ON A GENERALIZATION OF HENSTOCK-KURZWEIL INTEGRALS

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Abstract. We study a scale of integrals on the real line motivated by the MC_α integral by Ball and Preiss and some recent multidimensional constructions of integral. These integrals are non-absolutely convergent and contain the Henstock-Kurzweil integral. Most of the results are of comparison nature. Further, we show that our indefinite integrals are a.e. approximately differentiable. An example of approximate discontinuity of an indefinite integral is also presented.

Keywords: Henstock-Kurzweil integral

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1. Introduction

The Riemann approach to integration of a function $f\colon I \to \mathbb{R}$ is based on limits of sums

$$\sum_{i=1}^{m} f(x_i)(b_i - a_i),$$

where $\{[a_i, b_i], x_i\}_{i=1}^m$ is a complete tagged partition of the interval I. By this we mean that the intervals $[a_i, b_i]$ are nonoverlapping, their union is I and $x_i \in [a_i, b_i]$. The improvement by Henstock and Kurzweil consists in the requirement that the partitions are δ-fine for some gage δ. This trick makes the class of integrable functions much wider, in particular, the Henstock-Kurzweil integral extends the Lebesgue integral and integrates all derivatives.

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By the Saks-Henstock lemma, the corresponding indefinite integral F of f is characterized by smallness of the sums

$$\sum_{i=1}^{m} |F(b_i) - F(a_i) - f(x_i)(b_i - a_i)|,$$

for this $\{[a_i, b_i], x_i\}_{i=1}^m$ can be an "incomplete" partition, we omit the requirement concerning the union of the intervals $[a_i, b_i]$. Throughout this paper, the term partition will always refer to an incomplete partition.

The aim of this paper is to study a scale of non-absolutely convergent integrals which includes some integrals introduced recently. The common feature of these new integrals is that we estimate the expression $|F(y) - F(x_i) - f(x_i)(y - x_i)|$ on the partition intervals $[a_i, b_i]$ whereas their multiples denoted as $(\bar{a}_i, \bar{b}_i)$ are assumed to be pairwise disjoint.

A multidimensional modification of this idea is to estimate the expression $|F(y) - F(x_i) - f(x_i)(y - x_i)|$ on balls $B(x_i, r_i)$ and to assume that the multiples $B(x_i, \alpha r_i)$ are pairwise disjoint. This leads to the so called packing integrals in [28], [24], [15], [23], investigated in Euclidean or even metric spaces. A natural question arises what happens with these integrals if we consider them in the one-dimensional situation.

On the other hand, we want also to include a scale of one-dimensional monotonically controlled integrals studied by Ball and Preiss in [1]. The monotone control is a descriptive approach introduced in [2] which gives an alternative to Riemann-type constructive definitions.

We introduce the scales of HK_α^p integrals and centered HK_α^p integrals. They are based on partitions $\{[a_i, b_i], x_i\}_{i=1}^m$. The parameter α says that the α-multiples of the partition intervals are assumed to be pairwise disjoint. The parameter p is the Lebesgue exponent of the L^p-norm used to measure the p-oscillation of the expression $|F(y) - F(x_i) - f(x_i)(y - x_i)|$ in $[a_i, b_i]$. If the parameter p is skipped or is equal to the symbol C, it means that the supremum norm is used instead. Precise definitions are in Section 3.

All integrals considered here are investigated as indefinite integrals. Definite integrals can be introduced as increments of indefinite integrals.

We show that the HK_α integral is exactly the MC_α integral of Ball and Preiss (see [1], Theorem 4.1). Therefore the results of [1] formulated in terms of MC_α integrals can be applied to the scale of HK_α integrals as well. The centered HK_α integral is the one-dimensional α-packing BV integral from [23] (Theorem 8.13). The centered HK_α^1 integral is the α-packing Lip integral from [24] (Theorem 8.6).

Further, we show that the classes of HK_α^p integrable functions are distinct for different p (Theorem 5.5) and that the classes of centered HK_α^p integrable functions differ from uncentered ones (Theorem 5.7).

As shown in [1], the class of HK_α integrable functions contains the class of Henstock-Kurzweil integrable functions, and the inclusion is strict if $\alpha > 2$. Thus, also the classes of HK^p_α integrable functions contain the class of Henstock-Kurzweil integrable functions and the inclusion is strict if $\alpha > 2$, or $p > 1$, or the centered version is considered.

There is a huge variety of non-absolute convergent integrals which also contain the Henstock-Kurzweil integral strictly. The most famous of them is the Denjoy-Khintchine integral (see [7], [21]). Hence, it is interesting to compare the Denjoy-Khintchine integral with integrals of our scale. For the HK_α integrals it has been done in [1], we extend it to the entire scale. The result is that there is no inclusion between (centered or uncentered) HK^p_α integrable functions and Denjoy-Khintchine integrable function (with the exception of the case of uncentered HK_α for $\alpha \leqslant 2$). See Theorem 5.4.

The new non-inclusion is that the HK^p_α integral is not contained in the Denjoy-Khintchine integral. But much more is true. There is a variety of so called approximately continuous integrals with the property that the indefinite integral is approximately continuous, see e.g. [4], [22], [40], [9], [10]. Also these integrals do not contain the packing integral in view of our Theorem 7.2. It shows that there is a function f on $\mathbb{R}$ such that its indefinite HK^p_1 integral is not approximately continuous at the origin.

Most of our results concern comparison of various classes of integrable functions. To make the list of main results of the present paper complete, let us mention Theorem 6.3 which states that each (centered) HK^p_α integrable function f is at almost every point the approximate derivative of its indefinite (centered) HK^p_α integral.

The motivation to study non-absolutely convergent integrals originates from the task to integrate all derivatives and all Lebesgue integrable functions simultaneously. Similarly, the motivation for the multi-dimensional non-absolutely convergent integrals comes from the task to integrate all divergences or even “generalized divergences” and pass to an application to the divergence theorem. A brief account of the history is postponed to the last section.

2. Preliminaries

The open ball in $\mathbb{R}^n$ with the center at x and radius r is denoted by $B(x,r)$, whereas $\overline{B}(x,r)$ stands for the corresponding closed ball. If E is a set, χ_E denotes the characteristic function of E. The symbol $|E|$ means the (outer) Lebesgue measure of a set $E \subset \mathbb{R}^n$. The identity function $x \mapsto x$ on an interval I is denoted by Id. If $\Omega \subset \mathbb{R}^n$ is an open set, the symbol $\mathcal{D}(\Omega)$ stands for the set of all infinitely differentiable functions with compact support in Ω. A collection of intervals is said to be *nonoverlapping* if their interiors are pairwise disjoint.

2.1. Regulated functions. We say that $F\colon [a,b]\to\mathbb{R}$ is a *regulated function* if all one-sided limits of F exist and are finite. The space of all regulated functions equipped with the supremum norm is a Banach space. See [32] for details.

2.2. Approximate limit and derivative. We say that $x\in\mathbb{R}$ is a *density point* for a set $E\subset\mathbb{R}$ if

$$\lim_{r\to 0+}\frac{|(x-r,x+r)\setminus E|}{2r}=0.$$

Let $I\subset\mathbb{R}$ be an open interval. We say that a value $A\in\mathbb{R}$ is an *approximate limit* of a function $F\colon I\to\mathbb{R}$ at a point $x\in I$ if for each $\varepsilon>0$ there exists a set $E_\varepsilon\subset I$ such that x is a density point of E_ε and $|F-A|<\varepsilon$ on E_ε. Approximate derivative is defined as the approximate limit of difference quotients. See e.g. [41], Chapter VII.3 for details.

2.3. Denjoy-Khintchine integral. For the description of the Denjoy-Khintchine integral we use the equivalent definition according to [41], which follows the descriptive idea of Luzin (see [27]).

Definition 2.1. Let $I=(a,b)$ be an open interval. A function $F\colon I\to\mathbb{R}$ is said to be *absolutely continuous* (AC for short) on a set $E\subset I$ if for each $\varepsilon>0$ there exists $\delta>0$ such that for each finite sequence $([a_j,b_j])_{j=1}^m$ of nonoverlapping intervals with endpoints in E we have

$$\sum_{j=1}^m(b_j-a_j)<\delta\Rightarrow\sum_{j=1}^m|F(b_j)-F(a_j)|<\varepsilon.$$

We say that F is *generalized absolutely continuous* (ACG) on I if F is continuous on I and there exists a sequence $(E_k)_k$ of subsets of I such that $I=\bigcup\limits_k E_k$ and F is AC on each E_k.

Given a function $f\colon I\to\mathbb{R}$, we say that $F\colon I\to\mathbb{R}$ is an *indefinite Denjoy-Khintchine integral* of f if F is ACG on I and f is the approximate derivative of F a.e.

Remark 2.2. Every ACG function has an approximate derivative almost everywhere and therefore it acts as its indefinite Denjoy-Khintchine integral, see [41], Chapter VII, Theorem 4.3.

2.4. Oscillations.

Definition 2.3 (Oscillations). Let $[a,b] \subset \mathbb{R}$ be a closed interval and $p \in [1,\infty]$. We define the p-oscillation of a measurable function $F\colon [a,b] \to \mathbb{R}$ as

$$\operatorname{osc}_p(F,[a,b]) = (b-a)^{-1/p} \inf\{\|F-c\|_{L^p([a,b])}\colon\ c \in \mathbb{R}\}. \tag{1}$$

Here and in the sequel $1/p = 0$ if $p = \infty$.

The ordinary oscillation

$$\operatorname{osc}(F,[a,b]) = \operatorname{osc}_C(F,[a,b]) := \frac{1}{2} \sup_{x,y\in[a,b]} |F(y)-F(x)|$$

differs from $\operatorname{osc}_\infty$ in the aspect that it does not neglect Lebesgue null sets. The subscript C refers to the space of continuous functions and the somewhat unusual factor $\frac{1}{2}$ is an output of the usage of the supremum norm instead of the L^p-norm in (1). To simplify the presentation, we consider the symbol C as a possible value of p and $1/p$ is 0 for $p = C$. This convention will be used to include the choices of oscillation all at once.

Remark 2.4. Observe the elementary but useful inequality

$$[a,b] \subset [A,B] \Rightarrow (b-a)^{1/p} \operatorname{osc}_p(F,[a,b]) \leqslant (B-A)^{1/p} \operatorname{osc}_p(F,[A,B]) \tag{2}$$

which holds for a measurable function $F\colon [A,B] \to \mathbb{R}$ and $p \in [1,\infty] \cup \{C\}$.

Definition 2.5 (Median). Let F be a measurable function on an interval $[a,b]$ and $\mu \in \mathbb{R}^n$. We say that μ is a *median* of F in $[a,b]$ if there exists a measurable set $M \subset [a,b]$ such that $F \leqslant \mu$ on M, $F \geqslant \mu$ on $[a,b] \setminus M$ and $|M| = \frac{1}{2}(b-a)$. Each measurable function on $[a,b]$ has a median. On the other hand, its uniqueness is not guaranteed; it holds only under some additional assumptions like continuity of F. Medians give a useful choice of the constant c in (1). As shown in the following proposition, they yield a good estimate for all p and for $p = 1$ they are even minimizers.

Proposition 2.6. *Let μ be a median of F in $[a,b]$ and $p \in [1,\infty] \cup \{C\}$. Then*

$$\operatorname{osc}_p(F,[a,b]) \leqslant (b-a)^{-1/p}\|F-\mu\|_p \leqslant 2^{1-1/p} \operatorname{osc}_p(F,[a,b]).$$

In particular,

$$(b-a)^{-1}\|F-\mu\|_1 = \operatorname{osc}_1(F,[a,b]).$$

P r o o f. The first inequality is trivial, let us concentrate on the second one. We may assume $\mu = 0$. Consider a measurable set $M_+ \subset [a,b]$ such that $|M_+| = \frac{1}{2}(b-a)$ and $F \geqslant 0$ on M_+, $F \leqslant 0$ on $M_- := [a,b] \setminus M_+$. Choose $c \in \mathbb{R}$, e.g. $c \geqslant 0$. We have

$$\begin{aligned} |F(x)|^p &\leqslant 2^{p-1}(|F(x)-c|^p + c^p), & x \in M_+,\\ |F(x)|^p &\leqslant |c + |F(x)||^p - c^p \leqslant 2^{p-1}(|F(x)-c|^p - c^p), & x \in M_-. \end{aligned}$$

Integrating over $[a,b]$ we obtain

$$\int_{[a,b]} |F(x)|^p \,\mathrm{d}x \leqslant 2^{p-1} \int_{[a,b]} |F(x)-c|^p \,\mathrm{d}x,$$

as $|M_+| = |M_-|$. Taking the infimum over c we obtain

$$\|F\|_p^p \leqslant 2^{p-1}(b-a)(\operatorname{osc}_p F)^p$$

as required. □

3. The definition of integral

Definition 3.1. Let $\alpha \geqslant 1$, $p \in [1,\infty] \cup \{C\}$ and $I \subset \mathbb{R}$ be an open interval. A finite family $([a_i,b_i],x_i)_{i=1}^m$, where $[a_i,b_i] \subset I$ are closed intervals and $x_i \in [a_i,b_i]$, is called an *α-partition* in I if the intervals $(\bar{a}_i, \bar{b}_i)$, where

$$\bar{a}_i - x_i = \alpha(a_i - x_i), \quad \bar{b}_i - x_i = \alpha(b_i - x_i), \tag{3}$$

are subsets of I and pairwise disjoint. We say that a partition $([a_i,b_i],x_i)_{i=1}^m$ is *centered* if each x_i is the center of $[a_i,b_i]$. Let $\delta\colon I \to (0,\infty)$ be a *gage* (this means just a strictly positive function). We say that the α-partition is *δ-fine* if $[a_i,b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$, $i = 1,\dots,m$.

Definition 3.2 (HKS_α^p integrals)**.** Let $I \subset \mathbb{R}$ be an open interval, $p \in [1,\infty] \cup \{C\}$ and $\alpha \geqslant 1$. Let F, G, f be measurable functions on I. We say that F is an *indefinite* HKS_α^p *integral* (HKS refers to Henstock-Kurzweil-Stieltjes) of f with respect to G if for each $\varepsilon > 0$ there exists a gage $\delta\colon I \to (0,\infty)$ such that for each δ-fine α-partition $([a_i,b_i],x_i)_{i=1}^m$ in I we have

$$\sum_{i=1}^m \operatorname{osc}_p(F - f(x_i)G, [a_i,b_i]) < \varepsilon.$$

We denote $\mathrm{HKS}_\alpha = \mathrm{HKS}_\alpha^C$. We reduce the symbol to HK_α^p or HK_α, respectively, if G is the identity $\mathrm{Id}(x) = x$. We call the integral *centered* if the only centered α-partitions are taken into account. We say that F is a *free indefinite* HKS^p *integral* of f with respect to G if there exists $\alpha \geqslant 1$ such that F is an *indefinite* HKS_α^p *integral* of f with respect to G, similarly to the centered versions.

Remark 3.3. There are obvious inclusions between the classes of integrable functions. The class of HKS_α^p integrable functions increases with α and the class of all free HKS^p integrable functions is the union of the preceeding ones over α.

The centered version always leads to a wider class of integrable functions.

Using comparison of L^p norms, we also observe that the class of HKS_α^p integrable functions decreases with p.

The indefinite integrals to a function f are the same for all choices of α, p which make f integrable (with the exception that for $p = C$ only continuous representatives are valid).

Remark 3.4. Even if we do not assume that f is measurable in Definition 3.2, the measurability of f comes out as a consequence of HKS_α^p integrability (see [28], Theorem 5.3).

When defining a new notion of indefinite integral, it is desirable to show that this has the expected uniqueness behavior, namely that the indefinite integrals to the same functions differ only by an additive constant.

Theorem 3.5. *Let f, F_1, F_2, G be measurable functions on an open interval I. If F_1 and F_2 are indefinite (centered)* HKS_α^p *integrals of f with respect to G, then there is a constant $C \in \mathbb{R}$ such that $F_2 - F_1 = C$ a.e.*

Proof. By Remark 3.3, it is possible to reduce the question to the uniqueness of centered HKS_1^α. This follows from [24], Theorem 3.10 (see Theorem 8.6). $\square$

Theorem 3.6. *Let f, F_1, F_2, G be measurable functions on an open interval I, F_1, F_2, G be regulated. If F_1 and F_2 are indefinite (centered)* HKS_α *integrals of f with respect to G, then there is a constant $C \in \mathbb{R}$ such that $F_2 - F_1 = C$.*

Proof. Obviously $F_2 - F_1$ is an indefinite (centered) HK_α integral of 0 and hence by Theorem 3.5, $F_2 - F_1 = C$ a.e. Since $F_2 - F_1$ is regulated, the equality turns to hold everywhere. $\square$

Proposition 3.7. *Let $I \subset \mathbb{R}$ be an open interval and $F, f, G\colon I \to \mathbb{R}$ be measurable functions. Then F is an indefinite* HKS_1 *integral of f if and only if F is an indefinite Henstock-Kurzweil-Stieltjes integral of f.*

P r o o f. Suppose that F is an indefinite Henstock-Kurzweil-Stieltjes integral of f. By the Saks-Henstock lemma (see [32], Lemma 6.5.1), for each $\varepsilon > 0$ there exists a gage $\delta\colon I \to (0,\infty)$ such that for each δ-fine partition $\{([a_i,b_i],x_i)\}_{i=1}^m$ we have

$$\sum_{i=1}^m |F(b_i) - F(a_i) - f(x_i)(G(b_i) - G(a_i))| < \varepsilon.$$

Consider a δ-fine partition $\{([A_i,B_i],x_i)\}_{i=1}^m$. For each j we find $z_i \in [A_i,B_i]$ such that $z_i \neq x_i$ and

$$|F(z_i) - F(x_i) - f(x_i)(G(z_i) - G(x_i))| \geqslant \frac{1}{2}\operatorname{osc}(F - f(x_i)G, [A_i,B_i]).$$

Set

$$[a_i,b_i] = \begin{cases} [z_i,x_i], & z_i < x_i, \\ [x_i,z_i], & x_i < z_i. \end{cases}$$

Then $\{([a_i,b_i],x_i)\}_{i=1}^m$ is a δ-fine partition and thus

$$\sum_{i=1}^m \operatorname{osc}(F - f(x_i)G, [A_i,B_i]) \leqslant 2\sum_{i=1}^m |F(b_i) - F(a_i) - f(x_i)(G(b_i) - G(a_i))| < 2\varepsilon.$$

It follows that F is an indefinite HKS_1 integral of f. The converse implication is obvious. □

4. Monotone control

Let $I \subset \mathbb{R}$ be an interval and $p \in [1,\infty] \cup \{C\}$ be fixed. Let f, F, G be measurable functions on I. We say that an increasing function $\varphi\colon I \to \mathbb{R}$ is an *α-control function* for the triple (f,F,G) if for each $x \in I$ we have

$$\lim_{r\to 0+} \frac{\operatorname{osc}_p(F - f(x)G, [x,x+r])}{\varphi(x+\alpha r) - \varphi(x)} = \lim_{r\to 0+} \frac{\operatorname{osc}_p(F - f(x)G, [x-r,x])}{\varphi(x) - \varphi(x-\alpha r)} = 0.$$

We say that an increasing function $\varphi\colon I \to \mathbb{R}$ is a *centered α-control function* for the triple (f,F,G) and HK_α integration if for each $x \in I$ we have

$$\lim_{r\to 0+} \frac{\operatorname{osc}_p(F - f(x)G, [x-r,x+r])}{\varphi(x+\alpha r) - \varphi(x-\alpha r)} = 0.$$

Following Ball and Preiss in [1], we say that F is an indefinite MC_α integral of f if there exists an α-control function for (f,F,Id) and the choice $p = C$. In particular, the MC_1 integral is the MC integral of [2].

Theorem 4.1. *Let $I \subset \mathbb{R}$ be an interval and $p \in [1,\infty] \cup \{C\}$ be fixed. Let f, F, G be measurable functions on I. Then F is an indefinite HKS_α^p integral of f with respect to G if and only if there exists an α-control function for the triple (f,F,G).*

In particular, F is an indefinite HK_α integral of f if and only if F is an indefinite MC_α integral of f.

Proof. Suppose that the α-control function φ exists. We may assume that $|\varphi|$ is bounded by $\frac{1}{2}$ (otherwise φ can be replaced by $(\arctan\varphi)/\pi$). Given $\varepsilon > 0$, for each $x \in I$ we can find $\delta(x) > 0$ such that

$$x \in [a,b] \subset (x-\delta(x), x+\delta(x)) \Rightarrow \operatorname{osc}_p(F - f(x)G, [a,b]) < \varepsilon(\varphi(\bar{b}) - \varphi(\bar{a})),$$

where

$$\bar{a} = x + \alpha(a-x), \quad \bar{b} = x + \alpha(b-x).$$

Then δ is the desired gage. Indeed, if $([a_i,b_i],x_i)_{i=1}^m$ is a δ-fine α-partition, then

$$\sum_{i=1}^m \operatorname{osc}_p(F - f(x_i)G, [a_i,b_i]) \leqslant \varepsilon \sum_{i=1}^m (\varphi(\bar{b}_i) - \varphi(\bar{a}_i)) \leqslant \varepsilon \operatorname{osc}(\varphi, I) \leqslant \varepsilon.$$

For the reverse implication we introduce the following variation depending on an open interval $J \subset I$ and a gage δ:

$$V(J,\delta) = V(J,\delta,f,F,G) = \sup\Bigg\{ \sum_{i=1}^m \operatorname{osc}_p(F - f(x_i)G, [a_i,b_i]) \colon \\ ([a_i,b_i],x_i)_{i=1}^m \text{ is a } \delta\text{-fine } \alpha\text{-partition in } J \Bigg\}.$$

For each $k = 1,2,\dots$ we find a gage $\delta_k \colon I \to (0,\infty)$ such that

$$V(I,\delta_k) < 2^{-k}$$

and set

$$\varphi_k(x) = V(I \cap (-\infty,x), \delta_k), \quad \varphi(x) = x + \sum_{k=1}^\infty k\varphi_k(x).$$

Then $\varphi \colon I \to \mathbb{R}$ is a strictly increasing function. We want to show that φ is an α-control function to the triple (f,F,G). Fix $x \in I$ and choose $\varepsilon > 0$. Find $k \in \mathbb{N}$ such that $1/k < \varepsilon$. If $x \in [a,b] \subset (x-\delta(x), x+\delta(x))$, then for each α-partition $([a_i,b_i],x_i)_{i=1}^m$ in $I \cap (-\infty,\bar{a})$ we observe that $([a_i,b_i],x_i)_{i=1}^{m+1}$ is an α-partition in $I \cap (-\infty,\bar{b})$, where we set $([a_{m+1},b_{m+1}],x_{m+1}) = ([a,b],x)$. Hence

$$\varphi_k(\bar{a}) + \operatorname{osc}_p(F - f(x)G, [a,b]) \leqslant \varphi_k(\bar{b})$$

and thus

$$\frac{\operatorname{osc}_p(F - f(x)G, [a,b])}{\varphi(\bar b) - \varphi(\bar a)} \leqslant \frac{\varphi_k(\bar b) - \varphi_k(\bar a)}{\varphi(\bar b) - \varphi(\bar a)} \leqslant \frac{1}{k} < \varepsilon$$

as required. □

Theorem 4.2. *Let $I \subset \mathbb{R}$ be an interval $p \in [1,\infty] \cup \{C\}$. Let f, F, G be measurable functions on I. Then F is an indefinite centered* HKS^p_α *integral of f with respect to G if and only if there exists a centered α-control function for the triple f, F, G.*

P r o o f. The proof is almost the same as that of Theorem 4.1 with obvious modifications. □

5. Counterexamples

Definition 5.1. We denote by $\{0,1\}^k$ the family of all multiindices $s = (s_1,\dots,s_k)$, where $s_1,\dots,s_k \in \{0,1\}$. The set $\{0,1\}^0$ contains just one element denoted by o. We simplify the symbols $(0),(1) \in \{0,1\}^1$ to 0, 1. We denote

$$\mathbb{S} = \bigcup_{k=0}^{\infty} \{0,1\}^k.$$

If $s = (s_1,\dots,s_m) \in \{0,1\}^m$ and $t = (t_1,\dots,t_n) \in \{0,1\}^n$, we define the *concatenation* of s and t as

$$s \frown t = (s_1,\dots,s_m,t_1,\dots,t_n) \in \{0,1\}^{m+n}.$$

In particular, if $s \in \{0,1\}^k$, then $s \frown 0 = (s_1,\dots,s_k,0)$ and $s \frown 1 = (s_1,\dots,s_k,1)$. The *length* of $s \in \{0,1\}^k$ is $|s| := k$.

We define the relations $s \prec t$ and $s \succ t$: We write $s \prec t$ if there exists $u \in \mathbb{S}$ such that $t = s \frown u$; the symbol $s \succ t$ means $t \prec s$.

E x a m p l e 5.2. Set

$$\varrho = \frac{1}{2 + 4\alpha}. \tag{4}$$

We construct a Cantor type set in $[0,1]$. Let $P_o = [U_o, V_o] = [0,1]$. Let $s \in \{0,1\}^k$ and $P_s = [U_s, V_s]$ be an interval of the kth generation of length ϱ^k. We consider the concentric interval $Q_s = (u_s, v_s)$ of length $\varrho^k(1 - 2\varrho)$. Also, consider the intervals

$$P^*_s = [u_s, u_s + 2\varrho^{k+1}], \quad Q^*_s = (u_s + \varrho^{k+1}, u_s + 2\varrho^{k+1}).$$

The annulus $P_s \setminus Q_s$ splits into two intervals of $(k+1)$st generation of length ϱ^{k+1}, namely

$$P_{s\frown 0} = [U_{s\frown 0}, V_{s\frown 0}] := [U_s, u_s], \quad P_{s\frown 1} = [U_{s\frown 1}, V_{s\frown 1}] := [v_s, V_s].$$

Let $\eta\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function with support in $(0,1)$ such that

$$\sup_{x\in(0,1)} \eta(x) = 2 \quad \text{and} \quad \int_0^1 \eta(y)\,\mathrm{d}y = 1.$$

Consider sequences $(\lambda_k)_{k=0}^\infty$, $(\sigma_k)_{k=0}^\infty$ of positive real numbers such that $0 < \sigma_k \leqslant 1$, $k = 0, 1, \dots$ For each $k = 0, 1, \dots$ and $s \in \{0,1\}^k$ we set

$$\begin{aligned} &F_s\bigl(u_s + (1+\sigma_k x)\varrho^{k+1}\bigr) = \lambda_k \eta(x), && x \in \mathbb{R}, \\ &\beta_{k,p} = \operatorname{osc}_p(F_s, P_s^*), && p \in [1,\infty] \cup \{C\}. \end{aligned}$$

Then F_s is supported in Q_s^* and

$$\beta_{k,C} = \lambda_k, \quad \beta_{k,p} \approx \sigma_k^{1/p}\lambda_k,\ 1 \leqslant p \leqslant \infty. \tag{5}$$

We define the sets

$$K_k = \bigcup_{s\in\{0,1\}^k} P_s, \quad K = \bigcap_{k=1}^\infty K_k \quad \text{and} \quad F = \sum_{s\in\mathbb{S}} F_s.$$

We observe that K is a Cantor type set of measure 0. Further, F is smooth outside K as the intervals Q_s^* are pairwise disjoint and the support of each F_s is in Q_s^*. Finally, we set

$$f(x) = \begin{cases} F'(x), & x \in \mathbb{R} \setminus K, \\ 0, & x \in K. \end{cases}$$

Theorem 5.3. *Let $p \in [1,\infty] \cup \{C\}$.*

(a) *If*

$$\sum_{j=1}^\infty \beta_{j,p} = \infty,$$

then f does not have an indefinite centered HK_α^p integral.

(b) *If*

$$\sum_{j=1}^\infty 2^j \beta_{j,p} = \infty, \tag{6}$$

then F does not have an indefinite HK_α^p integral.

P r o o f. We use the Baire category theorem similarly to the usage for counterexamples in [1]. Consider a gage $\delta\colon \mathbb{R} \to (0,\infty)$ and denote

$$E_n = \Big\{x \in K\colon\ \delta(x) > \frac{1}{n}\Big\}.$$

Then, by the Baire category theorem, there exist $n \in \mathbb{N}$ and an open set $\Omega \subset \mathbb{R}$ such that $\Omega \cap K$ is nonempty and $\Omega \cap E_n$ is dense in $\Omega \cap K$. We find $k \in \mathbb{N}$ and a multiindex $t \in \{0,1\}^k$ such that $P_t \subset \Omega$ and $V_t - U_t = \varrho^k \leqslant 1/n$. We denote

$$[t,0] = t, \quad [t,1] = t \frown 1, \quad [t,2] = t \frown (1,1), \dots$$

Now, we distinguish the cases (a), (b).

(a) Assume that $\widetilde{F}$ is an indefinite centered HK^p_α integral of f. Let δ be chosen so that for each δ-fine centered α-partition $\{([a_i,b_i),x_i\}_{i=1}^m$ with $x_i \in K$ we have

$$\sum_{i=1}^m \operatorname{osc}_p(\widetilde{F},[a,b]) < 1. \tag{7}$$

The existence of such a gage is clear from the definition of the integral as $f = 0$ on K. For each $j = 0,1,2,\dots$ we find $x_j \in [U_{[t,j]}, u_{[t,j]}] \cap K$ such that $\delta(x_j) > 1/n$ and set

$$[a_j,b_j] = [x_j - 3\varrho^{k+j+1}, x_j + 3\varrho^{k+j+1}].$$

Let $\bar{a}_j, \bar{b}_j$ be as in (3). Since by (4)

$$\begin{aligned}\bar{b}_j &= x_j + 3\alpha\varrho^{k+j+1} \leqslant u_{[t,j]} + 3\alpha\varrho^{k+j+1} \leqslant v_{[t,j]} - \alpha\varrho^{k+j+1}\\ &= U_{[t,j+1]} - \alpha\varrho^{k+j+1} \leqslant x_{j+1} - 3\alpha\varrho^{k+j+2} = \bar{a}_{j+1},\end{aligned}$$

the intervals $(\bar{a}_j,\bar{b}_j)$, $j = 1,2\dots$, are pairwise disjoint and contained in P_t. Thus, $\{([a_j,b_j],x_j)\}_{j=1}^m$ is a δ-fine α-partition for each $m \in \mathbb{N}$. We observe that $[a_j,b_j] \supset P^*_{[t,j]}$. Since F is smooth in $Q_{[t,j]}$, both F and $\widetilde{F}$ are indefinite centered HK^p_α integrals of f in $Q_{[t,j]}$ and thus by uniqueness, $\widetilde{F} = F + C_j$ on $Q_{[t,j]}$ for some constant C_j. It follows that

$$\operatorname{osc}_p(\widetilde{F}, P^*_{[t,j]}) = \operatorname{osc}_p(F, P^*_{[t,j]}) = \beta_{k+j,p}. \tag{8}$$

Since $b_j - a_j = 6\varrho^{k+j+1}$ and the length of $P^*_{[t,j]}$ is $2\varrho^{k+j+1}$, by (2) we have

$$\operatorname{osc}_p(\widetilde{F},[a_j,b_j]) \geqslant \frac{1}{3^{1/p}} \operatorname{osc}_p(\widetilde{F}, P^*_{[t,j]}) = \frac{1}{3^{1/p}}\beta_{k+j,p}. \tag{9}$$

Since the sum $\sum\limits_j \beta_{j,p}$ diverges, we obtain a contradiction with (7).

(b) Assume that $\widetilde{F}$ is an indefinite $\mathrm{HK}_{\alpha,p}$ integral of f. Let δ be chosen so that for each δ-fine (uncentered) α-partition $\{([a_i,b_i),x_i)\}_{i=1}^m$ with $x_i\in K$ we have

$$\sum_{i=1}^{m} \operatorname{osc}_p(\widetilde{F},[a_j,b_j]) < 1. \tag{10}$$

As in (8) we obtain that

$$\operatorname{osc}_p(\widetilde{F},P_s^*) = \beta_{k+j,p}, \quad s\in\{0,1\}^{k+j}.$$

Hence

$$\sum_{j=1}^{\infty} \sum_{s\in\{0,1\}^{k+j}, s\succ t} \operatorname{osc}_p(\widetilde{F},P_s^*) = \sum_{j=1}^{\infty} 2^j\beta_{k+j,p} = \infty.$$

Find $m\in\mathbb{N}$ such that

$$\sum_{s\in S} \operatorname{osc}_p(\widetilde{F},P_s^*) > 2, \tag{11}$$

where

$$S := \bigcup_{j=1}^{m} \{s\in\{0,1\}^{k+j}, s\succ t\}.$$

Since the intervals $[u_s,v_s]$ are pairwise disjoint and S is finite, we can find $x_s\in[U_s,u_s]\cap K$ such that $\delta(x_s)>1/n$ and the intervals $[x_s,v_s]$ are still pairwise disjoint when s runs through S. Set

$$a_s = x_s, \quad b_s = u_s + 2\varrho^{|s|+1}.$$

As in (3), write

$$\bar{a}_s = x_s + \alpha(a_s - x_s) = a_s, \quad \bar{b}_s = x_s + \alpha(b_s - x_s) = a_s + \alpha(b_s - a_s).$$

Since

$$\bar{b}_s = a_s + \alpha(b_s - a_s) \leqslant u_s + \alpha(u_s + 2\alpha\varrho^{|s|+1} - U_s) = u_s + 3\alpha\varrho^{|s|+1} < v_s,$$

we have $[\bar{a}_s,\bar{b}_s]\subset[x_s,v_s]$, and thus the intervals $(\bar{a}_s,\bar{b}_s)$, $s\in S$, are pairwise disjoint and contained in P_t. It follows that $\{([a_s,b_s],x_s)\}_{s\in S}$ is a δ-fine α-partition in P_t. Since $[a_s,b_s]$ contains P_s^* for each $s\in S$, by (11) and (2) we obtain

$$\sum_{i=1}^{m} \operatorname{osc}_p(\widetilde{F},[a_i,b_i]) > 1,$$

which contradicts (10). □

5.1. Denjoy-Khintchine integrable function which is not HK_α^1 integrable. In [1], it is shown that for $\alpha > 2$ there exists a HK_α integrable function which is not Denjoy-Khintchine integrable. By Remark 3.3, such a function is also HK_α^p integrable. Also discontinuous HK_α^p integrable functions serve as examples of HK_α^p integrable functions which are not Denjoy-Khintchine integrable, see Example 7.1. We show that the converse inclusion also fails. We prove this for the widest class of our scale.

Theorem 5.4. *For each $\alpha \geqslant 1$ there exists a Denjoy-Khintchine integrable function which is not centered HK_α^1 integrable.*

Proof. Let F, f be as in Example 5.2. Set $\lambda_j = 1/j$ and $\sigma_j = 1$, so by (5) $\beta_{j,\infty} \to 0$ and $\sum_j \beta_{j,1}$ diverges. By Theorem 5.3, f does not have an indefinite centered HK_α^1 integral.

We show that F is the indefinite Denjoy-Khintchine integral of f. The function F is smooth, thus AC and the derivative of F is f on each Q_s. Further, $F = 0$ is AC on K and $|K| = 0$. It follows that F is ACG and a.e. differentiable in $\mathbb{R}$ and $f = F'$ a.e. Hence, F is an indefinite Denjoy-Khintchine integral of f on $\mathbb{R}$. □

5.2. Comparison of HK_α^p integral and HK_α^q integral.

Theorem 5.5. *For each $\alpha \geqslant 1$ and $1 \leqslant p < q \leqslant \infty$ there exists a HK_1^p integrable function which is not centered HK_α^q integrable.*

Proof. Let F, f be as in Example 5.2. Set

$$\lambda_j = 3^{jp/(q-p)}, \quad \sigma_j = 3^{-jpq/(q-p)},$$

so by (5) we have $\sum_j \beta_{j,q} = \infty$ and $\sum_j 2^j \beta_{j,p} < \infty$. Then by Theorem 5.3, f does not have an indefinite centered HK_α^q integral.

We will show that F is an indefinite HK_1^p integral of f. Choose $\varepsilon > 0$. Since $F' = f$ in $\mathbb{R} \setminus K$, for each $x \in \mathbb{R} \setminus K$ we can find $\delta(x) > 0$ such that

$$|y - x| < \delta(x) \Rightarrow |F(y) - F(x) - f(x)(y - x)| < \varepsilon|\arctan y - \arctan x|. \tag{12}$$

Find $k \in \mathbb{N}$ such that

$$\sum_{j>k} 2^j \beta_{j,p} < \varepsilon.$$

If $x \in K$, we can find $\delta(x) > 0$ such that the interval $(x - \delta(x), x + \delta(x))$ does not intersect any of the intervals Q_s^* with $|s| \leqslant k$. This defines a gage $\delta \colon \mathbb{R} \to (0, \infty)$.

Let $\{([a_i,b_i],x_i)\}_{i=1}^m$ be a δ-fine partition. Without loss of generality we may assume that $x_1,\dots,x_n \in K$ and $x_{n+1},\dots,x_m \notin K$. Set

$$S_i = \{s \in \mathbb{S} \colon\ [a_i,b_i] \cap Q_s^* \neq \emptyset\},\ i = 1,\dots,n, \quad S = \bigcup_{i=1}^m S_i.$$

Then $|s| > k$ for each $s \in S$. Fix $i \in \{1,\dots,n\}$. If $[a_i,b_i] \cap Q_s^* \neq \emptyset$, then $[a_i,b_i]$ contains either $[u_s, u_s + \varrho^{|s|+1}]$, or $[u_s + 2\varrho^{|s|+1}, v_s] \supset [v_s - \varrho^{|s|+1}, v_s]$. (The last inclusion follows from (4).) It follows that the length of $[a_i,b_i]$ is at least $\varrho^{|s|+1}$. Observe that 0 is a median of F in P_s^* for each s. In view of Proposition 2.6, we have

$$\int_{P_s^*} |F|^p \leqslant 2^{p-1} \operatorname{osc}_p(F, P_s^*)^p = 2^{p-1} \beta_{|s|,p}^p |P_s^*| = 2^p \varrho^{|s|+1} \beta_{|s|,p}^p.$$

Thus

$$\int_{a_i}^{b_i} |F(y)|^p \,\mathrm{d}y \leqslant \sum_{s\in S_i} 2^p \varrho^{|s|+1} \beta_{|s|,p}^p \leqslant 2^p (b_i - a_i) \sum_{s\in S_i} \beta_{|s|,p}^p \leqslant 2^p (b_i - a_i) \Big(\sum_{s \in S_i} \beta_{|s|,p}\Big)^p,$$

so

$$\operatorname{osc}_p(F, [a_i,b_i]) \leqslant 2 \sum_{s\in S_i} \beta_{|s|,p}.$$

Since each Q_s^* intersects at most two $[a_i,b_i]$, summing over $i = 1,\dots,n$ we obtain

$$\begin{aligned} \text{(13)} \qquad \sum_{i=1}^n \operatorname{osc}_p(F - f(x_i)\operatorname{Id}, [a_i,b_i]) &= \sum_{i=1}^n \operatorname{osc}_p(F, [a_i,b_i]) \\ &\leqslant 4 \sum_{s\in\mathbb{S}, |s|>k} \beta_{|s|,p} \leqslant 4 \sum_{j>k} 2^j \beta_{|s|,p} < 4\varepsilon. \end{aligned}$$

From (12) we obtain

$$\begin{aligned} \text{(14)} \qquad & \sum_{i=n+1}^m \operatorname{osc}_p(F - f(x_i)\operatorname{Id}, [a_i,b_i]) \\ &\leqslant \sum_{i=n+1}^m (b_i - a_i)^{-1/p} \Big(\int_{a_i}^{b_i} |F(y) - F(x) - f(x)(y-x)|^p \,\mathrm{d}y\Big)^{1/p} \\ &\leqslant \varepsilon \sum_{i=n+1}^m (\arctan b_i - \arctan a_i) \leqslant \pi\varepsilon. \end{aligned}$$

From (13) and (14) we conclude that F is an indefinite HK_1^p integral of f. $\square$

5.3. Centered HK_1 integrable function which is not HK^p_α integrable.

Lemma 5.6. *Let $S \subset \mathbb{S}$ be a finite set. For each $s \in S$ denote $T^0_s = \{t \in \mathbb{S}\colon s \frown 0 \frown t \in S\}$ and $T^1_s = \{t \in \mathbb{S}\colon s \frown 1 \frown t \in S\}$. Assume the following property:*

$$\forall\, s \in S \ \text{ either } \ T^0_s = \emptyset \ \text{ or } \ T^1_s = \emptyset. \tag{15}$$

Then

$$\sum_{s\in S} 2^{-|s|} < 2.$$

P r o o f. Denote by $\#S$ the number of elements of S. We prove by induction on $\#S$. The statement is true if S consists of one multiindex. Assume that the statement is true when $\#S \leqslant n$ and consider S with $\#S = n+1$. Consider $k = \min\{|s|\colon\ s \in S\}$ and $S_k = \{u \in S\colon\ |u| = k\}$. For each $u \in S_k$, T^0_u and T^1_u satisfy the property in consideration and $\#T^i_u \leqslant n$, $i = 0, 1$. Therefore

$$\sum_{t\in T^0_u} 2^{-|u\frown 0\frown t|} = 2^{-k-1} \sum_{t\in T^0_u} 2^{-|t|} < 2^{-k}$$

and similarly

$$\sum_{t\in T^1_u} 2^{-|u\frown 1\frown t|} < 2^{-k}.$$

Since at most one of the sets T^0_u, T^1_u is nonempty, we have

$$\sum_{s\in S\colon\ s\succ u} 2^{-|s|} < 2^{-|u|} + 2^{-k} = 2^{-k+1}, \quad u \in S_k.$$

Finally, as $\#S_k \leqslant 2^k$, we have

$$\sum_{s\in S} 2^{-|s|} < 2.$$

□

Theorem 5.7. *Let $p \in [1,\infty] \cup \{C\}$. Then for each $\alpha \geqslant 1$ there exists a centered HK_1 integrable function which is not HK^p_α integrable.*

P r o o f. Let F, f be as in Example 5.2. Set

$$\lambda_j = \frac{1}{j} 2^{-j}, \quad \sigma_j = 1,$$

so by (5) $\sum_j 2^j \beta_{j,p}$ diverges. Then by Theorem 5.3, f does not have an indefinite HK^p_α integral.

We will show that F is an indefinite centered HK_1 integral of f. Choose $\varepsilon > 0$. Since $F' = f$ in $\mathbb{R} \setminus K$, for each $x \in \mathbb{R} \setminus K$ we can find $\delta(x) > 0$ such that

$$|y - x| < \delta(x) \Rightarrow |F(y) - F(x) - f(x)(y - x)| < \varepsilon|\arctan y - \arctan x|. \tag{16}$$

Find $k \in \mathbb{N}$ such that $2/k < \varepsilon$. If $x \in K$, we can find $\delta(x) > 0$ such that the interval $(x - \delta(x), x + \delta(x))$ does not intersect any of the intervals Q_s^* with $|s| > k$. This defines a gage $\delta\colon \mathbb{R} \to (0, \infty)$. Let $\{([a_i, b_i], x_i)\}_{i=1}^m$ be a δ-fine partition. Without loss of generality we may assume that $x_1, \ldots, x_n \in K$ and $x_{n+1}, \ldots, x_m \notin K$. For each $i = 1, \ldots, n$ find $s_i \in \mathbb{S}$ such that

$$[a_i, b_i] \cap Q_{s_i}^* \neq \emptyset, \qquad \sup_{[a_i, b_i]} F = \sup_{[a_i, b_i] \cap Q_{s_i}^*} F.$$

Set

$$S = \{s_1, \ldots, s_n\}.$$

Assume that $s = s_i \in S$. If $x_i \leqslant u_s$, then $[x_i, b_i]$ contains $[u_s, u_s + \varrho^{|s|+1}]$, hence $x_i - a_i = b_i - x_i \geqslant \varrho^{|s|+1}$ and

$$a_i = x_i - (x_i - a_i) \leqslant u_s - \varrho^{|s|+1} = U_s = U_{s\frown 0}, \quad b_i \geqslant u_s = V_{s\frown 0}.$$

Therefore none of the intervals $[a_j, b_j]$, $j \neq i$, intersects $P_{s\frown 0}$. Similarly, if $x_i \geqslant v_s$, then none of the intervals $[a_j, b_j]$, $j \neq i$, intersects $P_{s\frown 1}$. It follows that S satisfies (15). We estimate

$$\sum_{i=1}^n \operatorname{osc}(F, [a_i, b_i]) \leqslant \sum_{i=1}^n \beta_{|s_i|, C} = \sum_{s \in S} \frac{1}{|s|} 2^{-|s|}.$$

Since $|s| > k$ for each $s \in S$, using Lemma 5.6 we can continue:

$$\sum_{i=1}^n \operatorname{osc}(F - f(x_i)\,\mathrm{Id}, [a_i, b_i]) = \sum_{i=1}^n \operatorname{osc}(F, [a_i, b_i]) \leqslant \frac{1}{k} \sum_{s \in S} 2^{-|s|} \leqslant \frac{2}{k} < \varepsilon. \tag{17}$$

From (16), as in (14) we obtain

$$\sum_{i=n+1}^m \operatorname{osc}(F - f(x_i)\,\mathrm{Id}, [a_i, b_i]) \leqslant \sum_{i=n+1}^m (\arctan b_i - \arctan a_i) \leqslant \pi\varepsilon. \tag{18}$$

From (17) and (18) we conclude that F is an indefinite centered HK_1 integral of f. □

R e m a r k 5.8. In all these constructions, the resulting function has the required non-integrability property with a fixed α. The construction can be easily modified to obtain the corresponding free non-integrability. It is enough to propose a function f which fails the $\alpha = n$ integrability property on $[1/(n+1), 1/n]$, $n = 1, 2, \ldots$, and multiply the function f on each $[1/(n+1), 1/n]$ by an appropriate constant c_n to keep control over the behavior at 0.

6. Differentiability and approximate differentiability

Lemma 6.1. *Let I be an open interval. Let $\alpha \geqslant 1$ and $p \in [1, \infty] \cup \{C\}$. Let F be an indefinite centered HK^p_α integral of f on I. Then*

$$\lim_{r \to 0_+} \frac{\operatorname{osc}_p(F - f(x)\,\mathrm{Id}, [x-r, x+r])}{r} = 0 \tag{19}$$

for a.e. $x \in I$.

P r o o f. From Theorem 4.2 we infer that there is a centered α-control function φ for the triple (f, F, Id) and centered HK^p_α integration. Since φ is monotone, it is a.e. differentiable. If x is a point where φ is differentiable, it is evident that (19) holds at x. □

Lemma 6.2. *Let I be an open interval and $F\colon I \to \mathbb{R}$ be a measurable function. Let $p \in [1, \infty]$. Let $r_0 > 0$ be such that $(x - r_0, x + r_0) \subset I$ and for each $r \in (0, r_0)$ let $\mu(r)$ be a median of F in $(x - r, x + r)$. Suppose that*

$$\lim_{r \to 0_+} \frac{\operatorname{osc}_p(F, [x-r, x+r])}{r} = 0. \tag{20}$$

Then there exists a limit

$$l = \lim_{r \to 0_+} \mu(r)$$

and

$$\lim_{r \to 0_+} \frac{\mu(r) - l}{r} = 0. \tag{21}$$

If, in addition, $F(x) = l$ (in particular, if F is approximately continuous at x), then 0 is the approximate derivative of F at x.

P r o o f. It is enough to consider the case $p = 1$. Pick $s, r \in (0, r_0)$ such that $s < r \leqslant 2s$. We claim that

$$|\mu(r) - \mu(s)| \leqslant 8\,\mathrm{osc}_1(F, [x - r, x + r]). \tag{22}$$

Assume that $\mu(r) \geqslant \mu(s)$. Find measurable sets E_s, E_r such that $E_r \subset (x - r, x + r)$, $E_s \subset (x - s, x + s)$, $F \leqslant \mu(s)$ on E_s, $F \geqslant \mu(r)$ on E_r, $|E_s| = s$ and $|E_r| = r$. Let $c \in \mathbb{R}$. If $c \leqslant \frac{1}{2}(\mu(s) + \mu(r))$, then

$$\int_{x-r}^{x+r} |F(y) - c|\,\mathrm{d}y \geqslant \int_{E_r} (\mu(r) - c) \geqslant \frac{r}{2}(\mu(r) - \mu(s)).$$

If $c \geqslant \frac{1}{2}(\mu(s) + \mu(r))$, then

$$\int_{x-r}^{x+r} |F(y) - c|\,\mathrm{d}y \geqslant \int_{E_s} (c - \mu(s)) \geqslant \frac{s}{2}(\mu(r) - \mu(s)) \geqslant \frac{r}{4}(\mu(r) - \mu(s)),$$

as we have assumed $r \leqslant 2s$. In both cases

$$\mu(r) - \mu(s) \leqslant \frac{4}{r} \int_{x-r}^{x+r} |F(y) - c|\,\mathrm{d}y \leqslant 8\,\mathrm{osc}_1(F, [x - r, x + r]). \tag{23}$$

The case $\mu(r) < \mu(s)$ is similar, so (23) is verified. Choose $\varepsilon > 0$ and find $\delta \in (0, r_0)$ such that

$$0 < r < \delta \Rightarrow 8\,\mathrm{osc}_1(F, [x - r, x + r]) \leqslant \varepsilon r.$$

Find $k \in \mathbb{N}$ such that $2^{-k} < \delta$. Then by (23)

$$\sum_{j=k+1}^{\infty} |\mu(2^{-j-1}) - \mu(2^{-j})| \leqslant \sum_{j=k+1}^{\infty} 2^{-j}\varepsilon = 2^{-k}\varepsilon, \tag{24}$$

similarly

$$2^{-k-1} \leqslant r \leqslant 2^{-k} \Rightarrow |\mu(r) - \mu(2^{-k-1})| \leqslant r\varepsilon. \tag{25}$$

We see that the sum

$$\mu(2^{-k}) + (\mu(2^{-k-1}) - \mu(2^{-k})) + (\mu(2^{-k-2}) - \mu(2^{-k-1})) + \dots$$

converges absolutely and thus it converges. Set

$$l = \lim_{k\to\infty} \mu(2^{-k}).$$

Then l makes sense. Now, for $r \in (0,\delta)$ we can find $k \in \mathbb{N}$ such that $2^{-k-1} < r \leqslant 2^{-k}$. By (24) and (25), for $j > k$ we have

$$\begin{aligned}|\mu(r) - \mu(2^{-j})| &\leqslant |\mu(r) - \mu(2^{-k-1})| + |\mu(2^{-k-2}) - \mu(2^{-k-1})| \\ &\quad + |\mu(2^{-k-2}) - \mu(2^{-k-3})| + \ldots \leqslant r\varepsilon + 2^{-k}\varepsilon \leqslant 3r\varepsilon.\end{aligned}$$

Letting $j \to \infty$ we obtain

$$|\mu(r) - l| \leqslant 3r\varepsilon \quad \text{for } 0 < r < \delta,$$

which verifies (21). Now, suppose that $F(x) = l$. Using Proposition 2.6 we estimate

$$\frac{1}{2r^2}\int_{x-r}^{x+r} |F(y) - F(x)|\,\mathrm{d}y \leqslant \frac{1}{2r^2}\int_{x-r}^{x+r} |F(y) - \mu(r)|\,\mathrm{d}y + \frac{1}{r}|\mu(r) - l| \to 0.$$

It is well known that this property implies that the approximate derivative at x is 0, see the proof of [8], Chapter 6.1, Theorem 4. $\square$

Theorem 6.3. *Let $I \subset \mathbb{R}$ be an open interval. Let $\alpha \geqslant 1$ and $p \in [1,\infty]$. Let F be an indefinite centered HK_α^p integral of f on I. Then f is the approximate derivative of F a.e. If F is an indefinite centered HK_α integral of f on I, then even f is the ordinary derivative of F a.e.*

P r o o f. Let x be a point where F is approximately continuous and (19) holds. (By Lemma 6.2 and the Denjoy-Stepanov theorem (see [41], Chapter IV, Theorem 10.6), almost every point $x \in \mathbb{R}$ satisfies these properties.) Set $\widetilde{F}(y) = F(y) - f(x)y$ and $\tilde{f}(y) = f(y) - f(x)$. Then $\widetilde{F}$ is an indefinite centered HK_α^p integral of $\tilde{f}$, $\widetilde{F}$ is approximately continuous at x and (19) holds for the pair $(\widetilde{F}, \tilde{f})$ at x as well. Thus, by Lemma 6.2, 0 is the approximate derivative of $\widetilde{F}$ at x, so $f(x)$ is the approximate derivative of F at x. The ordinary differentiability at a point where (19) holds with $p = C$ is obvious. $\square$

7. Discontinuity

E x a m p l e 7.1. Let $h\colon (0,\infty) \to \mathbb{R}$ be a smooth function such that $|h'| \leqslant 1$. Interesting choices are e.g. $h(t) = t$ of $h(t) = \sin t$. Set

$$F(x) = \begin{cases} h\Big(\log\log\dfrac{1}{|x|}\Big), & 0 < |x| < 1, \\ 0, & x = 0, \end{cases}$$

$$f(x) = \begin{cases} F'(x), & 0 < |x| < 1, \\ 0, & x = 0. \end{cases}$$

Theorem 7.2. *Let F, f be as in Example 7.1 and $p \in [1,\infty)$. Then*

(a) *F is an indefinite HK_1^p integral of f on $(-1,1)$,*

(b) *F has an approximate limit at 0 if and only if it has the ordinary limit at 0.*

Proof. (a) In view of Remark 3.3 we can restrict our attention to $p > 1$. Since F is continuously differentiable outside the origin, it is enough to verify that

$$\lim_{r\to 0_+} \operatorname{osc}_p(F,[0,r]) = \lim_{r\to 0_+} \operatorname{osc}_p(F,[-r,0]) = 0.$$

Since

$$\Bigl(y\Bigl(\log\log\frac{1}{y}\Bigr)^{p-1}\Bigr)' = \Bigl(\log\log\frac{1}{y}\Bigr)^{p-1}\Bigl(1-(p-1)\Bigl(\log\frac{1}{y}\log\log\frac{1}{y}\Bigr)^{-1}\Bigr),$$

there exists $\delta \in (0,1/\mathrm{e})$ such that

$$\Bigl(y\Bigl(\log\log\frac{1}{y}\Bigr)^{p-1}\Bigr)' \geqslant \frac{1}{2}\Bigl(\log\log\frac{1}{y}\Bigr)^{p-1}, \quad y \in (0,\delta),$$

and thus

$$\int_0^t \Bigl(\log\log\frac{1}{y}\Bigr)^{p-1}\,\mathrm{d}y \leqslant 2t\Bigl(\log\log\frac{1}{t}\Bigr)^{p-1}$$

for each $t \in (0,\delta)$. We can also assume that the function

$$t \mapsto \Bigl(\log\log\frac{1}{t}\Bigr)^{p-1}\Bigl(\log\frac{1}{t}\Bigr)^{-1}$$

is increasing on $(0,\delta)$. Pick $r \in (0,\delta)$. Since the Lipschitz constant of h does not exceed 1, we estimate

$$\begin{aligned}
(26)\qquad \int_0^r |F(y)-F(r)|^p\,\mathrm{d}y &\leqslant \int_0^r \Bigl(\log\log\frac{1}{y}-\log\log\frac{1}{r}\Bigr)^p\,\mathrm{d}y\\
&\leqslant \int_0^r \Bigl(\log\log\frac{1}{y}\Bigr)^{p-1}\Bigl(\int_y^r \Bigl(t\log\frac{1}{t}\Bigr)^{-1}\,\mathrm{d}t\Bigr)\,\mathrm{d}y\\
&= \int_0^r \Bigl(\int_0^t \Bigl(\log\log\frac{1}{y}\Bigr)^{p-1}\Bigl(t\log\frac{1}{t}\Bigr)^{-1}\,\mathrm{d}y\Bigr)\,\mathrm{d}t\\
&\leqslant 2\int_0^r \Bigl(\log\log\frac{1}{t}\Bigr)^{p-1}\Bigl(\log\frac{1}{t}\Bigr)^{-1}\,\mathrm{d}t\\
&\leqslant 2r\Bigl(\log\log\frac{1}{r}\Bigr)^{p-1}\Bigl(\log\frac{1}{r}\Bigr)^{-1}.
\end{aligned}$$

Similarly,

$$\int_{-r}^0 |F(y)-F(r)|\,\mathrm{d}y \leqslant 2r\Bigl(\log\log\frac{1}{r}\Bigr)^{p-1}\Bigl(\log\frac{1}{r}\Bigr)^{-1}.$$

It follows that

$$\lim_{r\to 0+} \operatorname{osc}_p(F,[0,r]) = \lim_{r\to 0+} \operatorname{osc}_p(F,[-r,0]) = 0.$$

(b) Assume that F has an approximate limit at 0. Then there exists the limit

$$\lim_{r\to 0+} \frac{1}{\mathrm{e}r-r}\int_r^{\mathrm{e}r} F(y)\,\mathrm{d}y.$$

Let $y\in[r,\mathrm{e}r]$. Then

$$\begin{aligned}|F(y)-F(r)| &\leqslant \log\log\frac{1}{r}-\log\log\frac{1}{y}\leqslant \log\log\frac{\mathrm{e}}{r}-\log\log\frac{1}{r}\\ &=\log\Bigl(1+\Bigl(\log\frac{1}{r}\Bigr)^{-1}\Bigr)\leqslant\Bigl(\log\frac{1}{r}\Bigr)^{-1}\to 0.\end{aligned}$$

It follows that

$$\lim_{r\to 0} F(r) = \lim \frac{1}{\mathrm{e}r-r}\int_r^{\mathrm{e}r} F(y)\,\mathrm{d}y.$$

□

R e m a r k 7.3. The choice $h(t)=t$ shows that the indefinite HK^p_α integral can be unbounded. If F does not have any limit at 0, as if, for example, $h(t)=\sin t$, then the "definite HK^p_α integral" of f does not make sense over any integral with endpoint at 0. The nonexistence of the approximate limit shows that even an attempt to define an "approximate definite integral" fails.

8. Notes and problems

8.1. The Henstock-Kurzweil integral. The first construction of an integral which integrates all derivatives and includes the Lebesgue integral at the same time was done by Denjoy (see [6]) in 1912, shortly followed by Luzin (see [27]) and Perron (see [34]).

In the fifties of the last century, Henstock (see [12]) and Kurzweil (see [25]) discovered independently that the Denjoy-Perron integral can be obtained by a minor, but ingenious, modification of the classical Riemann integral. The advantage of their approach is that it is more comprehensible than the former constructions and opens the possibility of multi-dimensional generalization.

8.2. Multi-dimensional analogues and the Pfeffer integral. Both Kurzweil and Henstock considered also multidimensional or abstract versions of their integral (see [13], [14], [26]). The fundamental issue in n-dimensional integration is what sets should act as counterparts of intervals in partitions. The choice of all n-dimensional intervals allows straightforward generalization of some one-dimensional ideas but is not suitable for applications. An important step forward has been done by Mawhin (see [31]), who brought the idea of regularity of the partition sets to n-dimensional integration resulting in integrability of all divergences. This idea has been further developed and improved e.g. in [30], [19], [16], [35], [18], [17], [36], [20], [33], see also [3] for a survey.

The most fruitful solution of the problem was to use partitions consisting of regular BV sets. This has been invented by Pfeffer [37], see also a presentation in [38], [39] and a generalization in [29]. The Pfeffer integral leads to a very general setting of the Gauss-Green divergence theorem.

8.3. General packing integrals. The packing integrals were introduced in [28], [24], [15] to define a class of integrals in $\mathbb{R}^n$ which can be applied to non-absolutely convergent integration with respect to distributions. They can be even generalized to metric measure spaces. One of main motivations was also to prove very general versions of the Gauss-Green divergence theorem.

Definition 8.1 (Packing). Let $\alpha \geqslant 1$. A system $\{B(x_i, r_i)\}_{i=1}^m$ of balls in $\mathbb{R}^n$ is called an *α-packing* if the balls $B(x_i, \alpha r_i)$ are pairwise disjoint. If $\delta\colon \mathbb{R}^n \to [0,\infty)$ is a nonnegative function, we say that $\{B(x_i, r_i)\}_{i=1}^m$ is *δ-fine* if $r_i < \delta(x_i)$ for each $i = 1, \dots, m$. If $\delta(x) = 0$, it has the effect that x cannot be any of x_i for the δ-fine α-packing. If $\mathcal{N}$ is a system of subsets of $\mathbb{R}^n$, we say that $\delta\colon \mathbb{R}^n \to [0,\infty)$ is an $\mathcal{N}$-gage if $\{x\colon\ \delta(x) = 0\} \in \mathcal{N}$.

Remark 8.2. Another application of α-packing related to absolute continuity has been studied by Hencl in [11].

Definition 8.3 (Packing integral). Let $(\mathcal{X}, \mathbf{p})$ be a structure which associates with any ball $B = B(x,r)$ a normed linear space $(\mathcal{X}(B), \mathbf{p}(\cdot, B))$ of distributions on B. Let $\mathcal{F}$, $\mathcal{G}$ be distributions on $\mathbb{R}^n$ which belong to $\mathcal{X}(B)$ for each ball $B \subset \mathbb{R}^n$. Let f be a function on $\mathbb{R}^n$. We say that $\mathcal{F}$ is an *indefinite α-packing $(\mathcal{X}, \mathbf{p})$ integral* of f with respect to $\mathcal{G}$ if for each $\varepsilon > 0$ there exists a gage $\delta\colon \mathbb{R}^n \to (0,\infty)$ such that for each δ-fine α-packing $\{B(x_i, r_i)\}_{i=1}^m$ we have

$$\sum_{i=1}^{m} \mathbf{p}(\mathcal{F} - f(x_i)\mathcal{G}), B(x_i, r_i)) < \varepsilon.$$

Remark 8.4. In [24], [15] we considered 1-packing and the norm has been read on the balls $B(x_i, \tau r_i)$ with $\tau \leqslant 1$. This is clearly equivalent to the setting above by the choice $\alpha = 1/\tau$. We have made the change for the purpose of compatibility with the approach of [1].

R e m a r k 8.5. This general notion of packing integral opens possibilities of further research. If we want to apply this general definition to the one-dimensional situation, it is useful to identify a locally integrable function F with its distributional derivative $\mathcal{F}$. We investigated the norms

$$\mathbf{p}(\mathcal{F}, [a,b]) = \operatorname{osc}_p(F, [a,b])$$

defined through the norms of L^p of C. There is a variety of further norms which could be taken into account, like Lorentz norms or Sobolev norms.

8.4. Lip**-packing integral and centered** HK^1_α **integral.** In [24] we have studied the case of

$$\mathbf{p}_{\mathrm{Lip}}(\mathcal{F}, B(x,r)) = \sup\Big\{\langle \mathcal{F}, \varphi\rangle\colon\ \varphi \in \mathrm{Lip}_0(B(x,r)),\ \mathrm{Lip}\,\varphi \leqslant \frac{1}{r}\Big\}, \tag{27}$$

where $\mathrm{Lip}_0(B(x,r))$ is the class of all Lipschitz continuous functions on $\mathbb{R}^n$ supported in $\overline{B}(x,r)$ normed by the Lipschitz constant. Let us label the resulting packing integral as the Lip *α-packing integral*. This choice is convenient for generalization to metric spaces and appears to be one of the most natural ones. The right space $\mathcal{X}(B)$ to be used here is the closure of $\mathcal{D}(B(x,r))$ in the dual space $\mathrm{Lip}_0(B(x,r))^*$ to $\mathrm{Lip}_0(B(x,r))$, see [28].

If $n = 1$ and $\mathcal{F}$ is the distributional derivative of a locally integrable function F, we observe that

$$\begin{aligned}\mathbf{p}_{\mathrm{Lip}}(\mathcal{F}, B(x,r)) &= \sup\Big\{\int_{x-r}^{x+r} F(y)\varphi'(y)\,\mathrm{d}y\colon\ \varphi \in \mathrm{Lip}_0(B(x,r)),\ |\varphi'| \leqslant \frac{1}{r}\Big\}\\ &= \sup\Big\{\int_{x-r}^{x+r} (F(y)-c)\varphi'(y)\,\mathrm{d}y\colon\ \varphi \in \mathrm{Lip}_0(B(x,r)),\ |\varphi'| \leqslant \frac{1}{r}\Big\}\\ &= \frac{1}{r}\int_{x-r}^{x+r} |F(y)-c|\,\mathrm{d}y,\end{aligned}$$

where c is a median of F in $(x-r, x+r)$. Indeed, the supremum is attained at a function φ with $\varphi'(y) = 1/r$ a.e. on E^+ and $\varphi'(y) = -1/r$ a.e. on E^-, where E_+ and E_- are disjoint, of equal measure, $E_+ \cup E_- = [x-r, x+r]$ and $F \geqslant c$ on E_+, $F \leqslant c$ on E_-. By Proposition 2.6,

$$\mathbf{p}_{\mathrm{Lip}}(\mathcal{F}, B(x,r)) = 2\operatorname{osc}_1(F, [x-r, x+r]).$$

Thus, we have verified the following theorem.

Theorem 8.6. *Let $f, F, G\colon \mathbb{R} \to \mathbb{R}$ be measurable functions and $\alpha \geqslant 1$. Then F is an indefinite α-packing* Lip *integral of f with respect to G if and only if F is the indefinite centered* HKS^1_α *integral of f with respect to G.*

8.5. BV-packing integral and centered HKS_α integral. The Lip_α packing integral does not include the Pfeffer integral. If we apply the Lip_α packing integral in [28], [24] to generalize the divergence theorem, we obtain new results but we miss the useful features of the Pfeffer integral. To share both advantages of the Pfeffer integral (see [38]) and of the packing approach, in [23] a new integral is introduced. To explain this integral we need first to introduce the notion of charge, which is fundamental also for the Pfeffer integral.

Definition 8.7. Recall that the space $BV(\mathbb{R}^n)$ is defined as the space of all L^1 functions u on $\mathbb{R}^n$ such that the distributional derivative Du of u is a $\mathbb{R}^n$-valued Radon measure. Then $\|Du\|$ is defined as the total variation of Du. The BV sets are sets E whose characteristic function χ_E is a BV function; perimeter of a BV-set E is $\|E\| := \|\chi_E\|$. Also, we denote the Lebesgue measure of E by $|E|$ and the diameter of E by $d(E)$. Then the *regularity* of a pair (E, x) is the number

$$r(E,x) = \begin{cases} \dfrac{|E|}{d(E \cup \{x\})\|E\|} & \text{if } |E| > 0, \\ 0 & \text{if } |E| = 0. \end{cases}$$

Definition 8.8 (Charge). Let $\mathcal{F}$ be a linear functional on $\mathcal{D}(\mathbb{R}^n)$. We say that $\mathcal{F}$ is a *charge* if for each $\varepsilon > 0$ there is $\theta > 0$ such that

$$\langle \mathcal{F}, \varphi \rangle \leqslant \theta \|\varphi\|_1 + \varepsilon(\|\nabla\varphi\|_1 + \|\varphi\|_\infty)$$

for each $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with support in $B(0, 1/\varepsilon)$, see [5]. We write

$$\mathcal{F}(E) := \mathcal{F}(\chi_E)$$

if E is a BV set.

Definition 8.9 (Norms on charges). Let $\mathcal{F}$ be a charge, $B(x,r)$ be a ball in $\mathbb{R}^n$ and $\varepsilon > 0$. We define

$$\mathbf{p}^\varepsilon_{BV}(\mathcal{F}, B(x,r)) = \sup\{\mathcal{F}(E)\colon\ E \subset B(x,r) \text{ is a } BV \text{ set},\ r(E,x) > \varepsilon\}. \tag{28}$$

Definition 8.10 (Packing BV integral). Let $\mathcal{N}$ be the class of all sets of σ-finite $(n-1)$-dimensional Hausdorff measure, see [38]. Let $\mathcal{F}$, $\mathcal{G}$ be charges and

$f\colon \mathbb{R}^n \to \mathbb{R}$ be a function. We say that $\mathcal{F}$ is an indefinite BV α-packing integral of f with respect to $\mathcal{G}$ if for each $\varepsilon > 0$ there is an $\mathcal{N}$-gage δ such that for each δ-fine α-packing $\{B(x_i, r_i)\}_{i=1}^m$ we have

$$\sum_{i=1}^m \mathbf{p}_{BV}^{\varepsilon}(\mathcal{F} - f(x_i)\mathcal{G}, B(x_i, r_i)) < \varepsilon.$$

Remark 8.11. The integral defined in Definition 8.10 follows the philosophy of packing integrals, but it does not fall to the category of general packing integrals of Definition 8.3 as the seminorm depends on ε and the system $\mathcal{N}$ of exceptional sets is considered. However, we did not want to give the general definition more complicated for the sake of one example.

8.6. BV **packing integral in** $\mathbb{R}$ **and centered** HKS_α **integral.** In the one-dimensional setting things simplify a lot.

First, BV sets can be represented by figures. These are defined as finite unions of bounded closed intervals. The representation means that the BV set E differs from its representing figure E' only by a Lebesgue null set, thus χ_E and $\chi_{E'}$ represent the same element of the BV function space.

Second, charges are represented by continuous functions: a distribution $\mathcal{F}$ on $\mathbb{R}$ is a charge if and only if there is a continuous function $F\colon \mathbb{R} \to \mathbb{R}$ such that for any closed interval $[a, b]$ we have

$$\mathcal{F}([a, b]) = F(b) - F(a). \tag{29}$$

Then $\mathcal{F}$ acts on test functions as the distributional derivative of F.

Third, if $B(x, r) = (x - r, x + r)$ is a ball in $\mathbb{R}$, $E \subset B(x, r)$ is a figure of the form $\bigcup_{j=1}^{k} [u_j, v_j]$, where the intervals $[u_j, v_j]$ are pairwise disjoint, (E, x) is ε-regular and $\mathcal{F}$ is a charge given by (29), then $\|E\| = 2k$, $|E| \leqslant d(E \cup \{x\})$, and thus $2k < 1/\varepsilon$ and

$$\mathcal{F}(E) \leqslant 2k \operatorname{osc}(F, B(x, r)) < \frac{1}{\varepsilon} \operatorname{osc}(F, B(x, r)).$$

It follows that

$$\mathbf{p}_{BV}^{\varepsilon}(B(x, r)) \leqslant \frac{1}{\varepsilon} \operatorname{osc}(F, B(x, r)). \tag{30}$$

On the other hand, the regularity of any interval $x \in [a, b] \subset (x - r, x + r)$ is $r([a, b], x) = \frac{1}{2}$, so

$$0 < \varepsilon < \tfrac{1}{2} \Rightarrow \operatorname{osc}(F, B(x, r)) \leqslant \mathbf{p}_{BV}^{\varepsilon}(B(x, r)). \tag{31}$$

Fourth, the exceptional sets are just the countable sets. So, we consider $\mathcal{N}$-gages, where $\mathcal{N}$ is the family of all countable subsets of $\mathbb{R}$.

We can then reformulate the definition of the BV α-packing integral from [23] for the one-dimensional case as follows:

Definition 8.12. Let $f, F, G\colon \mathbb{R} \to \mathbb{R}$ be measurable functions and $\alpha \geqslant 1$. Assume that F, G are continuous. Let $\mathcal{N}$ be the family of all countable subsets of $\mathbb{R}$. Then F is the indefinite α-packing BV integral of f with respect to G if for each $\varepsilon > 0$ there exists an $\mathcal{N}$-gage $\delta\colon \mathbb{R} \to [0, \infty)$ such that for each δ-fine centered α-partition $\{([a_i, b_i], x_i\}$ in $\mathbb{R}$ we have

$$\sum_{i=1}^{m} \mathbf{p}_{BV}^{\varepsilon}(F - f(x_i)G, [a_i, b_i]) < \varepsilon.$$

Theorem 8.13. *Let $f, F, G\colon \mathbb{R} \to \mathbb{R}$ be measurable functions. Assume that F, G are continuous. Then F is an indefinite α-packing BV integral of f with respect to G if and only if F is the indefinite centered* HKS_α *integral of f with respect to G.*

P r o o f. Let F be an indefinite centered HKS_α integral of f with respect to G. Choose $\varepsilon > 0$. We can find a gage $\delta > 0$ such that for each δ-fine centered α-partition $\{([a_i, b_i], x_i\}$ in $\mathbb{R}$ we have

$$\sum_{i=1}^{m} \operatorname{osc}(F - f(x_i)G, [a_i, b_i]) < \varepsilon^2.$$

Using (30) we obtain

$$\sum_{i=1}^{m} \mathbf{p}_{BV}^{\varepsilon}(F - f(x_i)G, [a_i, b_i]) \leqslant \frac{1}{\varepsilon} \sum_{i=1}^{m} \operatorname{osc}(F - f(x_i)G, [a_i, b_i]) < \varepsilon.$$

Thus, F is an indefinite α-packing BV integral of f with respect to G.

Conversely, if F is an indefinite α-packing BV integral of f with respect to G and $\varepsilon \in (0, \frac{1}{2})$, then there is an $\mathcal{N}$-gage $\delta > 0$ such that for each δ-fine centered α-partition $\{([a_i, b_i], x_i\}$ in $\mathbb{R}$ we have

$$\sum_{i=1}^{m} \mathbf{p}_{BV}^{\varepsilon}(F - f(x_i)G, [a_i, b_i]) < \varepsilon.$$

Let $N = \{x \in \mathbb{R}\colon \delta(x) = 0\}$. Since N is countable, there exists $\xi\colon N \to (0, \infty)$ such that

$$\sum_{x \in N} \xi(x) < \varepsilon.$$

Using continuity of F and G, for each $x \in N$ we find $\hat{\delta}(x) > 0$ such that for each $y \in \mathbb{R}$ we have

$$|y - x| < \hat{\delta}(x) \Rightarrow |F(y) - F(x) - f(x)(G(y) - G(x))| < \xi(x). \tag{32}$$

We define a gage $\bar{\delta} : \mathbb{R} \to (0, \infty)$ as

$$\bar{\delta}(x) = \begin{cases} \delta(x), & x \in \mathbb{R} \setminus N, \\ \hat{\delta}(x), & x \in N. \end{cases}$$

Now let us fix a $\bar{\delta}$-fine α partition $\{[a_i, b_i], x_i\}_{i=1}^m$ in $\mathbb{R}$. Without loss of generality we may assume that $x_1, \dots, x_k \notin N$ and $x_{k+1}, \dots, x_m \in N$ for some $k \in \{0, 1, \dots, m\}$. Then $\{[a_i, b_i], x_i\}_{i=1}^k$ is δ-fine and thus by (31)

$$\sum_{i=1}^{k} \operatorname{osc}(F - f(x_i)G, [a_i, b_i]) < \varepsilon.$$

For $i > k$ we estimate

$$\sum_{i=k+1}^{m} \operatorname{osc}(F - f(x_i)G, [a_i, b_i]) \leqslant \sum_{i=k+1}^{m} \xi(x_i) \leqslant \varepsilon.$$

Together

$$\sum_{i=1}^{m} \operatorname{osc}(F - f(x_i)G, [a_i, b_i]) \leqslant 2\varepsilon.$$

Therefore F is the indefinite centered HKS_α integral of f with respect to G. □

8.7. Open problem: dependence on α. In [1], it is shown that each HK_2 integrable function on an open interval $I \subset \mathbb{R}$ is HK_1 integrable, but for each $2 \leqslant \alpha < \beta$ there is a HK_β integrable function f which is not α integrable. The characterization of pairs (α, β) such that $\mathrm{HK}_\alpha^p = \mathrm{HK}_\beta^p$ (here the symbol represents the class of integrable functions) is known neither for $p > 1$, nor for the centered version.

8.8. Open problem: approximate nondifferentiability. It would be interesting to know how large can be the set of approximate nondifferentiability of an indefinite (centered) HK_α^p integral.

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