SOME REMARKS ON DESCRIPTIVE CHARACTERIZATIONS OF THE STRONG MCSHANE INTEGRAL

Sokol Bush Kaliaj, Elbasan

Received September 5, 2017. Published online August 9, 2018.
Communicated by Jiří Spurný

Dedicated to the memory of Štefan Schwabik

Abstract. We present the full descriptive characterizations of the strong McShane integral (or the variational McShane integral) of a Banach space valued function $f\colon W \to X$ defined on a non-degenerate closed subinterval W of $\mathbb{R}^m$ in terms of strong absolute continuity or, equivalently, in terms of McShane variational measure $V_{\mathcal{M}}F$ generated by the primitive $F\colon \mathcal{I}_W \to X$ of f, where $\mathcal{I}_W$ is the family of all closed non-degenerate subintervals of W.

Keywords: strong McShane integral; McShane variational measure; Banach space, m-dimensional Euclidean space; compact non-degenerate m-dimensional interval

MSC 2010: 28B05, 26A46, 46B25, 46G10, 28A35

1. Introduction and preliminaries

In the monograph [21] of Štefan Schwabik and Ye Guoju, a full characterization of the strong McShane integral of Banach-space valued functions defined on a compact non-degenerate subinterval of $\mathbb{R}$ is given, see Theorem 7.4.14. There is also a full descriptive characterization of the variational McShane integral in [12], Theorem 2.5.

In [13], Yeong gives some full characterizations of the strong McShane integral of Banach-space valued functions defined on a compact non-degenerate subinterval of $\mathbb{R}^m$.

In this paper, we present the full descriptive characterizations of the strong McShane integral of a Banach space valued function $f\colon W \to X$ defined on a non-degenerate closed subinterval W of $\mathbb{R}^m$ in terms of strong absolute continuity or, equivalently, in terms of McShane variational measure $V_{\mathcal{M}}F$ generated by the primitive $F\colon \mathcal{I}_W \to X$ of f, see Theorem 2.8.

DOI: 10.21136/MB.2018.0100-17

Throughout this paper, X denotes a real Banach space with the dual X^* and W denotes a compact non-degenerate subinterval of the m-dimensional Euclidean space $\mathbb{R}^m$. The Euclidean space $\mathbb{R}^m$ is equipped with the maximum norm. $B_m(t,r)$ is the open ball in $\mathbb{R}^m$ with center t and radius $r > 0$. We denote by $\mathscr{B}(\mathbb{R}^m)$ the Borel σ-algebra on $\mathbb{R}^m$ and by $\mathcal{L}(\mathbb{R}^m)$ the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^m$. We put

$$\mathcal{L}(W) = \{W \cap E\colon\ E \in \mathcal{L}(\mathbb{R}^m)\} \quad \text{and} \quad \mathscr{B}(W) = \{W \cap B\colon\ B \in \mathscr{B}(\mathbb{R}^m)\}.$$

The Lebesgue measure on $\mathcal{L}(W)$ is denoted by λ and the Lebesgue measure of a Lebesgue measurable set $E \in \mathcal{L}(W)$ is denoted by $|E|$. The phrase "at almost all" always referes to λ.

If μ is a measure on $\mathcal{L}(W)$, then by $\mu \ll \lambda$ we mean that $|E| = 0$ implies $\mu(E) = 0$. A vector measure $\nu\colon\ \mathcal{L}(W) \to X$ is said to be a *countable additive vector measure* if ν is countable additive in the norm topology of X. A countable additive vector measure ν is said to be λ-continuous if $|E| = 0$ implies $\nu(E) = 0$. The *variation* of a countable additive vector measure ν is denoted by $|\nu|$; ν is said to be of *bounded variation* on W if $|\nu|(W) < \infty$.

Let $\alpha = (a_1, \ldots, a_m)$ and $\beta = (b_1, \ldots, b_m)$ with $-\infty < a_j < b_j < \infty$ for $j = 1, \ldots, m$. The set $[\alpha, \beta] = \prod_{j=1}^{m} [a_j, b_j]$ is called a *closed non-degenerate interval* in $\mathbb{R}^m$, while $[\alpha, \beta) = \prod_{j=1}^{m} [a_j, b_j)$ is said to be a *half-closed interval* (*or brick*) in $\mathbb{R}^m$. By $\mathscr{B}_r(\mathbb{R}^m)$, the family of all bricks in $\mathbb{R}^m$ is denoted. In particular, if $b_1 - a_1 = \ldots = b_m - a_m$, then $I = [\alpha, \beta]$ is called a *cube* and we set $l_I = b_1 - a_1$. In this case, $|I| = (l_I)^m$. We denote by $\mathcal{I}_W$ the family of all closed non-degenerate subintervals of W.

Two intervals $I, J \in \mathcal{I}_W$ are said to be *non-overlapping* if $I^\circ \cap J^\circ = \emptyset$, where I° denotes the *interior* of I. A function $F\colon\ \mathcal{I}_W \to X$ is said to be an *additive interval function* if for each pair of non-overlapping intervals $I, J \in \mathcal{I}_W$ with $I \cup J \in \mathcal{I}_W$, we have

$$F(I \cup J) = F(I) + F(J).$$

Definition 1.1. An additive interval function $F\colon\ \mathcal{I}_W \to X$ is said to be *strongly absolutely continuous* (sAC) on W if for each $\varepsilon > 0$ there exists $\eta > 0$ such that for each finite collection $\{I_1, \ldots, I_p\}$ of pairwise non-overlapping subintervals in $\mathcal{I}_W$ we have

$$\sum_{i=1}^{p} |I_i| < \eta \Rightarrow \sum_{i=1}^{p} \|F(I_i)\| < \varepsilon.$$

Replacing the last inequality with $\left\| \sum_{i=1}^{p} F(I_i) \right\| < \varepsilon$, we obtain the notion of *absolute continuity* (AC) on W.

Definition 1.2. A finite collection $\{I_1, \dots, I_p\}$ of pairwise non-overlapping intervals in $\mathcal{I}_W$ is said to be a *division* of W if $\bigcup_{i=1}^{p} I_i = W$. $\mathscr{D}_W$ denotes the family of all divisions of interval W. The *total variation* $V_F(W)$ of an additive interval function $F\colon \mathcal{I}_W \to X$ on W is defined as

$$V_F(W) = \sup\Big\{\sum_{J\in\mathscr{D}} \|F(J)\|\colon \mathscr{D} \in \mathscr{D}_W\Big\}.$$

If $V_F(W) < \infty$, then F is said to be of *bounded variation on* W.

The following lemma has been proven in [14], Lemma 10.3.7 for the real valued functions, but the proof works also for Banach-space valued functions after trivial changes.

Lemma 1.3. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function. If F is sAC on W, then F is of bounded variation on W.*

Definition 1.4. Assume that a point $t \in W$ and a function $F\colon \mathcal{I}_W \to X$ are given. We set

$$\mathcal{I}_W(t) = \{I \in \mathcal{I}_W\colon\ t \in I,\ I \text{ is a cube}\}.$$

We say that F is *cubic derivable* at t if there exists a vector $F'_{\rm c}(t) \in X$ such that

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to 0}} \frac{F(I)}{|I|} = F'_{\rm c}(t).$$

$F'_{\rm c}(t)$ is said to be the *cubic derivative* of F at t. The *cubic derivative* of F at t is a generalization of the derivative $F'(t)$ defined in [13], Definition 3.2.

A function $f\colon\ \mathbb{R}^m \to \mathbb{R}$ is called *locally integrable* if f is Borel measurable function and

$$\int_K |f(s)|\,\mathrm{d}\lambda < \infty \text{ for every bounded measurable set } K \in \mathscr{B}(\mathbb{R}^m).$$

The following theorem is the Lebesgue Differentiation Theorem, c.f. Theorem 3.21 in [7].

Theorem 1.5. *If a function $f\colon\ \mathbb{R}^m \to \mathbb{R}$ is locally integrable, then there exists $Z \in \mathscr{B}(\mathbb{R}^m)$ with $|Z| = 0$ such that*

$$\lim_{r\to 0} \frac{1}{|E_r|} \int_{E_r} |f(s) - f(t)|\,\mathrm{d}\lambda(s) = 0 \quad \textit{for all } t \in \mathbb{R}^m \setminus Z$$

whenever $(E_r)_{r>0}$ is a family that shrinks nicely to t.

A family $(E_r)_{r>0}$ of Borel subsets of $\mathbb{R}^m$ is said to *shrink nicely* to $t \in \mathbb{R}^m$ if

▷ $E_r \subset B_m(t,r)$ for each r,

▷ there is a constant $\alpha > 0$, independent of r, such that $|E_r| > \alpha |B_m(t,r)|$,

c.f. [7], page 98.

A pair (t, I) of a point $t \in W$ and an interval $I \in \mathcal{I}_W$ is called an *$\mathcal{M}$-tagged interval* in W, t is the tag of I. A finite collection $\{(t_i, I_i)\colon\ i = 1,\dots,p\}$ of $\mathcal{M}$-tagged intervals in W is called an *$\mathcal{M}$-partition* in W if $\{I_i\colon\ i = 1,\dots,p\}$ is a collection of pairwise non-overlapping intervals in $\mathcal{I}_W$. Given $Z \subset W$, a positive function $\delta\colon\ Z \to (0,\infty)$ is called a *gauge* on Z. We say that an $\mathcal{M}$-partition $\pi = \{(t_i, I_i)\colon\ i = 1,\dots,p\}$ in W is

▷ a partition of W if $\bigcup\limits_{i=1}^{p} I_i = W$;

▷ Z-tagged if $\{t_1,\dots,t_p\} \subset Z$;

▷ δ-fine if for each $(t,I) \in \pi$ we have $I \subset B_m(t, \delta(t))$.

Definition 1.6. A function $f\colon\ W \to X$ is said to be *McShane integrable* on W if there is a vector $x_f \in X$ such that for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\Bigl\| \sum_{(t,I)\in\pi} f(t)|I| - x_f \Bigr\| < \varepsilon.$$

In this case, the vector x_f is said to be the *McShane integral* of f on W and we set $x_f = (M)\int_W f\,\mathrm{d}\lambda$. The function f is said to be *McShane integrable* over a subset $A \subset W$ if the function $f \cdot \mathbb{1}_A\colon\ W \to X$ is McShane integrable on W, where $\mathbb{1}_A$ is the characteristic function of the set A. The McShane integral of f over A will be denoted by $(M)\int_A f\,\mathrm{d}\lambda$. If $f\colon\ W \to X$ is McShane integrable on W, then by Theorem 4.1.6 in [21] the function f is McShane integrable on each $E \in \mathcal{L}(W)$.

Definition 1.7. The function $f\colon\ W \to X$ is said to be *variationally McShane integrable* (*or strongly McShane integrable*) on W if there exists an additive interval function $F\colon\ \mathcal{I}_W \to X$ such that for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\sum_{(t,I)\in\pi} \|f(t)|I| - F(I)\| < \varepsilon.$$

The function F is said to be the primitive of f. Clearly, if f is variationally McShane integrable with the primitive F, then f is McShane integrable, and by Proposition 3.6.16 in [21] we also have

$$F(I) = (M)\int_I f\,\mathrm{d}\lambda \quad \text{for every } I \in \mathcal{I}_W.$$

For more information about the McShane integral we refer to [21], [25], [5], [8], [9]–[11], [16], [15], [26] and [1].

Definition 1.8. Given an additive interval function $F\colon \mathcal{I}_W \to X$, a subset $Z \subset W$ and a gauge δ on Z, we define

$$V_{\mathcal{M}}F(Z,\delta) = \sup\Bigl\{\sum_{(t,I)\in\pi} \|F(I)\|\colon \pi \text{ is a } Z\text{-tagged } \delta\text{-fine } \mathcal{M}\text{-partition in } W\Bigr\}.$$

Then we set

$$V_{\mathcal{M}}F(Z) = \inf\{V_{\mathcal{M}}F(Z,\delta)\colon \delta \text{ is a gauge on } Z\}.$$

The set function $V_{\mathcal{M}}F$ is said to be the *McShane variational measure generated by* F.

The set function $V_{\mathcal{M}}F$ is a Borel metric outer measure on W, see [4] or [23]. The McShane variational measure have been used extensively for studying the primitives (indefinite integrals) of real functions. See e.g. the paper [4] of Di Piazza, the book [14] of Lee Tuo-Yeong, [20] of Pfeffer for relations to integration and the fundamental general work [24] of Thomson. The following lemma has been proven by Di Piazza in [4], Proposition 1 (there she considers real valued functions, but the proof works also for vector valued functions, after trivial changes).

Lemma 1.9. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function. Then the following statements are equivalent:*

(i) *F is sAC on W;*

(ii) $V_{\mathcal{M}}F \ll \lambda$.

A function $f\colon W \to X$ is said to be *weakly measurable* if for each $x^* \in X^*$ the real function $x^* \circ f$ is Lebesgue measurable; f is said to be *measurable* if there is a sequence $f_n\colon W \to X$ of simple measurable functions such that

$$\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0 \quad \text{at almost all } t \in W.$$

The function $f\colon W \to X$ is said to be *Bochner integrable* on W if f is measurable and there exists a sequence (f_n) of simple measurable functions such that

$$\lim_{n\to\infty} \int_W \|f(t) - f_n(t)\| \,\mathrm{d}\lambda = 0.$$

In this case, $(B)\int_E f\,\mathrm{d}\lambda$ is defined for each Lebesgue measurable set $E \in \mathcal{L}(W)$ as

$$(B)\int_E f\,\mathrm{d}\lambda = \lim_{n\to\infty} (B)\int_E f_n\,\mathrm{d}\lambda,$$

where $(B)\int_E f_n\,\mathrm{d}\lambda$ is defined in the usual way.

The function $f\colon W\to X$ is said to be *Pettis integrable* on W if $x^*\circ f$ is Lebesgue integrable on W for each $x^*\in X^*$ and for every Lebesgue measurable set $E\in\mathcal{L}(W)$ there is a vector $\nu(E)\in X$ such that

$$x^*(\nu(E))=\int_E(x^*\circ f)\,\mathrm{d}\lambda\quad\text{for all } x^*\in X^*.$$

The vector $\nu(E)$ is then called the *Pettis integral* of f over E and we set $\nu(E)=(P)\int_E f\,\mathrm{d}\lambda$. We refer to [3], [17]–[19], [22] and [2] for Pettis integral.

2. The main result

The main result is Theorem 2.8. Let us start with some auxiliary lemmas.

Lemma 2.1. *If a function $f\colon W\to\mathbb{R}$ is Lebesgue integrable on W, then*

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to0}}\frac{1}{|I|}\int_I|f(s)-f(t)|\,\mathrm{d}\lambda(s)=0\quad\textit{for almost all } t\in W.$$

Consequently,

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to0}}\frac{1}{|I|}\int_I f(s)\,\mathrm{d}\lambda(s)=f(t)\quad\textit{for almost all } t\in W. \tag{2.1}$$

P r o o f. Since f is Lebesgue integrable on W, there exists a Borel measurable function $h\colon W\to\mathbb{R}$ such that it is Lebesgue integrable on W and $h(t)=f(t)$ for almost all $t\in W$. Consider a function $g\colon\mathbb{R}^m\to\mathbb{R}$ defined as

$$g(t)=\begin{cases}h(t) & \text{if } t\in W,\\ 0 & \text{if } t\in\mathbb{R}^m\setminus W.\end{cases}$$

Since g is locally integrable, by Theorem 1.5 there exists $Z\in\mathscr{B}(\mathbb{R}^m)$ with $|Z|=0$ such that

$$\lim_{r\to0}\frac{1}{|E_r|}\int_{E_r}|g(s)-g(t)|\,\mathrm{d}\lambda(s)=0\quad\text{for all } t\in\mathbb{R}^m\setminus Z,$$

whenever $(E_r)_{r>0}$ is a family that shrinks nicely to t.

Fix an arbitrary $t\in W\setminus Z$. For each real positive number $r>0$ we can choose an arbitrary cube $I_r\in\mathcal{I}_W(t)$ such that $r=l(I_r)$. Note that

$$I_r\subset B(t,r)\quad\text{and}\quad |I_r|=r^m>\frac{1}{2^{m+1}}|B_m(t,r)|$$

whenever $r > 0$. Thus, the family $(I_r)_{r>0}$ shrinks nicely to t. Therefore

$$\lim_{r\to 0} \frac{1}{|I_r|} \int_{I_r} |g(s) - g(t)| \,\mathrm{d}\lambda(s) = 0,$$

and since t and $(I_r)_{r>0}$ are arbitrary, it follows that

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to 0}} \frac{1}{|I|} \int_I |g(s) - g(t)| \,\mathrm{d}\lambda(s) = 0 \quad \text{for all } t \in W \setminus Z.$$

Hence,

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to 0}} \frac{1}{|I|} \int_I |h(s) - h(t)| \,\mathrm{d}\lambda(s) = 0 \quad \text{for all } t \in W \setminus Z.$$

Further, since $h(t) = f(t)$ for almost all $t \in W$, it follows that

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to 0}} \frac{1}{|I|} \int_I |f(s) - f(t)| \,\mathrm{d}\lambda(s) = 0 \quad \text{for almost all } t \in W.$$

The last result together with

$$\left| \frac{1}{|I|} \int_I f(s) \,\mathrm{d}\lambda(s) - f(t) \right| \leqslant \frac{1}{|I|} \int_I |f(s) - f(t)| \,\mathrm{d}\lambda(s)$$

yields (2.1), and this ends the proof. □

As in [6], page 156, define a function $\varrho\colon\ \mathcal{L}(W) \times \mathcal{L}(W) \to [0, \infty)$ by

$$\varrho(U, V) = |U \Delta V| \quad \text{for each } (U, V) \in \mathcal{L}(W) \times \mathcal{L}(W).$$

It is not difficult to check that ϱ is a semimetric in $\mathcal{L}(W)$, i.e. ϱ satisfies the following conditions:

▷ $\varrho(U, U) = 0$,
▷ $\varrho(U, V) = \varrho(V, U)$,
▷ $\varrho(U, V) \leqslant \varrho(U, H) + \varrho(H, V)$,

whenever $U, V, H \in \mathcal{L}(W)$.

Lemma 2.2. *If $\nu\colon\ \mathcal{L}(W) \to X$ is a countably additive λ-continuous vector measure, then*

$$\nu(\mathcal{I}_W) = \{\nu(I)\colon\ I \in \mathcal{I}_W\}$$

is a separable set in X.

P r o o f. We denote by $\mathcal{Q}_W$ the family of all intervals in $\mathcal{I}_W$ with vertices having rational coordinates. It is easy to see that

$$\mathcal{I}_W \subset \overline{\mathcal{Q}}_W^{\varrho}, \tag{2.2}$$

where $\overline{\mathcal{Q}}_W^{\varrho}$ is the closure of $\mathcal{Q}_W$ in the semimetric space $(\mathcal{L}(W), \varrho)$. We are going to show that

$$\nu(\mathcal{I}_W) \subset \overline{\nu(\mathcal{Q}_W)}^{\|\cdot\|}, \tag{2.3}$$

where

$$\nu(\mathcal{Q}_W) = \{\nu(I)\colon\ I \in \mathcal{Q}_W\}$$

and $\overline{\nu(\mathcal{Q}_W)}^{\|\cdot\|}$ is the closure of $\nu(\mathcal{Q}_W)$ in the Banach space X. To see this, let $\nu(I) \in \nu(\mathcal{I}_W)$. Then by (2.2), there exists a sequence $(I_k) \subset \mathcal{Q}_W$ such that

$$\lim_{k\to\infty} (|I \setminus I_k| + |I_k \setminus I|) = \lim_{k\to\infty} \varrho(I_k, I) = 0$$

and therefore by Theorem I.2.1 in [3], we obtain

$$\lim_{k\to\infty} \nu(I \setminus I_k) = 0 \quad \text{and} \quad \lim_{k\to\infty} \nu(I_k \setminus I) = 0. \tag{2.4}$$

Since

$$I = (I \setminus I_k) \cup (I \cap I_k) \quad \text{and} \quad I_k = (I_k \setminus I) \cup (I \cap I_k),$$

it follows that

$$\|\nu(I) - \nu(I_k)\| = \|\nu(I \setminus I_k) - \nu(I_k \setminus I)\| \leqslant \|\nu(I \setminus I_k)\| + \|\nu(I_k \setminus I)\|.$$

The last result together with (2.4) yields that

$$\lim_{k\to\infty} \|\nu(I) - \nu(I_k)\| = 0.$$

This means that (2.3) holds, and this ends the proof. □

The next lemma is proved by using Caratheodory-Hahn-Kluvanek Extension theorem, see Theorem I.5.2 in [3]. We recall that a collection $\mathcal{E}$ of subsets of W is said to be an elementary family if

- ▷ $\emptyset \in \mathcal{E}$,
- ▷ if $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$,
- ▷ if $E \in \mathcal{E}$, then $E^{\mathrm{c}} = W \setminus E$ is a finite disjoint union of members of $\mathcal{E}$,

c.f. [7], page 23.

Lemma 2.3. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function. If F is AC on W, then there exists a unique countably additive λ-continuous vector measure $F_{\mathcal{L}}\colon \mathcal{L}(W) \to X$ such that*

$$F(I) = F_{\mathcal{L}}(I) \quad \text{for all } I \in \mathcal{I}_W.$$

Moreover, if F is sAC on W, then $F_{\mathcal{L}}$ is of bounded variation on W.

Proof. We set

$$\mathscr{B}_r(W) = \{W \cap B_r\colon\ B_r \in \mathscr{B}_r(\mathbb{R}^m)\}.$$

It is easy to see that $\mathcal{E} = \mathscr{B}_r(W) \cup \{\emptyset\}$ is an elementary family. Therefore, by Proposition 1.7 in [7], it follows that the collection $\mathscr{A}$ of finite disjoint unions of members of $\mathcal{E}$ is an algebra. Since

$$\mathscr{B}(W) = \sigma(\mathscr{A}),$$

where $\sigma(\mathscr{A})$ is the σ-algebra generated by $\mathscr{A}$, and since the closure of $\mathscr{A}$ with respect to ϱ is a σ-algebra, it follows that $\mathscr{A}$ is a dense subset of $\mathscr{B}(W)$ with respect to ϱ.

Assume that an arbitrary nonempty set $A \in \mathscr{A}$ is given. If $\{I_1, \ldots, I_p\}$ and $\{J_1, \ldots, J_q\}$ are finite collections of pairwise disjoint bricks in $\mathscr{B}_r(W)$ such that

$$A = I_1 \cup \ldots \cup I_p = J_1 \cup \ldots \cup J_q,$$

then

$$B = \{I_i \cap J_j\colon\ I_i \cap J_j \neq \emptyset,\ i = 1, \ldots, p,\ j = 1, \ldots, q\}$$

is a finite collection of pairwise disjoint bricks in $\mathscr{B}_r(W)$ and $A = \bigcup\limits_{I \in B} I$. Then, since F is additive and any two representations of A as a finite disjoint union of bricks have a common refinement, the sum

$$F(\overline{I}_1) + \ldots + F(\overline{I}_p)$$

is independent of the particular family $\{I_1, \ldots, I_p\}$ of pairwise disjoint bricks whose union is A, where $\overline{I}_i$ is the closure of I_i. Thus, we can define vector $F_{\mathscr{A}}(A)$ by equation

$$F_{\mathscr{A}}(A) = F(\overline{I}_1) + \ldots + F(\overline{I}_p).$$

In particular, we define $F_{\mathscr{A}}(\emptyset) = 0$.

From the fact that F is AC it follows that

$$\lim_{\substack{(A \in \mathscr{A})\\ |A| \to 0}} F_{\mathscr{A}}(A) = 0.$$

Hence, $F_{\mathscr{A}}$ is a strongly additive and countably additive vector measure on $\mathscr{A}$. Therefore by Caratheodory-Hahn-Kluvanek Extension theorem, Theorem I.5.2 in [3], $F_{\mathscr{A}}$ has a unique countable additive λ-continuous extension $F_{\mathscr{B}}\colon \mathscr{B}(W) \to X$, and since

$$F_{\mathscr{B}}(B') - F_{\mathscr{B}}(B'') = F_{\mathscr{B}}(B' \setminus B'') - F_{\mathscr{A}}(B'' \setminus B'), \quad B', B'' \in \mathscr{B}(W),$$

it follows that $F_{\mathscr{B}}$ is uniformly continuous on $\mathscr{B}(W)$ with respect to ϱ.

Since $F_{\mathscr{B}}$ is a countably additive λ-continuous vector measure on $\mathscr{B}(W)$, it has a unique countable additive λ-continuous extension $F_{\mathcal{L}}\colon \mathcal{L}(W) \to X$.

We now assume that F is sAC on W. It is enough to show that $F_{\mathscr{B}}$ is of bounded variation on W. To see this, let us consider a finite collection $\{B_i\colon i = 1, 2, \ldots, p\}$ of pairwise disjoint members of $\mathscr{B}(W)$. Since $F_{\mathscr{B}}$ is uniformly continuous with respect to ϱ on $\mathscr{B}(W)$, given $0 < \varepsilon < 1$ there exists $\delta > 0$ such that for each $B, B' \in \mathscr{B}(W)$ we have

$$\varrho(B, B') = |B \Delta B'| < \delta \Rightarrow \|F_{\mathscr{B}}(B) - F_{\mathscr{B}}(B')\| < \frac{\varepsilon}{2p^2}.$$

Since $\mathscr{A}$ is dense in $\mathscr{B}(W)$ with respect to ϱ, for each B_i there exists an $A_i \in \mathscr{A}$ such that

$$\varrho(B_i, A_i) = |B_i \Delta A_i| < \frac{\delta}{2},$$

and since

$$(A_i \cap A_j) \setminus B_i \subset A_i \Delta B_i, \quad (A_i \cap A_j) \setminus B_j \subset A_j \Delta B_j$$

and

$$A_i \cap A_j \subset ((A_i \cap A_j) \setminus B_i) \cup ((A_i \cap A_j) \setminus B_j),$$

it follows that

$$\varrho((A_i \cap A_j), \emptyset) = |A_i \cap A_j| < \delta, \quad i \neq j.$$

Therefore, if we set

$$C_1 = A_1, \quad C_2 = A_2 \setminus A_1, \quad \ldots, \quad C_p = A_p \setminus \bigcup_{k=1}^{p-1} A_k,$$

then

$$\begin{aligned}
\sum_{i=1}^{p} \|F_{\mathscr{B}}(B_i)\| &\leqslant \sum_{i=1}^{p} \|F_{\mathscr{B}}(B_i) - F_{\mathscr{B}}(A_i)\| + \sum_{i=1}^{p} \|F_{\mathscr{B}}(A_i)\| < \sum_{i=1}^{p} \|F_{\mathscr{A}}(A_i)\| + \frac{\varepsilon}{2} \\
&\leqslant \sum_{i=1}^{p} \|F_{\mathscr{A}}(C_i)\| + \sum_{\substack{i \neq j \\ i,j}} \|F_{\mathscr{A}}(A_i \cap A_j)\| + \frac{\varepsilon}{2} \\
&< V_F(W) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = V_F(W) + \varepsilon < V_F(W) + 1.
\end{aligned}$$

Since F is sAC on W, the last result together with Lemma 1.3 yields

$$|F_{\mathscr{B}}|(W) \leqslant V_F(W) + 1 < \infty.$$

Thus, $F_{\mathscr{B}}$ is of bounded variation on W, and this ends the proof. □

The next lemma gives full descriptive characterizations of Lebesgue integral.

Lemma 2.4. *Let $F\colon \mathcal{I}_W \to \mathbb{R}$ be an additive interval function and let $f\colon W \to \mathbb{R}$ be a function. Then the following statements are equivalent:*

(i) *F is AC on W;*
(ii) *F is sAC on W;*
(iii) *$V_{\mathcal{M}} \ll \lambda$;*
(iv) *F is AC on W, $F'_{\rm c}(t)$ exists and $F'_{\rm c}(t) = f(t)$ for almost all $t \in W$;*
(v) *F is sAC on W, $F'_{\rm c}(t)$ exists and $F'_{\rm c}(t) = f(t)$ for almost all $t \in W$;*
(vi) *$V_{\mathcal{M}} \ll \lambda$, $F'_{\rm c}(t)$ exists and $F'_{\rm c}(t) = f(t)$ for almost all $t \in W$;*
(vii) *f is Lebesgue integrable on W with the primitive F, i.e.*

$$F(I) = \int_I f \,\mathrm{d}\lambda \quad \text{for all } I \in \mathcal{I}_W.$$

P r o o f. Since F is a real valued function, it is easy to see that if F is AC on W, then F is sAC on W. Therefore (i) ⇔ (ii) and (iv) ⇔ (v). By virtue of Lemma 1.9 it follows that (ii) ⇔ (iii) and (v) ⇔ (vi).

(ii) ⇒ (vii): Assume that F is sAC on W. Then by Lemma 2.3 there exists a unique countably additive λ-continuous vector measure $F_{\mathcal{L}}\colon \mathcal{L}(W) \to \mathbb{R}$ of bounded variation on W such that $F_{\mathcal{L}}(I) = F(I)$ for all $I \in \mathcal{I}_W$. Therefore, by Lebesgue-Radon-Nikodym theorem, see Theorem 3.8 in [7], there exists a Lebesgue integrable function $f\colon W \to \mathbb{R}$ such that $F_{\mathcal{L}}(E) = \int_E f \,\mathrm{d}\lambda$ for all $E \in \mathcal{L}(W)$. In particular, we have $F(I) = \int_I f \,\mathrm{d}\lambda$ for all $I \in \mathcal{I}_W$.

(vii) ⇒ (iv): Assume that (vii) holds. Then by Corollary 3.6 in [7], F is AC on W. Also, since $F(I) = \int_I f \,\mathrm{d}\lambda$ for all $I \in \mathcal{I}_W$, by Lemma 2.1 it follows that $F'_{\rm c}(t)$ exists and $F'_{\rm c}(t) = f(t)$ for almost all $t \in W$.

Clearly, (iv) ⇒ (i), and this ends the proof. □

We now show full descriptive characterizations of Pettis integral.

Lemma 2.5. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function and let $f\colon W \to X$ be a function. Then the following statements are equivalent:*

(i) *f is Pettis integrable on W with the primitive F, i.e.*

$$F(I) = (P)\int_I f\,\mathrm{d}\lambda \quad \textit{for all } I \in \mathcal{I}_W;$$

(ii) *F is AC on W and for each $x^* \in X^*$, $(x^* \circ F)'_{\mathrm{c}}(t)$ exists and*

$$(x^* \circ F)'_{\mathrm{c}}(t) = (x^* \circ f)(t) \quad \textit{for almost all } t \in W$$

(the exceptional set may vary with x^*).

Proof. (i) $\Rightarrow$ (ii): Assume that (i) holds. Then each $(x^* \circ f)$ is Lebesgue integrable on W with the primitive $(x^* \circ F)$. Therefore for each $x^* \in X^*$, by Lemma 2.4, $(x^* \circ F)'_{\mathrm{c}}(t)$ exists and $(x^* \circ F)'_{\mathrm{c}}(t) = (x^* \circ f)(t)$ for almost all $t \in W$.

Since f is Pettis integrable on W, by Theorem II.3.5 in [3], the vector measure $\nu\colon \mathcal{L}(W) \to X$ defined as

$$\nu(E) = (P)\int_E f\,\mathrm{d}\lambda \quad \text{for all } E \in \mathcal{L}(W)$$

is a countably additive λ-continuous vector measure on $\mathcal{L}(W)$, and since λ is a finite measure on $\mathcal{L}(W)$, we obtain by Theorem I.2.1 in [3] that F is AC.

(ii) $\Rightarrow$ (i): Assume that (ii) holds. Then by Lemma 2.4, each $(x^* \circ f)$ is Lebesgue integrable on W with the primitive $(x^* \circ F)$, i.e.

$$(x^* \circ F)(I) = \int_I (x^* \circ f)\,\mathrm{d}\lambda \quad \text{for all } I \in \mathcal{I}_W.$$

Since F is AC on W, by Lemma 2.3 there exists a unique countably additive λ-continuous vector measure $\nu\colon \mathcal{L}(W) \to X$ such that $F(I) = \nu(I)$ for all $I \in \mathcal{I}_W$. It follows that for each $x^* \in X^*$ we have

$$x^*(\nu(I)) = \int_I (x^* \circ f)\,\mathrm{d}\lambda \quad \text{for all } I \in \mathcal{I}_W.$$

It is easy to see that the family

$$\mathcal{C} = \left\{B \in \mathscr{B}(W)\colon\ \forall\, x^* \in X^*,\ \left[x^*(\nu(B)) = \int_B (x^* \circ f)\,\mathrm{d}\lambda\right]\right\}$$

is a σ-algebra such that

$$\mathcal{I}_W \subset \mathcal{C} \subset \mathscr{B}(W),$$

and since $\mathscr{B}(W) = \sigma(\mathcal{I}_W)$, it follows that $\mathcal{C} = \mathscr{B}(W)$. Thus, for each $B \in \mathscr{B}(W)$ we have

$$x^*(\nu(B)) = \int_B (x^* \circ f)\,\mathrm{d}\lambda \quad \text{for all } x^* \in X^*.$$

Hence, since ν is λ-continuous, for each $E \in \mathcal{L}(W)$ we have

$$x^*(\nu(E)) = \int_E (x^* \circ f)\,\mathrm{d}\lambda \quad \text{for all } x^* \in X^*.$$

This means that f is Pettis integrable on W, and this ends the proof. □

By Theorem 3.5 in [13] it follows that if $V_{\mathcal{M}}F \ll \lambda$, $F'(t)$ exists and $F'(t) = f(t)$ for almost all $t \in W$, then $f\colon W \to X$ is variationally McShane integrable on W with the primitive $F\colon \mathcal{I}_W \to X$. Since $F'_{\mathrm{c}}(t)$ is a generalization of $F'(t)$, we need to prove the following theorem.

Theorem 2.6. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function and let $f\colon W \to X$ be a function. Assume that F is sAC on W, $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t) = f(t)$ for almost all $t \in W$. Then f is variationally McShane integrable function with the primitive F, i.e.*

$$F(I) = (M)\int_I f\,\mathrm{d}\lambda \quad \text{for all } I \in \mathcal{I}_W.$$

Proof. By hypothesis, for all $x^* \in X^*$ we have $(x^* \circ F)'_{\mathrm{c}}(t)$ exists and

$$(x^* \circ F)'_{\mathrm{c}}(t) = (x^* \circ f)(t) \quad \text{for almost all } t \in W.$$

Therefore, by Lemma 2.5, f is Pettis integrable on W with the primitive F. Hence, by Theorem II.3.5 in [3], the vector measure $\nu\colon \mathcal{L}(W) \to X$ defined by

$$\nu(E) = (P)\int_E f\,\mathrm{d}\lambda \quad \text{for all } E \in \mathcal{L}(W)$$

is a countably additive λ-continuous vector measure. Since F is sAC on W and since

$$\nu(I) = F(I) \quad \text{for all } I \in \mathcal{I}_W,$$

we obtain by Lemma 2.3 that ν is of bounded variation.

We obtain by Lemma 2.2 that $Y_0 = \{F(I)\colon I \in \mathcal{I}_W\}$ is a separable subset of X. If Y is the closed linear subspace spanned by Y_0, then Y is also a separable subset of X. Since $F(I)/|I| \in Y$ for all $I \in \mathcal{I}_W(t)$, we obtain that $f(t) \in Y$ for almost all $t \in W$. Hence, f is λ-essentially separably valued. Since f is Pettis integrable on W, we have also that f is weakly measurable. Therefore by Theorem II.1.2 in [3], the function f is measurable. Hence, by Remark 4.1 in [18] it follows that

$$|\nu|(E) = \int_E \|f(t)\| \,\mathrm{d}\lambda \quad \text{for each } E \in \mathcal{L}(W),$$

and since ν is of bounded variation, the function $\|f(\cdot)\|$ is Lebesgue integrable on W. Further, by Theorem II.2.2 in [3], function f is Bochner integrable on W. Since the Bochner and Pettis integrals coincide whenever they coexist, we have $F(I) = (B)\int_I f \,\mathrm{d}\lambda$ for all $I \in \mathcal{I}_W$. Thus, function f is Bochner integrable and therefore by Theorem 5.1.4 in [21], f is variationally McShane integrable on W with the primitive F, and this ends the proof. □

According to Theorem 3.1 in [13], if $F\colon \mathcal{I}_W \to X$ is the primitive of a variationally McShane integrable function $f\colon W \to X$, then $V_{\mathcal{M}}F \ll \lambda$. Therefore, to prove (i) ⇒ (ii) in Theorem 2.8, it is enough to prove that if F is the primitive of a variationally McShane integrable function f, then $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t) = f(t)$ for almost all $t \in W$.

Theorem 2.7. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function. Assume that a function $f\colon W \to X$ is variationally McShane integrable on W with the primitive F, i.e.*

$$F(I) = (M)\int_I f \,\mathrm{d}\lambda \quad \textit{for all } I \in \mathcal{I}_W.$$

Then $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t) = f(t)$ for almost all $t \in W$.

P r o o f. By Theorem 5.1.4 in [21], f is Bochner integrable on W and

$$F(I) = (B)\int_I f \,\mathrm{d}\lambda \quad \text{for all } I \in \mathcal{I}_W.$$

Since f is measurable, we assume without loss of generality that f is separably valued. Then there exists a countable set

$$Y = \{x_k \in X\colon\ k \in \mathbb{N}\}$$

such that Y is a dense subset of $f(W)$. By virtue of Theorem II.2.2 in [3], $\|f(\cdot) - x_k\|$ is Lebesgue integrable on W. Hence, by Lemma 2.1 there exists a subset $Z_k \subset W$ with $|Z_k| = 0$ such that for all $t \in W \setminus Z_k$ we have

$$\lim_{\substack{I \in \mathcal{I}_W(t) \\ |I| \to 0}} \frac{1}{|I|} \int_I \|f(s) - x_k\| \,\mathrm{d}\lambda(s) = \|f(t) - x_k\|.$$

Fix an arbitrary $t \in W \setminus Z$, where $Z = \bigcup_{k=1}^{\infty} Z_k$. Since

$$\frac{1}{|I|}\int_I \|f(s) - f(t)\| \, \mathrm{d}\lambda(s) \leqslant \frac{1}{|I|}\int_I \|f(s) - x_k\| \, \mathrm{d}\lambda(s) + \|x_k - f(t)\|,$$

we obtain

$$\limsup_{\substack{I \in \mathcal{I}_W(t) \\ |I| \to 0}} \frac{1}{|I|}\int_I \|f(s) - f(t)\| \, \mathrm{d}\lambda(s) \leqslant 2\|x_k - f(t)\| \quad \text{for all } k \in \mathbb{N}.$$

The last inequality together with the fact that Y is a dense subset of $f(W)$ yields

$$\limsup_{\substack{I \in \mathcal{I}_W(t) \\ |I| \to 0}} \frac{1}{|I|}\int_I \|f(s) - f(t)\| \, \mathrm{d}\lambda(s) = 0$$

and therefore

$$\lim_{\substack{I \in \mathcal{I}_W(t) \\ |I| \to 0}} \frac{1}{|I|}\int_I \|f(s) - f(t)\| \, \mathrm{d}\lambda(s) = 0.$$

The last result together with

$$\left\| \frac{1}{|I|}(B)\int_I f(s) \, \mathrm{d}\lambda(s) - f(t) \right\| \leqslant \frac{1}{|I|}\int_I \|f(s) - f(t)\| \, \mathrm{d}\lambda(s)$$

yields

$$\lim_{\substack{I \in \mathcal{I}_W(t) \\ |I| \to 0}} \frac{1}{|I|}(B)\int_I f(s) \, \mathrm{d}\lambda(s) = f(t).$$

Since t is arbitrary, the last equality holds at all $t \in W \setminus Z$. Thus, $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t) = f(t)$ for almost all $t \in W$, and this ends the proof. $\square$

We are now ready to present the main result.

Theorem 2.8. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function and let $f\colon W \to X$ be a function. Then the following statements are equivalent:*

(i) *f is variationally McShane integrable on W with the primitive F, i.e.*

$$F(I) = (M)\int_I f \, \mathrm{d}\lambda \quad \textit{for all } I \in \mathcal{I}_W;$$

(ii) *F is sAC on W, $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t) = f(t)$ for almost all $t \in W$;*

(iii) *$V_{\mathcal{M}}F \ll \lambda$, $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t) = f(t)$ for almost all $t \in W$.*

P r o o f. By virtue of Lemma 1.9, we obtain immediately that (ii) ⇔ (iii). By Theorem 2.6 it follows that (ii) ⇒ (i). Theorem 2.7 together with Theorem 3.1 in [13] yields that (i) ⇒ (iii), and this ends the proof. □

References

[1] *D. Candeloro, L. Di Piazza, K. Musiał, A. R. Sambucini*: Gauge integrals and selections of weakly compact valued multifunctions. J. Math. Anal. Appl. *441* (2016), 293–308. zbl MR doi

[2] *D. Candeloro, L. Di Piazza, K. Musiał, A. R. Sambucini*: Relations among gauge and Pettis integrals for $cwk(X)$-valued multifunctions. Ann. Mat. Pura Appl. (4) *197* (2018), 171–183. zbl MR doi

[3] *J. Diestel, J. J. Uhl*, Jr.: Vector Measures. Mathematical Surveys 15. AMS, Providence, 1977. zbl MR

[4] *L. Di Piazza*: Variational measures in the theory of the integration in $\mathbb{R}^m$. Czech. Math. J. *51* (2001), 95–110. zbl MR doi

[5] *L. Di Piazza, K. Musiał*: A characterization of variationally McShane integrable Banach-space valued functions. Ill. J. Math. *45* (2001), 279–289. zbl MR

[6] *N. Dunford, J. T. Schwartz*: Linear Operators I. General Theory. Pure and Applied Mathematics. Vol. 7. Interscience Publishers, New York, 1958. zbl MR

[7] *G. B. Folland*: Real Analysis. Modern Techniques and Their Applications. Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs, and Tracts. Wiley, New York, 1999. zbl MR

[8] *D. H. Fremlin*: The generalized McShane integral. Ill. J. Math. *39* (1995), 39–67. zbl MR

[9] *R. A. Gordon*: The Denjoy extension of the Bochner, Pettis, and Dunford integrals. Stud. Math. *92* (1989), 73–91. zbl MR doi

[10] *R. A. Gordon*: The McShane integral of Banach-valued functions. Ill. J. Math. *34* (1990), 557–567. zbl MR

[11] *R. A. Gordon*: The Integrals of Lebesgue, Denjoy, Perron, and Henstock. Graduate Studies in Mathematics 4. AMS, Providence, 1994. zbl MR doi

[12] *S. B. Kaliaj*: Descriptive characterizations of Pettis and strongly McShane integrals. Real Anal. Exch. *39* (2014), 227–238. zbl MR doi

[13] *T.-Y. Lee*: Some full characterizations of the strong McShane integral. Math. Bohem. *129* (2004), 305–312. zbl MR

[14] *T. Y. Lee*: Henstock-Kurzweil Integration on Euclidean Spaces. Series in Real Analysis 12. World Scientific, Hackensack, 2011. zbl MR doi

[15] *V. Marraffa*: The variational McShane integral in locally convex spaces. Rocky Mt. J. Math. *39* (2009), 1993–2013. zbl MR doi

[16] *E. J. McShane*: Unified Integration. Pure and Applied Mathematics 107. Academic Press, Orlando (Harcourt Brace Jovanovich, Publishers), 1983. zbl MR

[17] *K. Musiał*: Vitali and Lebesgue convergence theorems for Pettis integral in locally convex spaces. Atti Semin. Mat. Fis. Univ. Modena *35* (1987), 159–165. zbl MR

[18] *K. Musiał*: Topics in the theory of Pettis integration. Rend. Ist. Math. Univ. Trieste *23* (1991), 177–262. zbl MR

[19] *K. Musiał*: Pettis integral. Handbook of Measure Theory. Vol. I. and II. (E. Pap, ed.). North-Holland, Amsterdam, 2002, pp. 531–586. zbl MR doi

[20] *W. F. Pfeffer*: Derivation and Integration. Cambridge Tracts in Mathematics 140. Cambridge University Press, Cambridge, 2001. zbl MR doi

[21] *Š. Schwabik, G. Ye*: Topics in Banach Space Integration. Series in Real Analysis 10. World Scientific, Hackensack, 2005. zbl MR doi

[22] *M. Talagrand*: Pettis integral and measure theory. Mem. Am. Math. Soc. *51* (1984), 224 pages. zbl MR doi
[23] *B. S. Thomson*: Derivates of interval functions. Mem. Am. Math. Soc. *452* (1991), 96 pages. zbl MR doi
[24] *B. S. Thomson*: Differentiation. Handbook of Measure Theory. Volume I. and II. (E. Pap, ed.). North-Holland, Amsterdam, 2002, pp. 179–247. zbl MR doi
[25] *C. Wu, X. Yao*: A Riemann-type definition of the Bochner integral. J. Math. Study *27* (1994), 32–36. zbl MR
[26] *G. Ye*: On Henstock-Kurzweil and McShane integrals of Banach space-valued functions. J. Math. Anal. Appl. *330* (2007), 753–765. zbl MR doi

Author's address: *Sokol Bush Kaliaj*, Department of Mathematics, Faculty of Natural Science, Aleksander Xhuvani University, Rruga Rinia, Elbasan, Albania, e-mail: `sokolkaliaj@yahoo.com`.