145 (2020) MATHEMATICA BOHEMICA No. 1, 33–43

LUCAS FACTORIANGULAR NUMBERS

Bir Kafle, Westville, Florian Luca, Ostrava,
Alain Togbé, Westville

Received February 17, 2018. Published online November 28, 2018.
Communicated by Dagmar Medková

Abstract. We show that the only Lucas numbers which are factoriangular are 1 and 2.

Keywords: Lucas number; factoriangular number

MSC 2010: 11A25, 11B39, 11J86

1. Introduction

Let $(L_m)_{m\geqslant 0}$ be the Lucas sequence given by $L_0 = 2$, $L_1 = 1$ and $L_m = L_{m-1} + L_{m-2}$ for $m \geqslant 2$. The first few terms of this sequence are

$$\mathbf{2}, \mathbf{1}, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \ldots$$

For the beauty and rich applications of Lucas numbers, one can see Koshy's book [6].

Recently, Castillo in [4] dubbed a number of the form $Ft_n = n! + \frac{1}{2}n(n+1)$ for $n \geqslant 0$, a *factoriangular number*. The first few factoriangular numbers are

$$\mathbf{1}, \mathbf{2}, 5, 12, 34, 135, 741, 5068, 40356, 362925, \ldots$$

In [5], Luca and Gómez-Ruiz proved that the only Fibonacci factoriangular numbers are $2, 5$ and 34. This settled a conjecture of Castillo from [4]. Luca, Odjoumani and Togbé in [7] proved that the only Pell factoriangular numbers are 2, 5 and 12.

In this paper, we prove the following related result.

Theorem 1. *The only Lucas numbers which are factoriangular are* 1 *and* 2.

DOI: 10.21136/MB.2018.0021-18

Our method is similar to the one from [5]. First, we assume that $L_m = Ft_n$ for positive integers n and m. Then we use linear forms in p-adic logarithms to find some bounds on n and m. The resulting bounds are large, so we run a calculation to reduce these bounds. This computation is highly nontrivial and relates on reducing the Diophantine equation $L_m = Ft_n$ modulo the primes from a carefully selected finite set of suitable primes.

2. p-adic linear forms in logarithms

Our main tool is an upper bound for a nonzero p-adic linear form in two logarithms of algebraic numbers due to Bugeaud and Laurent (see [2]). Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(x) = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive and $\eta^{(i)}$, $i = 1, \ldots, d$ are the conjugates of η. The *logarithmic height* of η is given by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \max\{1, \log|\eta^{(i)}|\}\right).$$

Let $\mathbb{K}$ be an algebraic number field of degree $d_{\mathbb{K}}$. Let $\eta_1, \eta_2 \in \mathbb{K} \setminus \{0, 1\}$ and b_1, b_2 be positive integers. We put

$$\Lambda = \eta_1^{b_1} - \eta_2^{b_2}.$$

For a prime ideal π of the ring $\mathcal{O}_{\mathbb{K}}$ of algebraic integers in $\mathbb{K}$ and $\eta \in \mathbb{K}$, we denote by $\mathrm{ord}_\pi(\eta)$ the order at which π appears in the prime factorization of the principal fractional ideal $\eta\mathcal{O}_{\mathbb{K}}$ generated by η in $\mathbb{K}$. When η is an algebraic integer, $\eta\mathcal{O}_{\mathbb{K}}$ is an ideal of $\mathcal{O}_{\mathbb{K}}$. When $\mathbb{K} = \mathbb{Q}$, π is just a prime number. Let e_π and f_π be the ramification index and the inertial degree of π, respectively, and let $p \in \mathbb{Z}$ be the only prime number such that $\pi \mid p$. Then

$$p\mathcal{O}_{\mathbb{K}} = \prod_{i=1}^{k} \pi_i^{e_{\pi_i}}, \quad \left|\frac{\mathcal{O}_{\mathbb{K}}}{\pi}\right| = p^{f_{\pi_i}}, \quad d_{\mathbb{K}} = \sum_{i=1}^{k} e_{\pi_i} f_{\pi_i},$$

where $\pi_1 := \pi, \ldots, \pi_k$ are prime ideals in $\mathcal{O}_{\mathbb{K}}$. We set $D := d_{\mathbb{K}}/f_\pi$. Let A_1, A_2 be positive real numbers such that

$$\log A_i \geqslant \max\left\{h(\eta_i), \frac{\log p}{D}\right\}, \quad i = 1, 2.$$

Further, let

$$b' := \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

With the above notations, Bugeaud and Laurent proved the following result (see Corollary 1 of Theorem 3 in [2]).

Theorem 2. *Assume that η_1, η_2 are algebraic integers which are multiplicatively independent and that π does not divide $\eta_1\eta_2$. Then*

$$\begin{aligned}\operatorname{ord}_\pi(\Lambda) \leqslant & \frac{24p(p^{f_\pi}-1)}{(p-1)(\log p)^4} D^5 \log A_1 \log A_2 \\ & \times \Bigl(\max\Bigl\{\log b' + \log(\log p) + \frac{4}{10}, \frac{10\log p}{D}, 10\Bigr\}\Bigr)^2.\end{aligned}$$

In the actual statement of [2], there is only a dependence of D^4 on the right-hand side of the above inequality, but there all the valuations are normalized. Since we work with the actual order $\operatorname{ord}_\pi(\Lambda)$, we must multiply the upper bound of [2] by another factor of $d_{\mathbb{K}}/f_\pi = D$.

3. Proof of Theorem 1

Recall that if k is any nonnegative integer, then

$$L_k = \alpha^k + \beta^k, \tag{1}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2} \tag{2}$$

are the solutions of the quadratic equation $x^2 - x - 1 = 0$. Equation (1) is known as Binet's formula for Lucas numbers.

Lemma 1. *The inequalities*

$$\alpha^{k-1} \leqslant L_k \leqslant \alpha^{k+1} \tag{3}$$

hold for all $k \geqslant 1$.

P r o o f. The proof follows immediately by induction on k. See [1]. □

We now study the Diophantine equation

$$L_m = Ft_n. \tag{4}$$

Further, the inequalities

$$\left(\frac{n}{\mathrm{e}}\right)^n \leqslant n! + \frac{n(n+1)}{2} \leqslant n^n$$

hold for $n \geqslant 3$, see [5]. By taking the logarithms, we get

$$n(\log n - 1) < \log\Big(n! + \frac{n(n+1)}{2}\Big) < n \log n \tag{5}$$

for all $n \geqslant 3$. Also, inequalities (3) yield

$$(m-1)\log\alpha \leqslant \log L_m \leqslant (m+1)\log\alpha. \tag{6}$$

Combining inequalities (5) and (6), we get

$$n(\log n - 1) < (m+1)\log\alpha \quad \text{and} \quad (m-1)\log\alpha < n\log n.$$

Hence,

$$\frac{n(\log n - 1)}{\log\alpha} - 1 < m < \frac{n\log n}{\log\alpha} + 1. \tag{7}$$

If $n \leqslant 200$, then the above inequality implies that $m \leqslant 2204$. We listed all Lucas numbers L_m with $m \leqslant 2204$ and all factoriangular numbers Ft_n with $n \leqslant 200$ and intersected these two lists. The only solutions of (4) in this range are the ones listed in Theorem 1.

Our next goal is to find an upper bound for n and we assume that $n > 200$. We rewrite the Diophantine equation (4) using the Binet formula (1) as

$$\alpha^m + \beta^m = n! + \frac{n(n+1)}{2}.$$

Now, using the fact that $\beta = -\alpha^{-1}$, the above equation yields

$$n! = \alpha^{-m}\Big(\alpha^{2m} - \frac{n(n+1)}{2}\alpha^m + \varepsilon\Big),$$

where $\varepsilon = (-1)^{m+1} = \pm 1$. We note that

$$\alpha^{-m}\Big(\alpha^{2m} - \frac{n(n+1)}{2}\alpha^m + \varepsilon\Big) = \alpha^{-m}(\alpha^m - z_1)(\alpha^m - z_2),$$

where

$$z_{1,2} = \frac{n(n+1) \pm \sqrt{n^2(n+1)^2 - 16\varepsilon}}{4}$$

are the roots of the polynomial

$$z^2 - \frac{n(n+1)}{2} z + \varepsilon.$$

Therefore, equation (4) is equivalent to

$$n! = \alpha^{-m}(\alpha^m - z_1)(\alpha^m - z_2). \tag{8}$$

Let $\mathbb{L} = \mathbb{Q}(z_1)$ and π be a prime ideal lying above 2 in $\mathcal{O}_{\mathbb{K}}$. From equation (8), we have

$$\mathrm{ord}_2(n!) \leqslant \mathrm{ord}_\pi(\alpha^m - z_1) + \mathrm{ord}_\pi(\alpha^m - z_2). \tag{9}$$

We use Theorem 2 to get an upper bound on $\mathrm{ord}_\pi(\alpha^m - z_i)$ for $i = 1, 2$. We fix $i \in \{1, 2\}$ and take

$$\eta_1 = \alpha, \quad \eta_2 = z_i, \quad b_1 = m, \quad b_2 = 1 \quad \text{and} \quad \Lambda_i = \alpha^m - z_i.$$

Note that $z_1 z_2 = \varepsilon$ and $z_1 + z_2 = \frac{1}{2} n(n+1)$. In particular, z_1, z_2 and α are all units, so π does not divide any one of them and all these numbers are in $\mathbb{L}$. Next, we need to check that α and z_i are multiplicatively independent. Since $z_2 = \pm z_1^{-1}$, it suffices to show that this is so only for $i = 1$. Let d be that squarefree integer such that for some positive integer u we have

$$n^2(n+1)^2 - 16\varepsilon = du^2.$$

Clearly, $d > 0$ as $n > 200$. Since the left-hand side above is a multiple of 4 and d is squarefree, it follows that u is even and

$$\left(\frac{n(n+1)}{2}\right)^2 - 4\varepsilon = d\left(\frac{u}{2}\right)^2.$$

Next, $d \neq 1$. Indeed, if $d = 1$, then

$$\left(\frac{n(n+1)}{2}\right)^2 - 4\varepsilon = \left(\frac{u}{2}\right)^2.$$

Hence, $(x, y) = (\frac{1}{2}u, \frac{1}{2}n(n+1))$ is a positive integer solution of the equation

$$x^2 - y^2 = \pm 4,$$

giving us $(x+y)(x-y)=\pm 4$, which implies both $x+y=\pm 2$ and $x-y=\pm 2$. So,

$$x = L_1 = 0, \quad 2 = \frac{n(n+1)}{2}$$

and $n > 200$, which is impossible. Next, let $d = 5$. We get

$$\Big(\frac{n(n+1)}{2}\Big)^2 - 5\Big(\frac{u}{2}\Big)^2 = \pm 4.$$

It is well-known that all positive integer solutions (x, y) of $x^2 - 5y^2 = \pm 4$ are of the form $(x, y) = (L_k, F_k)$ for some positive integer k. Hence, $L_k = \frac{1}{2}n(n+1)$ is a triangular number. Ming in [8] proved that the largest triangular Lucas number is 5778, which gives us $n \leqslant 107$, contradicting our hypothesis that $n > 200$. Thus, $(\frac{1}{2}n(n+1))^2 - 4\varepsilon = d(\frac{1}{2}u)^2$ holds with some squarefree integer $d > 1$, $d \neq 5$. Since α is a unit in $\mathbb{Q}(\sqrt{5})$ and z_i is a unit in $\mathbb{Q}(\sqrt{d})$ while $d \neq 1, 5$, it follows that α and z_i cannot be multiplicatively dependent.

Next, we calculate the upper bounds for the logarithmic heights of α and z_i. The minimal polynomial of α over the integers is $x^2 - x - 1$ and $h(\alpha) = \frac{1}{2}\log\alpha$. So, we take $\log A_1 = \frac{1}{2}\log\alpha$. For the logarithmic height of z_i we note that the minimal polynomial of z_i over the integers is

$$z^2 - \frac{n(n+1)}{2} z \pm 1.$$

Next, each z_i has degree 2 and its conjugates are

$$z_i^{(j)} = \frac{\pm n(n+1) \pm \sqrt{n^2(n+1)^2 - 16\varepsilon}}{4}$$

satisfying

$$|z_i^{(j)}| \leqslant \frac{n(n+1)}{4} + \sqrt{\Big(\frac{n(n+1)}{4}\Big)^2 + 1} < n^{21/10}$$

for $n > 200$. Hence, we get

$$h(z_i) = \frac{1}{2}\sum_{j=1}^{2} \log\max\{|z_i^{(j)}|, 1\} \leqslant \frac{1}{2}\sum_{j=1}^{2} \log n^{21/10} < \frac{21}{10}\log n$$

for $i = 1, 2$. We take $\log A_2 = \frac{21}{10}\log n$, and therefore

$$b' = \frac{m}{\frac{42}{10}\log n} + \frac{1}{\log\alpha}.$$

From inequality (7) we have

$$m < \frac{n \log n}{\log \alpha} + 1 < \frac{21}{10} n \log n$$

for $n > 200$. We then get

$$b' = \frac{n}{2} + \frac{1}{\log \alpha} < \frac{6}{10} n$$

for $n > 200$. Thus,

$$\log b' + \log(\log 2) + \frac{4}{10} < \log\left(\frac{6}{10} n\right) + \log(\log 2) + \frac{4}{10} < \log n.$$

We deduce that

$$\max\left\{\log b' + \log(\log 2) + \frac{4}{10}, \frac{10 \log 2}{2}, 10\right\}$$

equals

$$\max\{\log n, 10\}$$

because $5 \log 2 < \log n$ for $n > 200$. By Theorem 2, we get

$$(10) \qquad \begin{aligned} \mathrm{ord}_\pi(\Lambda_i) &< \frac{24 \times 2 \times 3}{(\log 2)^4} \times 2^5 \times 0.5 \log \alpha \times \frac{21}{10} \log n \times (\max\{\log n, 10\})^2 \\ &< 10087 (\max\{\log n, 10\})^3 \end{aligned}$$

for $i = 1, 2$. We now return to inequalities (9), and give a lower bound to $\mathrm{ord}_2(n!)$. It is well known that for any prime p we have

$$\mathrm{ord}_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \ldots + \left\lfloor \frac{n}{p^t} \right\rfloor + \ldots$$

Hence,

$$\mathrm{ord}_2(n!) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \ldots + \left\lfloor \frac{n}{2^t} \right\rfloor + \ldots$$

Since $n > 2^k$, we have

$$\left\lfloor \frac{n}{2^k} \right\rfloor \geqslant \frac{n}{2^k} - \frac{2^k - 1}{2^k}.$$

We now conclude, using the fact that $n > 200 > 2^5$,

$$(11) \qquad \mathrm{ord}_2(n!) \geqslant \sum_{k=1}^{5} \left(\frac{n}{2^k} - \frac{2^k - 1}{2^k} \right) = \frac{31n - 129}{32} > \frac{15n}{16}.$$

Assume further that $\log n > 10$ (that is, $n > 22027$). Combining inequalities (9), (10) and (11), we obtain

$$n < 21519 (\log n)^3,$$

which leads to $n \leqslant 1.5 \times 10^8$.

In summary, we proved the following result.

Lemma 2. *Let* (n,m) *be a solution of Diophantine equation* (4) *with* $n > 200$. *Then the inequalities*

$$\frac{n(\log n-1)}{\log\alpha}-\frac{\log 2}{\log\alpha}<m<\frac{n\log n}{\log\alpha}+1 \quad \textit{and} \quad n\leqslant 1.5\times 10^8$$

hold.

Let $\lfloor x\rceil$ denote the nearest integer to the real number x. The range for which we search the positive integer solutions (n,m) of the Diophantine equation (4) with $n>200$ is

$$(n,m)\in[201,1.5\times 10^8]\times\Big[\Big\lfloor\frac{n(\log n-1)}{\log\alpha}-\frac{\log 2}{\log\alpha}\Big\rceil,\Big\lfloor\frac{n\log n}{\log\alpha}+1\Big\rceil\Big].$$

The bounds for n and m are too large for our Diophantine equation (4) to be verified, even computationally. To reduce these bounds, we use the procedure described in [5]. We first write equation (4) as

$$L_m=n!\Big(1+\frac{n+1}{2(n-1)!}\Big).$$

Put

$$\nu:=1+\frac{n+1}{2(n-1)!}.$$

From inequalities (3) we get

$$\alpha^{m-1}\leqslant \nu n!\leqslant 2\alpha^m,$$

which leads to

$$\frac{\log n!+\log\nu}{\log\alpha}-\frac{\log 2}{\log\alpha}\leqslant m\leqslant\frac{\log n!+\log\nu}{\log\alpha}+1. \tag{12}$$

By Stirling's theorem for $n!$ (see [9]),

$$n!=\sqrt{2\pi n}\,\frac{n^n}{\mathrm{e}^n}\mathrm{e}^{\lambda_n},\qquad \text{where } \frac{1}{12n+1}<\lambda_n<\frac{1}{12n}.$$

We write inequalities (12) as

$$\frac{(12n+1)^{-1}+\log\nu}{\log\alpha}-\frac{\log 2}{\log\alpha}\leqslant m-\frac{\log\sqrt{2\pi}+\big(n+\frac12\big)\log n-n}{\log\alpha}\leqslant\frac{\frac{1}{12}n^{-1}+\log\nu}{\log\alpha}+1. \tag{13}$$

Hence, we conclude here that if (n, m) is a solution of Diophantine equation (4) with $n > 200$, then

$$m = \left\lfloor \frac{\log \sqrt{2\pi} + \left(n + \frac{1}{2}\right) \log n - n}{\log \alpha} \right\rceil + \delta \quad \text{with } \delta \in \{-2, 1\}. \tag{14}$$

We consider two cases for $n \in [201, 1.5 \times 10^8]$.

Case 1: $n \in [201, 2.5 \times 10^5]$. For each n in this interval we generate the list of $L_m \equiv n \pmod{10^{20}}$, that is, we take only last 20 digits of the Lucas numbers L_m, where m is given by the last equation (14). Since $n! \equiv 0 \pmod{10^{20}}$, we explored the congruence

$$\frac{n(n+1)}{2} \equiv L_m \pmod{10^{20}}. \tag{15}$$

A simple calculation in Maple shows that the above equation has no solutions in this range. This proves that equation (4) has no solutions in this range.

Case 2: $n \in (2.5 \times 10^5, 1.5 \times 10^8]$. The Lucas sequence is periodic modulo any positive integer. For a prime number q, let l be the period of $L_m \pmod q$. Then

$$l \mid q - 1 \quad \text{if } q \equiv \pm 1 \pmod 5,$$

or

$$l \mid 2q + 2 \quad \text{if } q \equiv \pm 2 \pmod 5$$

(see [3]).

We set $A := 2^4 \times 3^2 \times 5^2 \times 7 \times 11$. We found all the primes $q \equiv 1 \pmod 5$ such that $q - 1 \mid A$. They are

$$11, 31, 41, 61, 71, 101, 151, 181, 211, 241, 281, 331, 401, 421, 601, 631, 661, 701, 881,$$
$$991, 1051, 1201, 1321, 1801, 2311, 2521, 2801, 3301, 3851, 4201, 4621, 4951, 6301,$$
$$9241, 9901, 11551, 12601, 15401, 18481, 19801, 34651, 55441, 92401.$$

For each prime q in the list above, L_m is periodic modulo q and the period of the Lucas sequence modulo q divides A. Hence, if (n, m) is a solution of the Diophantine equation (4) with $n > 2.5 \times 10^5$, then $n! \equiv 0 \pmod q$. Further,

$$L_m \equiv \frac{n(n+1)}{2} \pmod q,$$

which is equivalent to

$$8L_m + 1 \equiv (2n+1)^2 \pmod q.$$

However, a quick search in Maple shows that for each $m \in [1, A]$ there is a prime q in the above list such that the Legendre symbol

$$\left(\frac{8L_m + 1}{q}\right) = -1$$

except for $m \in \{1, 2, 3, 4, 6\}$.

Hence, we conclude that the only possible values of $n \in (3.6 \times 10^5, 1.8 \times 10^8]$, which can be the solutions of the Diophantine equation (4), satisfy the conditions

$$\frac{n(n+1)}{2} \equiv L_{m_0} \pmod{A} \quad \text{for } m_0 = 1, 2, 3, 4, 6. \tag{16}$$

We generate the set N_{m_0} of residue classes for $n \pmod{A}$ of equation (16) for the corresponding values of m_0.

m_0	N_{m_0}
1	1, 92399
2	2, 92398
3	40999, 51401
4	6742, 85658
6	17382, 75018

So, we have the following result.

Lemma 3. *If $n \in (2.5 \times 10^5, 1.5 \times 10^8]$ and (m, n) is a solution of Diophantine equation (4), then*

$$n \equiv n_0 \pmod{A},$$

where $A := 2^4 \times 3^2 \times 5^2 \times 7 \times 11$ and $n_0 \in N_{m_0}$ for $m_0 = 1, 2, 3, 4, 6$. Furthermore,

$$m = \left\lfloor \frac{\log\sqrt{2\pi} + \left(n + \frac{1}{2}\right)\log n - n}{\log\alpha} \right\rceil + \delta$$

with $\delta \in \{-2, 1\}$.

Next, we computationally analysed equation (4) with the restrictions

$$n = n_0 + t \times A \quad \text{with } 1 \leqslant t \leqslant \left\lfloor \frac{1.5 \times 10^8}{A} \right\rfloor, \; n_0 \in N_{m_0},$$

where $m_0 \in \{1, 2, 3, 4, 6\}$. We compare last 20 digits of the Lucas numbers and the factoriangular numbers in pairs (m, n) satisfying the above restrictions. An extensive computational search with Maple shows that equation (15) has no solutions other than the ones from the statement of Theorem 1.

This completes the proof of Theorem 1. $\square$

A c k n o w l e d g e m e n t s . The authors thank the referee for the useful comments which helped to improve this paper. Florian Luca was supported in parts by Grants CPRR160325161141 of NRF (South Africa), CGA 17-02804S (Czech Republic). This paper was completed during a visit of Florian Luca at Purdue University Northwest in February, 2018. This author thanks this institution for the hospitality and the support.

References

[1] *J. J. Bravo, F. Luca*: Powers of two as sums of two Lucas numbers. J. Integer Seq. *17* (2014), Article No. 14.8.3, 12 pages. zbl MR
[2] *Y. Bugeaud, M. Laurent*: Effective lower bound for the p-adic distance between powers of algebraic numbers. J. Number Theory *61* (1996), 311–342. (In French.) zbl MR doi
[3] *Y. Bugeaud, F. Luca, M. Mignotte, S. Siksek*: Almost powers in the Lucas sequence. J. Théor. Nombres Bordx. *20* (2008), 555–600. zbl MR doi
[4] *R. C. Castillo*: On the sum of corresponding factorials and triangular numbers: Some preliminary results. Asia Pac. J. Multidisciplinary Research *3* (2015), 5–11.
[5] *C. A. Gómez Ruiz, F. Luca*: Fibonacci factoriangular numbers. Indag. Math., New Ser. *28* (2017), 796–804. zbl MR doi
[6] *T. Koshy*: Fibonacci and Lucas Numbers with Applications. Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs, and Tracts. Wiley-Interscience, New York, 2001. zbl MR doi
[7] *F. Luca, J. Odjoumani, A. Togbé*: Pell factoriangular numbers. To appear in Publ. Inst. Math. Beograd, N.S.
[8] *L. Ming*: On triangular Lucas numbers. Applications of Fibonacci Numbers. Vol. 4. Proc. 4th Int. Conf. on Fibonacci Numbers and Their Applications, Wake Forest University, Winston-Salem, USA, 1990 (G. E. Bergun et al., eds.). Kluwer Academic Publishers, Dordrecht, 1991, pp. 231–240. zbl MR doi
[9] *H. Robbins*: A remark on Stirling's formula. Am. Math. Mon. *62* (1955), 26–29. zbl MR doi

Authors' addresses: *Bir Kafle*, Department of Mathematics, Statistics and Computer Science, Purdue University Northwest, 1401 S, U.S. 421, Westville IN 46391, USA, e-mail: `bkafle@pnw.edu`; *Florian Luca*, School of Mathematics, University of the Witwatersrand, Private Bag X3, Wits 2050, South Africa, and School of Mathematics, Faculty of Sciences, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic, e-mail `florian.luca@wits.ac.za`; *Alain Togbé*, Department of Mathematics, Statistics and Computer Science, Purdue University Northwest, 1401 S, U.S. 421, Westville IN 46391, USA, e-mail: `atogbe@pnw.edu`.