the topology induced by spectral topology on $\mathrm{Min}(A)$ and show that the spectral topology on $\mathrm{Min}(A)$ is zero-dimensional Hausdorff topology. Next, we prove that the spectral topology on $\mathrm{Min}(A)$ is finer than the inverse topology on $\mathrm{Min}(A)$. Also, we show that the inverse topology on $\mathrm{Min}(A)$ is a Hausdorff space, compact space, T_0-space and T_1-space on $\mathrm{Min}(A)$.

We recollect some definitions and results which will be used in the following.

Definition 1.1 ([3]). An MV-algebra is a structure $(A, \oplus, *, 0)$, where $\oplus$ is a binary operation, $*$ is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in A$:
(MV1) $(A, \oplus, 0)$ is an Abelian monoid,
(MV2) $(a^*)^* = a$,
(MV3) $0^* \oplus a = 0^*$,
(MV4) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Take $1 = 0^*$ and define the auxiliary operation $\odot$ as:

$$x \odot y = (x^* \oplus y^*)^*.$$

We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Definition 1.2 ([4]). An ideal of an MV-algebra A is a nonempty subset I of A satisfying the following conditions:
(I1) If $x \in I$, $y \in A$ and $y \leqslant x$, then $y \in I$.
(I2) If $x, y \in I$, then $x \oplus y \in I$.

We denote by $\mathrm{Id}(A)$ the set of all ideals of an MV-algebra A.

Definition 1.3 ([4]). Let I be an ideal of an MV-algebra A. Then I is proper if $I \neq A$.

- ▹ A proper ideal I of an MV-algebra A is called a prime ideal if whenever $x \wedge y \in I$ for all $x, y \in A$, then $x \in I$ or $y \in I$.

We denote the set of all prime ideals of an MV-algebra A by $\mathrm{Spec}(A)$.

▷ An ideal I of an MV-algebra A is called a minimal prime ideal of A if:

(1) $I \in \mathrm{Spec}(A)$;

(2) If there exists $Q \in \mathrm{Spec}(A)$ such that $Q \subseteq I$, then $I = Q$.

We denote the set of all prime minimal ideals of an MV-algebra A by $\mathrm{Min}(A)$.

Definition 1.4 ([8])**.** Let X be a nonempty subset of an MV-algebra A and $\mathrm{Ann}_A(X)$ be the annihilator of X defined as

$$\mathrm{Ann}_A(X) = \{a \in A\colon\ a \wedge x = 0\ \forall\, x \in X\}.$$

2. Inverse topology in MV-algebras

In the sequel section, $(A, \oplus, *, 0)$ or simply A is an MV-algebra.

Theorem 2.1. *Let A be an* MV*-algebra and $P \in \mathrm{Spec}(A)$. Then $P \in \mathrm{Min}(A)$ if and only if for each $x \in P$ there exists $r \in A - P$ such that $x \wedge r = 0$.*

P r o o f. Let $P \in \mathrm{Min}(A)$. Suppose that there exists $x \in P$ such that for each $r \in A - P$, $x \wedge r \neq 0$. Obviously, $T = \{r \wedge x\colon\ r \in A - P\} \cup \{1\}$ is a $\wedge$-closed system of A. Then there exists $Q \in \mathrm{Spec}(A)$ such that $Q \cap T = \emptyset$. Consider two cases:

Case 1. Let $Q \subseteq P$. Since $P \in \mathrm{Min}(A)$, $Q = P$, hence $x \in Q$. Since $1 \wedge x = x$, $Q \cap T \neq \emptyset$, which is a contradiction.

Case 2. Let $Q \nsubseteq P$. Hence, there exists $u \in Q - P$. Since $u \wedge x \leqslant u$ and $u \in Q$, we get $u \wedge x \in Q$. Also, we have $u \wedge x \in T$, hence $Q \cap T \neq \emptyset$, which is a contradiction.

Conversely, let for all $x \in P$, there exist $r \in A - P$ such that $r \wedge x = 0$. We show that $P \in \mathrm{Min}(A)$. Let $K \in \mathrm{Spec}(A)$ such that $K \subsetneqq P$. Hence, there exist $x \in P - K$ and $r \in A - P$ such that $r \wedge x = 0$. Thus $0 = r \wedge x \in K$, since $x \notin K$, hence $r \in K$, which is a contradiction. Thus $P \in \mathrm{Min}(A)$. $\square$

Theorem 2.2. *Let A be an* MV*-algebra, $P \in \mathrm{Min}(A)$ and I be a finitely generated ideal. Then $I \subseteq P$ if and only if $\mathrm{Ann}_A(I) \nsubseteq P$.*

P r o o f. Let $I = (a_1, a_2, \ldots, a_n]$ and $I \subseteq P$. By Theorem 2.1, for all $a_i \in P$, $1 \leqslant i \leqslant n$, there exists $u_i \in A - P$ such that $u_i \wedge a_i = 0$. Take $u = u_1 \wedge u_2 \wedge \ldots \wedge u_n$. Obviously, $u \in A - P$. Since for all $x \in I$ we have $x \leqslant a_1 \oplus \ldots \oplus a_n$, we get $u \wedge x \leqslant u \wedge (a_1 \oplus a_2 \oplus \ldots \oplus a_n) \leqslant u \wedge a_1 \oplus u \wedge a_2 \oplus \ldots \oplus u \wedge a_n = 0$. Hence $u \wedge x = 0$ for all $x \in I$. Therefore $u \in \mathrm{Ann}_A(I)$. This implies $\mathrm{Ann}_A(I) \nsubseteq P$.

Conversely, let $\mathrm{Ann}_A(I) \nsubseteq P$. Then there exists $x \in \mathrm{Ann}_A(I) - P$, so $x \wedge a_i = 0$ for all $a_i \in I$. Since $x \wedge a_i = 0 \in P$ and $x \notin P$, we get $a_i \in P$ for all $1 \leqslant i \leqslant n$. Therefore $I \subseteq P$. $\square$

Lemma 2.3. *Let A be an* MV*-algebra. If $0 \neq x \in A$, then there exists $P \in \mathrm{Min}(A)$ such that $x \notin P$.*

Proof. Let $0 \neq x \in A$. Assume that for all $P \in \mathrm{Min}(A)$, $1 \neq x \in P$. So $x \in \bigcap_{P\in \mathrm{Min}(A)} P$. Also, we have $\bigcap_{P\in \mathrm{Min}(A)} P = \bigcap_{P\in \mathrm{Spec}(A)} P = 0$. Hence $x = 0$, which is a contradiction. □

Note: Let A be an MV-algebra. Since $\mathrm{Min}(A) \subseteq \mathrm{Spec}(A)$, we consider $\mathrm{Min}(A)$ as the topology induced by the spectral topology. Thus, for any ideal $I \subseteq A$ and $a \in A$ let us define

$$V_A(I) = \mathrm{Min}(A) \cap v_A(I), \quad U_A(I) = \mathrm{Min}(A) \cap u_A(I),$$

where $v_A(I) = \{P \in \mathrm{Spec}(A)\colon I \subseteq P\}$ and $u_A(I) = \{P \in \mathrm{Spec}(A)\colon I \nsubseteq P\}$. It follows that the family $\{V_A(I)\}_{I\subseteq A}$ is the family of closed sets of the spectral topology on $\mathrm{Min}(A)$, the family $\{U_A(I)\}_{I\subseteq A}$ is the family of open sets of the spectral topology on $\mathrm{Min}(A)$ and the family $\{U_A(a)\}_{a\in A}$ is a basis for the topology of $\mathrm{Min}(A)$.

Lemma 2.4. *Let A be an* MV*-algebra. Suppose that $a, b \in A$ and $I, J \in \mathrm{Id}(A)$. Then the following holds:*
(1) $U_A(a) \cap U_A(b) = U_A(a \wedge b)$,
(2) $U_A(I) \cup U_A(J) = U_A(I \vee J)$,
(3) $U_A(a) = \emptyset$ *if and only if* $a = 0$,
(4) $U_A(I) \cap U_A(J) = U_A(I \wedge J)$,
(5) $U_A(a) \cup U_A(b) = U_A(a \vee b) = U_A(a \oplus b)$,
(6) $V_A(I) \cap V_A(J) = V_A(I \vee J)$,
(7) $V_A(a) \cup V_A(b) = V_A(a \wedge b)$,
(8) $V_A(a) \cap V_A(b) = V_A(a \vee b) = V_A(a \oplus b)$,
(9) $V_A(I) \cup V_A(J) = V_A(I \wedge J)$.

Proof. (1) We have

$$\begin{aligned} P \in U_A(a \wedge b) &\Leftrightarrow P \in \mathrm{Min}(A),\ a \wedge b \notin P \\ &\Leftrightarrow P \in \mathrm{Min}(A),\ a \notin P \text{ and } b \notin P \\ &\Leftrightarrow P \in U_A(a) \cap U_A(b). \end{aligned}$$

(2) Let $P \in U_A(I) \cup U_A(J)$. Consider two cases, $P \in U_A(I)$ or $P \in U_A(J)$.
Case 1. Let $P \in U_A(I)$.

$$\begin{aligned} P \in U_A(I) &\Rightarrow P \in \mathrm{Min}(A),\ I \nsubseteq P \\ &\Rightarrow \exists\, t \in I \text{ such that } t \notin P \end{aligned}$$

$$\begin{aligned}&\Rightarrow \exists t \in I \subseteq I \vee J \text{ such that } t \notin P\\&\Rightarrow I \vee J \nsubseteq P\\&\Rightarrow P \in U_A(I \vee J).\end{aligned}$$

Case 2. Let $P \in U_A(J)$. It is similar to Case 1. Then $U_A(I) \cup U_A(J) \subseteq U_A(I \vee J)$. Let $P \in U_A(I \vee J)$ but $P \notin U_A(I) \cup U_A(J)$. We have

$$\begin{aligned}P \notin U_A(I) \cup U_A(J) &\Rightarrow P \in \mathrm{Min}(A),\ I \subseteq P \text{ and } J \subseteq P\\&\Rightarrow P \in \mathrm{Min}(A),\ I \vee J \subseteq P \vee P = P\\&\Rightarrow P \notin U_A(I \vee J),\end{aligned}$$

which is a contradiction. So $U_A(I \vee J) \subseteq U_A(I) \cup U_A(J)$. Therefore $U_A(I \vee J) = U_A(I) \cup U_A(J)$.

(3) Since for all $P \in \mathrm{Min}(A)$ we have $0 \in P$.

(4) We have

$$\begin{aligned}P \in U_A(I \cap J) &\Rightarrow P \in \mathrm{Min}(A),\ I \cap J \nsubseteq P\\&\Rightarrow P \in \mathrm{Min}(A),\ I \nsubseteq P \text{ and } J \nsubseteq P\\&\Rightarrow P \in U_A(I) \text{ and } P \in U_A(J)\\&\Rightarrow P \in U_A(I) \cap U_A(J).\end{aligned}$$

Thus $U_A(I \cap J) \subseteq U_A(I) \cap U_A(J)$. Now, suppose that

$$\begin{aligned}P \in U_A(I) \cap U_A(J) &\Rightarrow P \in U_A(I) \text{ and } P \in U_A(J)\\&\Rightarrow P \in \mathrm{Min}(A),\ I \nsubseteq P \text{ and } J \nsubseteq P\\&\Rightarrow \exists t \in I \text{ such that } t \notin P \text{ and } \exists x \in J \text{ such that } x \notin P\\&\Rightarrow t \wedge x \leqslant t \in I,\ x \wedge t \leqslant x \in J \text{ and } x \wedge t \notin P\\&\Rightarrow t \wedge x \in I \cap J \text{ and } x \wedge t \notin P\\&\Rightarrow I \cap J \nsubseteq P\\&\Rightarrow P \in U_A(I \cap J).\end{aligned}$$

Thus $U_A(I) \cap U_A(J) \subseteq U_A(I \cap J)$. Therefore $U_A(I) \cap U_A(J) = U_A(I \cap J)$.

(5) We have for any ideal I of A, $a \notin I$ or $b \notin I$ if and only if $a \oplus b \notin I$ if and only if $a \vee b \notin I$. For any prime ideal P we have $P \in U_A(a) \cup U_A(b)$ if and only if $P \in U_A(a \vee b)$ if and only if $P \in U_A(a \oplus b)$. Hence $U_A(a) \cup U_A(b) = U_A(a \vee b) = U_A(a \oplus b)$.

(6) We have by (2)

$$\begin{aligned}V_A(I \vee J) &= \mathrm{Min}(A) - U_A(I \vee J) = \mathrm{Min}(A) - (U_A(I) \cup U_A(J))\\&= U_A^{\mathrm{c}}(I) \cap U_A^{\mathrm{c}}(J) = V_A(I) \cap V_A(J).\end{aligned}$$

(7) We have

$$\begin{aligned} P \in V_A(a) \cup V_A(b) &\Leftrightarrow P \in V_A(a) \text{ or } P \in V_A(b) \\ &\Leftrightarrow a \in P \text{ or } b \in P \text{ (since } a \wedge b \leqslant a \in P \text{ or } a \wedge b \leqslant b \in P) \\ &\Leftrightarrow a \wedge b \in P \text{ (since } P \in \operatorname{Spec}(A)) \\ &\Leftrightarrow P \in V_A(a \wedge b). \end{aligned}$$

(8) By (5) we have

$$\begin{aligned} V_A(a) \cap V_A(b) &= U_A^{\mathrm{c}}(a) \cap U_A^{\mathrm{c}}(b) = (U_A(a) \cup U_A(b))^{\mathrm{c}} \\ &= (U_A(a \vee b))^{\mathrm{c}} = U_A^{\mathrm{c}}(a \oplus b)) = V(a \oplus b). \end{aligned}$$

(9) By (4) we have

$$\begin{aligned} V_A(I) \cup V_A(J) &= U_A^{\mathrm{c}}(I) \cup U_A^{\mathrm{c}}(J) = (U_A(I) \cap U_A(J))^{\mathrm{c}} \\ &= U_A^{\mathrm{c}}(I \wedge J) = V_A(I \wedge J). \end{aligned}$$

□

Theorem 2.5. *Let A be an* MV*-algebra. The spectral topology on* $\operatorname{Min}(A)$ *is a zero-dimensional Hausdorff topology.*

Proof. We know that the family $\{U_A(a)\}_{a \in A}$ is a basis for spectral topology on $\operatorname{Min}(A)$. We claim that $U_A(a) = V_A(\operatorname{Ann}_A(a))$. By Theorem 2.2, we obtain

$$\begin{aligned} P \in U_A(a) &\Rightarrow P \in \operatorname{Min}(A),\ a \notin P \\ &\Rightarrow \operatorname{Ann}_A(a) \subseteq P \\ &\Rightarrow P \in V_A(\operatorname{Ann}_A(a)) \\ &\Rightarrow U_A(a) \subseteq V_A(\operatorname{Ann}_A(a)). \end{aligned}$$

It follows from Theorem 2.2 that

$$\begin{aligned} P \in V_A(\operatorname{Ann}_A(a)) &\Rightarrow P \in \operatorname{Min}(A),\ \operatorname{Ann}_A(a) \subseteq P \\ &\Rightarrow a \notin P \\ &\Rightarrow P \in U_A(a) \\ &\Rightarrow V_A(\operatorname{Ann}_A(a)) \subseteq U_A(a). \end{aligned}$$

Therefore $V_A(\operatorname{Ann}_A(a)) = U_A(a)$.

Now, we show that the spectral topology on $\mathrm{Min}(A)$ is a Hausdorff topology. Let P_1 and P_2 be two distinct minimal prime ideals of A. Since $P_1 \neq P_2$, we get $P_1 \nsubseteq P_2$ or $P_2 \nsubseteq P_1$. Without loss of generality, we suppose that $P_1 \nsubseteq P_2$. Then there exists $a \in P_1$ such that $a \notin P_2$. Take $U = U_A(a)$ and $V = U_A^{\mathrm{c}}(a) = V_A(a) = U_A(\mathrm{Ann}_A(a))$. Hence $P_2 \in U$ and $P_1 \in V$. We have $U \cap V = U_A(a) \cap U_A(\mathrm{Ann}_A(a)) = U_A(a) \cap U_A^{\mathrm{c}}(a) = \emptyset$. We conclude that the spectral topology on $\mathrm{Min}(A)$ is a Hausdorff topology. □

Lemma 2.6. *Let A be a nonempety* MV*-algebra. The collection $\beta = \{V_A(I)\colon I \in \mathrm{Id}(A)\}$ is a base for a topology on* $\mathrm{Min}(A)$.

P r o o f. For all $P \in \mathrm{Min}(A)$, $I_0 = \{0\}$ is an ideal of an MV-algebra A such that $I_0 \subseteq P$. So $P \in V_A(I_0)$. Let $V_A(I), V_A(J) \in \beta$. It follows from Lemma 2.4 (6) that $V_A(I) \cap V_A(J) = V_A(I \vee J)$. Therefore, this collection is a base for the topology on $\mathrm{Min}(A)$. □

R e m a r k 2.7. The induced topology of base

$$\beta = \{V_A(I)\colon\ I \text{ is finitely generated ideal of } A\}$$

is called the inverse topology. When equipped with the inverse topology on $\mathrm{Min}(A)$, we shall write $\mathrm{Min}^{-1}(A)$.

R e m a r k 2.8. Collection $\{V_A(a)\colon\ a \in A\}$ forms a subbase for a topology on $\mathrm{Min}^{-1}(A)$.

P r o o f. Obviously, $\mathrm{Min}(A) = \bigcup\limits_{a \in A} V_A(a)$. By Theorem 2.4 (6), we have

$$V(I) = V((a_1, \ldots, a_n]) = V((a_1] \vee \ldots \vee (a_n]) = V((a_1]) \cap \ldots \cap V((a_n]) = \bigcap_{i=1}^{n} V(a_i).$$

□

Lemma 2.9. *The spectral topology on* $\mathrm{Min}(A)$ *is finer than the inverse topology on* $\mathrm{Min}(A)$.

P r o o f. It follows from Theorem 2.5, Lemma 2.4 (6) and (4) that

$$\begin{aligned} V(I) = V((a_1, a_2, \ldots, a_n]) &= V\left(\bigvee_{i=1}^{n} (a_i]\right) = \bigcap_{i=1}^{n} V(a_i) \\ &= \bigcap_{i=1}^{n} U(\mathrm{Ann}_A(a_i)) = U\left(\bigwedge_{i=1}^{n} \mathrm{Ann}_A(a_i)\right). \end{aligned}$$

For any finitely generated ideal I of A we have $V_A(I)$ an open set in the spectral topology on $\operatorname{Min}(A)$. We conclude that the spectral topology is finer than the inverse topology on $\operatorname{Min}(A)$. $\square$

Remark 2.10. Let I and J be finitely generated ideals of an MV-algebra A. Then the following holds:
(1) $I \wedge J$ is a finitely generated ideal.
(2) $I \vee J$ is a finitely generated ideal.

Proof. Let $I = (a_1, a_2, \ldots, a_n]$ and $J = (b_1, b_2, \ldots, b_m]$. We have

$$\begin{aligned}(a_1, a_2, \ldots, a_n] \cap (b_1, b_2, \ldots, b_m] &= ((a_1] \vee (a_2] \vee \ldots (a_n]) \cap ((b_1] \vee (b_2] \vee \ldots \vee (b_m]) \\ &= (a_1 \oplus a_2 \oplus \ldots \oplus a_n] \cap (b_1 \oplus b_2 \oplus \ldots \oplus b_m] \\ &= ((a_1 \oplus \ldots \oplus a_n) \wedge (b_1 \oplus \ldots \oplus b_m)].\end{aligned}$$

Thus, $I \wedge J$ is a finitely generated ideal of A.

(2) Suppose that $I = (a_1, a_2, \ldots, a_n]$ and $J = (b_1, b_2, \ldots, b_m]$. We have

$$\begin{aligned}I \vee J = (a_1, a_2, \ldots, a_n] \vee (b_1, b_2, \ldots, b_m] &= \bigvee_{i=1}^{n} (a_i] \vee \bigvee_{i=1}^{m} (b_i] \\ &= (a_1 \oplus a_2 \oplus \ldots \oplus a_n] \vee (b_1 \oplus b_2 \oplus \ldots \oplus b_m] \\ &= (a_1 \oplus \ldots \oplus a_n \oplus b_1 \oplus \ldots \oplus b_m].\end{aligned}$$

Hence, $I \vee J$ is a finitely generated ideal of A. $\square$

Theorem 2.11. *Let A be an* MV*-algebra. If for any $a \in A$ there exists a finitely generated ideal I of A such that $I \subseteq \operatorname{Ann}_A(a)$ and $\operatorname{Ann}_A((a] \vee I) = \{0\}$, then the spectral topology and the inverse topology on* $\operatorname{Min}(A)$ *are equal.*

Proof. By Lemma 2.9, we have that the spectral topology on $\operatorname{Min}(A)$ is finer than the inverse topology on $\operatorname{Min}(A)$. It is enough to show that for any $a \in A$ there exists a finitely generated ideal I of A such that $U_A(a) = V_A(I)$. We have for any $a \in A$ that there exists a finitely generated ideal I of A such that $I \subseteq \operatorname{Ann}_A(a)$ and $\operatorname{Ann}_A((a] \vee I) = 0$. Let $P \in U_A(a)$. Then $a \notin P$. If $x \in I$, then $x \wedge a = 0$ and we get $x \in P$. Thus $I \subseteq P$, so $U_A(a) \subseteq V_A(I)$. Let $V_A(I) \nsubseteq U_A(a)$. Then there exists $P \in \operatorname{Min}(A)$ such that $I \subseteq P$ and $a \in P$. We show that $(a] \vee I \subseteq P$.

$$\begin{aligned}t \in (a] \vee I &\Rightarrow t \leqslant na \oplus b \ \text{ such that } \ b \in I \subseteq P \\ &\Rightarrow t \in P \\ &\Rightarrow (a] \vee I \subseteq P.\end{aligned}$$

But we have $\operatorname{Ann}_A((a] \vee I) = 0$, then $\operatorname{Ann}_A((a] \vee I) \subseteq P$, which contradicts Theorem 2.2. $\square$

We recall that for any $a \in A$, $U_A(a)$ is compact in Spec(A) (see [1]). It follows form Theorem 2.4 (2) that $U_A(I) = U_A\Big(\bigvee_{i=1}^{n}(a_i]\Big) = \bigcup_{i=1}^{n} U_A((a_i]) = \bigcup_{i=1}^{n} U_A(a_i)$. We conclude that $U(I)$ is compact.

Theorem 2.12. *Let A be an* MV*-algebra. Then for any $a \in A$, $V_A(a)$ is compact in* $\mathrm{Min}(A)^{-1}$.

P r o o f. It is sufficient to show that any cover of $V_A(a)$ with open basis sets contains a finite cover of $V_A(a)$. By Theorem 2.5 we have

$$U_A(\mathrm{Ann}_A(a)) = V_A(a) \subseteq \bigcup_{i\in I} V_A(a_i) = \bigcup_{i\in I} U_A(\mathrm{Ann}_A(a_i)).$$

Since $U_A(\mathrm{Ann}_A(a))$ is compact in spectral topology on $\mathrm{Min}(A)$, there exists a finite subset J of I such that $V(a) \subseteq \bigcup_{i\in J} U_A(\mathrm{Ann}_A(a_i)) = \bigcup_{i=1}^{n} V_A(a_i)$. This implies that $V_A(a)$ is a compact set in $\mathrm{Min}^{-1}(A)$. $\square$

R e m a r k 2.13. Let A be an MV-algebra. For any ideal I of A, $\overline{U}_A(I) = V_A(\mathrm{Ann}_A(I))$ in the spectral topology on $\mathrm{Min}^{-1}(A)$.

P r o o f. Let $P \in V_A(\mathrm{Ann}_A(I))$. Then $P \in \mathrm{Min}(A)$ and $\mathrm{Ann}_A(I) \subseteq P$. Let $t \in A$ and $U_A(t)$ be an open set such that $P \in U_A(t)$. We show that $U_A((t]) \cap U_A(I) \neq \emptyset$. Let $U_A((t]) \cap U_A(I) = \emptyset$. By Lemma 2.4 (4), we have $U_A((t]) \cap U_A(I) = U_A((t] \wedge I) = \emptyset$. It follows from Lemma 2.4 (3) that $(t] \wedge I = 0$. As $t \in \mathrm{Ann}_A(I)$, we get $t \in P$, which is a contradiction. Thus $V_A(\mathrm{Ann}_A(I)) \subseteq \overline{U}_A(I)$.

Let $P \in U_A(I)$. Then $P \in \mathrm{Min}(A)$, $I \nsubseteq P$. Now, we have

$$\begin{aligned} x \in \mathrm{Ann}_A(I) &\Rightarrow x \wedge a = 0 \ \ \forall a \in I \\ &\Rightarrow x \in P \text{ (since P is a prime ideal)} \\ &\Rightarrow \mathrm{Ann}_A(I) \subseteq P \\ &\Rightarrow P \in V_A(\mathrm{Ann}_A(I)) \\ &\Rightarrow U_A(I) \subseteq V_A(\mathrm{Ann}_A(I)) \\ &\Rightarrow \overline{U}_A(I) \subseteq \overline{V}_A(\mathrm{Ann}_A(I)) \\ &\Rightarrow \overline{U}_A(I) \subseteq V_A(\mathrm{Ann}_A(I)). \end{aligned}$$

Therefore $\overline{U}_A(I) = V_A(\mathrm{Ann}_A(I))$. $\square$

Theorem 2.14. $\mathrm{Min}^{-1}(A)$ *is compact, T_0-space and T_1-space.*

P r o o f. We have $\mathrm{Min}(A) = V_A(0) = \{P \in \mathrm{Min}(A)\colon 0 \in P\}$. It follows from Theorem 2.12 that $\mathrm{Min}^{-1}(A)$ is compact. For $P_1, P_2 \in \mathrm{Min}(A)$ such that $P_1 \neq P_2$, where $P_1 \nsubseteq P_2$ or $P_2 \nsubseteq P_1$. Without loss of generality, we suppose that $P_1 \nsubseteq P_2$. Then there exists $a \in P_1$ such that $a \notin P_2$. Taking $U = V_A(a)$, then $P_1 \in U$ and $P_2 \notin U$. Hence $\mathrm{Min}^{-1}(A)$ is a T_0-space. Let $P, Q \in \mathrm{Min}(A)$ be distinct minimal prime ideals and let $a \in P - Q$. By Lemma 2.1, there is an $x \notin P$ such that $a \wedge x = 0$. It follows that $a \wedge x \in Q$ and so $x \in Q - P$. Notice that $P \in V_A(a) - V_A(x)$ and $Q \in V_A(x)$, so $V_A(x) \nsubseteq V_A(a)$. Hence, the inverse topology is a T_1-space. □

We note that a topological space X is connected if and only if it has only A and $\emptyset$ as clopen subsets (see [7]).

Corollary 2.15. *If A is an* MV*-algebra and $A \neq \{0,1\}$, then $\mathrm{Min}^{-1}(A)$ is disconnected.*

P r o o f. Since $A \neq \{0,1\}$, there exists $a \in A$ such that $a \neq 0, 1$. By Theorem 2.5, $V_A(a) = U_A(\mathrm{Ann}_A(a))$ is a nonempty clopen set. Therefore $\mathrm{Min}^{-1}(A)$ is disconnected. □

Theorem 2.16. *$\mathrm{Min}^{-1}(A)$ is a Hausdorff topological space.*

P r o o f. Let P and Q be two distinct minimal prime ideals of A. Since $P \neq Q$, there are $a \in P - Q$ and $b \in Q - P$. By Theorem 2.2, since $(a] \subseteq P$ and $(b] \subseteq Q$ and $P, Q \in \mathrm{Min}(A)$, we get $\mathrm{Ann}_A((a]) \nsubseteq P$ and $\mathrm{Ann}_A((b]) \nsubseteq Q$. Hence, $P \in U_A(\mathrm{Ann}_A(a))$, $Q \in U_A(\mathrm{Ann}_A(b))$, $P \in V_A(a)$ and $Q \in V_A(b)$. By Theorem 2.5, since the spectral topology on $\mathrm{Min}(A)$ is Hausdorff, we have $V_A(a) \cap V_A(b) = U_A(\mathrm{Ann}_A(a)) \cap U_A(\mathrm{Ann}_A(b)) = \emptyset$. □

Lemma 2.17. *$H \subseteq \mathrm{Min}^{-1}(A)$ is clopen if and only if there exist finitely generated ideals I and J of A such that $V(I) = H$, $I \wedge J = \{0\}$ and $\mathrm{Ann}(I \vee J) = \{0\}$.*

P r o o f. Suppose H is a clopen subset of $\mathrm{Min}^{-1}(A)$. By Theorem 2.14, the inverse topology on $\mathrm{Min}(A)$ is compact. It follows that H is compact. So H is a union of base open sets. Now, by Lemma 2.4 (9), we have $H = \bigcup\limits_{i=1}^{n} V_A(I_i) = V_A(I_1 \wedge I_2 \wedge \ldots \wedge I_n)$ and by Remark 2.10 (1), there exists a finitely generated ideal I of A such that $V_A(I_1 \wedge I_2 \wedge \ldots \wedge I_n) = V_A(I)$. Since the complement of H is also clopen, we conclude that for some finitely generated ideal, $\mathrm{Min}(A) - H = V_A(J)$. Thus, by Lemma 2.4 (6), we have $\emptyset = V_A(I) \cap V_A(J) = V_A(I \vee J)$. So for every $P \in \mathrm{Min}(A)$ we get $I \vee J \nsubseteq P$. By Remark 2.10 (2), $I \vee J$ is a finitely generated ideal. Now, by Theorem 2.2, for every $P \in \mathrm{Min}(A)$, $\mathrm{Ann}_A(I \vee J) \subseteq P$. We have

$\mathrm{Ann}_A(I \vee J) \subseteq \bigcap_{P \in \mathrm{Min}(A)} P = 0$. Hence $\mathrm{Ann}_A(I \vee J) = 0$. Finally, by Lemma 2.4 (9), we get $V_A(I) \cup V_A(J) = V_A(I \wedge J) = \mathrm{Min}(A)$. Since $I \wedge J \subseteq P$ for all $P \in \mathrm{Min}(A)$, $I \wedge J = \{0\}$. Conversely, since $\mathrm{Ann}(I \vee J) = \{0\}$ for every $P \in \mathrm{Min}(A)$ we have $\mathrm{Ann}_A(I \vee J) \subseteq P$. By Remark 2.16 and Theorem 2.2, we get $I \vee J \nsubseteq P$. Now, by Lemma 2.4 (2), for every $P \in \mathrm{Min}(A)$ we have

$$P \in U_A(I \vee J) \Rightarrow P \in U_A(I) \cap U_A(J) \Rightarrow P \in V_A^{\mathrm{c}}(I) \cup V_A^{\mathrm{c}}(J).$$

We obtain $V_A^{\mathrm{c}}(I) \cup V_A^{\mathrm{c}}(J) = \mathrm{Min}(A)$. Hence $V_A(I) \cap V_A(J) = \emptyset$. Now by Lemma 2.4 (9), we have

$$V_A(I) \cup V_A(J) = V_A(I \wedge J) = V_A(0) = \mathrm{Min}(A).$$

Then $V_A(I)$ is a complement of $V_A(J)$. Thus $V_A(I) = H$ is clopen. $\square$

N o t a t i o n. We recall that let A and B be disjoint compact subspaces of the Hausdorff space X. Then there exist disjoint open sets U and V containing A and B, respectively (see [7]).

Theorem 2.18. $\mathrm{Min}(A)^{-1}$ *is a compact zero-dimensional Hausdorff space if and only if for every $a, b \in A$ such that $a \wedge b = 0$ there exist finitely generated ideals I, J of A such that $a \in I$, $b \in J$, $I \wedge J = \{0\}$ and* $\mathrm{Ann}_A(I \vee J) = \{0\}$.

P r o o f. Suppose that $\mathrm{Min}^{-1}(A)$ is a compact zero-dimensional Hausdorff space and $a, b \in A$ such that $a \wedge b = 0$. By Lemma 2.4 (1), we have $U_A(a) \cap U_A(b) = U_A(a \wedge b) = U_A(0) = \emptyset$ and since $U_A(a)$ and $U_A(b)$ are closed, they are compact subsets of $\mathrm{Min}(A)$. By the above notation and since $\mathrm{Min}^{-1}(A)$ is zero-dimensional, there exists a clopen set $H \subseteq \mathrm{Min}(A)$ such that $U_A(a) \subseteq H$, and $H \cap U_A(b) = \emptyset$. By Lemma 2.17, there exist finitely generated ideals I, J containing a and b, respectively, such that $H = V_A(I)$ and $V_A(J) = \mathrm{Min}(A) - H$ such that ($a \in I$, $b \in J$), $I \wedge J = \{0\}$, and $\mathrm{Ann}_A(I \vee J) = \{0\}$.

Conversely, we show that $\mathrm{Min}^{-1}(A)$ has a base of clopen sets. It is sufficient to show that given $P \in V_A(a)$, there is a clopen subset $H \subseteq \mathrm{Min}(A)$ for which $P \in H \subseteq V_A(a)$. Let $P \in V_A(a)$. Then $P \in \mathrm{Min}(A)$ and $a \in P$. By Theorem 2.1, there exists $b \in A - P$ such that $a \wedge b = 0$. Now, by hypothesis, there exist finitely generated ideals I, J of A such that $a \in I$, $b \in J$, $I \wedge J = \{0\}$ and $\mathrm{Ann}_A(I \vee J) = \{0\}$. By Theorem 2.17, we define $H = V_A(I)$. It is a clopen subset of $\mathrm{Min}^{-1}(A)$. Since $a \in I$, it follows that $V_A(I) \subseteq V_A(a)$. Now, we show that $P \in V_A(I)$. Let $P \notin V_A(I)$. Then $I \nsubseteq P$. Since $I \wedge J = \{0\} \subseteq P$, we get $J \subseteq P$. It follows that $b \in P$, which is a contradiction. Therefore $\mathrm{Min}^{-1}(A)$ is zero-dimensional. By Theorems 2.16 and 2.14, we obtain that $\mathrm{Min}^{-1}(A)$ is a Hausdorff space and it is compact. $\square$

Theorem 2.19. *Let A be a nontrivial* MV*-algebra. For $a, b \in A$ define $a \sim b$ if and only if $V_A(a) = V_A(b)$. Hence, $a \sim b$ if and only if for any $P \in \mathrm{Min}(A)$, $a \in P$ if and only if $b \in P$.*

Proof. Obviously, $\sim$ is an equivalence relation on A. Let $a, b, c, d \in A$ such that $a \sim b$ and $c \sim d$. We will prove that $a \oplus c \sim b \oplus d$, $a^* \sim b^*$. Suppose that $P \in \mathrm{Min}(A)$. Then $a, c \leqslant a \oplus c \in P$ if and only if $a \in P$ and $c \in P$ if and only if $b \in P$ and $d \in P$ if and only if $b \oplus d \in P$. That is, $a \oplus c \in P$ if and only if $b \oplus d \in P$. Hence $a \oplus c \sim b \oplus d$. Similarly, $a \vee c \sim b \vee d$. Since P is a prime ideal, we get $a \wedge c \in P$ if and only if $a \in P$ or $c \in P$ if and only if $b \in P$ or $d \in P$ if and only if $b \wedge d \in P$. Thus $a \wedge c \sim b \wedge d$. Let $a \sim b$. Then $a \in P$ if and only if $b \in P$. We show that $a^* \in P$ if and only if $b^* \in P$. We have

$$\begin{aligned} a^* \in P &\Rightarrow a \odot a^* \leqslant a^* \in P \Rightarrow a \odot a^* \in P \\ &\Rightarrow a^* \oplus (a \odot a^*) \in P \Rightarrow a \leqslant a \vee a^* \in P \\ &\Rightarrow a \in P \Rightarrow b \in P \\ &\Rightarrow b \odot b^* \leqslant b \in P \Rightarrow b \odot b^* \in P \\ &\Rightarrow b \oplus (b \odot b^*) \in P \Rightarrow b^* \leqslant b \vee b^* \in P \\ &\Rightarrow b^* \in P. \end{aligned}$$

Similarly, we obtain that if $b^* \in P$, then $a^* \in P$. Hence $a^* \in P$ if and only if $b^* \in P$. Thus $a^* \sim b^*$. The congruence class of x with respect to $\sim$ will be denoted by $[x]$, i.e. $[x] = \{y \in A \colon x \sim y\}$. Let $A/{\sim}$ be the quotient set. Since $\sim$ is a congruence on A, the algebra $(A/{\sim}, \oplus, *, [0])$ is an MV-algebra, where $[a] \oplus [b] = [a \oplus b]$ and $[a]^* = [a^*]$. $\square$

Theorem 2.20. *Let $a, b \in A$. Then we have:*

(1) *$[a] \leqslant [b]$ if and only if $V_A(b) \subseteq V_A(a)$.*
(2) *$[a] = [b]$ if and only if $(a] = (b]$.*
(3) *$[a] = [1]$ if and only if $\mathrm{ord}(a) \leqslant \infty$.*
(4) *$[a] = [0]$ if and only if $a = 0$.*
(5) *$[a \vee b] = [a \oplus b]$.*
(6) *$[na] = [a]$ for some $n \in \mathbb{N}$.*

Proof. (1) By Lemma 2.4 (7), we have

$$\begin{aligned} [a] \leqslant [b] &\Leftrightarrow [a] \wedge [b] = [a] \Leftrightarrow [a \wedge b] = [a] \Leftrightarrow V_A(a \wedge b) = V_A(a) \\ &\Leftrightarrow V_A(a) \cup V_A(b) = V_A(a) \Leftrightarrow V_A(b) \subseteq V_A(a). \end{aligned}$$

(2) We have $[a]=[b]$ if and only if $V_A(a)=V_A(b)$. It follows that

$$\begin{aligned}(a] &= \bigcap\{P\in \mathrm{Min}(A)\colon\ a\in P\} = \bigcap\{P\in \mathrm{Min}(A)\colon\ P\in V_A(a)\}\\ &= \bigcap\{P\in \mathrm{Min}(A)\colon\ P\in V_A(b)\} = \bigcap\{P\in \mathrm{Min}(A)\colon\ b\in P\} = (b].\end{aligned}$$

(3) By (2), we have $[a]=[1]$ if and only if $(a]=(1]=A$. Hence, $1\in(a]$ if and only if $1\leqslant na$ for some $n\in\mathbb{N}$. We get $na=1$ for some $n\in\mathbb{N}$, that is, $\mathrm{ord}(a)\leqslant\infty$.

(4) By (2), we obtain $[a]=[0]$ if and only if $(a]=(0]=\{0\}$ if and only if $a=0$.

(5) It follows from Lemma 2.4 (8) that $V_A(a\vee b)=V_A(a\oplus b)$. Hence $[a\vee b]=[a\oplus b]$.

(6) By (5), we get $[na]=[a\oplus a\oplus\ldots\oplus a]=[a\vee a\vee\ldots\vee a]=[a]$. $\square$

A c k n o w l e d g e m e n t. The authors are very indebted to the referees for valuable suggestions that improved the readability of the paper.

References

[1] *L. P. Belluce, A. Di Nola, S. Sessa*: The prime spectrum of an MV-algebra. Math. Log. Q. *40* (1994), 331–346. zbl MR doi

[2] *P. Bhattacharjee, K. M. Drees, W. W. McGovern*: Extensions of commutative rings. Topology Appl. *158* (2011), 1802–1814. zbl MR doi

[3] *C. C. Chang*: Algebraic analysis of many valued logics. Trans. Am. Math. Soc. *88* (1958), 467–490. zbl MR doi

[4] *R. L. O. Cignoli, I. M. L. D'Ottaviano, D. Mundici*: Algebraic Foundations of Many-Valued Reasoning. Trends in Logic-Studia Logica Library 7. Kluwer Academic Publishers, Dordrecht, 2000. zbl MR doi

[5] *E. Eslami*: The prime spectrum on BL-algebras and MV-algebras. Siminar Algebra. Tarbiat Moallem University, 2009, pp. 58–61. (In Persian.)

[6] *F. Forouzesh, E. Eslami, A. Borumand Saeid*: Spectral topology on MV-modules. New Math. Nat. Comput. *11* (2015), 13–33. zbl MR doi

[7] *J. R. Munkres*: Topology. Prentice Hall, Upper Saddle River, 2000. zbl MR

[8] *D. Piciu*: Algebras of Fuzzy Logic. Editura Universitaria din Craiova, Craiova, 2007. (In Romanian.)

Authors' address: *Fereshteh Forouzesh*, *Farhad Sajadian*, *Mahta Bedrood*, Faculty of Mathematics and Computing, Higher Education Complex of Bam, Khalij Fars street, 76613-14477 Bam, Kerman, Iran, e-mail: `frouzesh@bam.ac.ir`, `fsajadian@bam.ac.ir`, `bedrood.m@gmail.com`.