MULTIPLICITY OF POSITIVE SOLUTIONS FOR SECOND ORDER QUASILINEAR EQUATIONS

Dahmane Bouafia, Toufik Moussaoui, Kouba,
Donal O'Regan, Galway

Received April 20, 2018. Published online June 13, 2019.
Communicated by Jiří Šremr

Abstract. We discuss the existence and multiplicity of positive solutions for a class of second order quasilinear equations. To obtain our results we will use the Ekeland variational principle and the Mountain Pass Theorem.

Keywords: critical point; Ekeland variational principle; Mountain Pass Theorem; Palais-Smale condition; positive solution

MSC 2010: 35A15, 35B38, 30E25, 58E30, 49K35

1. Introduction

Our aim in this paper is to obtain at least two positive solutions for the problem

$$\begin{cases} -u'' + u = \lambda h(x)|u|^{\beta-2}u + q(x)f(u), & x \in (0,\infty), \\ u(0) = u(\infty) = 0, \end{cases} \tag{1}$$

where $f \in C(\mathbb{R},\mathbb{R})$, β and λ are real parameters with $1 < \beta < 2$ and $\lambda > 0$.

Throughout this paper we assume the following hypotheses are satisfied:

(H_0) h and $q\colon [0,\infty) \to (0,\infty)$ belong to $L^1(0,\infty) \cap L^\infty(0,\infty)$;

(H_1) there is a continuously differentiable and bounded function $p\colon [0,\infty) \to (0,\infty)$ belonging to $L^1(0,\infty) \cap L^\infty(0,\infty)$ such that the functions q/p, q/p^2, q/p^β, $q/p^{\beta+1}$, $h/p^{\beta-1}$ and h/p^β all belong to $L^1(0,\infty)$;

(H_2) $M = \max(\|p\|_{L^2}, \|p'\|_{L^2}) < \infty$,

$$M_{r,g} = \|p\|_\infty^{1/2}\biggl(\int_0^\infty g(x)\biggl(\int_0^x \frac{\mathrm{d}s}{p(s)}\biggr)^{r/2}\,\mathrm{d}x\biggr)^{1/r} < \infty$$

DOI: 10.21136/MB.2019.0051-18

for all $r \in \{\beta, 2, \beta+1\}$ and all $g \in \{q, h\}$ and

$$M_{2,q} = \|p\|_\infty^{1/2}\left(\int_0^\infty q(x)\left(\int_0^x \frac{\mathrm{d}s}{p(s)}\right)\mathrm{d}x\right)^{1/2} < \frac{1}{\sqrt{A}},$$

where the constant A satisfies

(H$_3$) $\lim_{u\to 0^+} f(u)/|u| = A \in (0, \overline{\lambda}_{2,q}^2)$ and $\lim_{u\to\infty} f(u)/|u|^\beta = B \in (\overline{\lambda}_{2,q}^2, \infty)$, where $\overline{\lambda}_{2,q}$ is the first eigenvalue of problem (2) which is defined in Lemma 1.3;

(H$_4$) there exists $\mu > \beta + 1$ such that

$$F(s) \leqslant \frac{1}{\mu} s f(s) \quad \forall\,|s| > 0, \text{ where } F(s) = \int_0^s f(t)\,\mathrm{d}t.$$

Now we introduce the Hilbert space $H_0^1(0,\infty)$ which is suitable for the study of our problem. Let

$$H_0^1(0,\infty) = \{u \text{ measurable: } u, u' \in L^2(0,\infty),\ u(0) = u(\infty) = 0\}$$

equipped with the norm

$$\|u\| = \left(\int_0^\infty |u'(x)|^2\,\mathrm{d}x + \int_0^\infty |u(x)|^2\,\mathrm{d}x\right)^{1/2}$$

and endowed with the inner product

$$(u,v) = \int_0^\infty u'(x)\cdot v'(x)\,\mathrm{d}x + \int_0^\infty u(x)\cdot v(x)\,\mathrm{d}x.$$

We consider the spaces $L_g^r(0,\infty)$ which are defined by

$$L_g^r(0,\infty) = \left\{u\colon (0,\infty) \to \mathbb{R} \text{ measurable such that } \int_0^\infty g(x)|u(x)|^r\,\mathrm{d}x < \infty\right\}$$

for all $r \in \{\beta, 2, \beta+1\}$ and all $g \in \{h, q\}$ equipped, respectively, with the norms

$$\|u\|_{r,g} = \left(\int_0^\infty g(x)|u(x)|^r\,\mathrm{d}x\right)^{1/r}.$$

Let the space $C_{l,p}[0,\infty)$ be defined by

$$C_{l,p}[0,\infty) = \left\{u \in C([0,\infty),\mathbb{R})\colon \lim_{x\to\infty} p(x)u(x) \text{ exists}\right\}.$$

The corresponding norm is defined by

$$\|u\|_{\infty,p} = \sup_{x\in[0,\infty)} p(x)|u(x)|.$$

Now we give some necessary lemmas and corollaries, which are used below.

Lemma 1.1 ([5])**.** *$H_0^1(0,\infty)$ embeds continuously and compactly in $C_{l,p}[0,\infty)$, i.e.*

$$\|u\|_{\infty,p} \leqslant \sqrt{2}M\|u\| \quad \forall\, u \in H_0^1(0,\infty).$$

Lemma 1.2 ([2])**.** *$C_{l,p}[0,\infty)$ is continuously embedded in $L_g^r(0,\infty)$ for all $r \in \{\beta, 2, \beta+1\}$ and all $g \in \{h, q\}$.*

Corollary 1.1 ([2])**.** *$H_0^1(0,\infty)$ embeds continuously and compactly in $L_g^r(0,\infty)$ with the embedding constant $M_{r,g}$.*

Let $\overline{\lambda}_{r,g}$ be the first eigenvalue of the problem

$$\begin{cases} -u''(x) + u(x) = \lambda g(x)|u(x)|^{r-2}u(x), \quad x > 0, \\ u(0) = u(\infty) = 0, \end{cases} \tag{2}$$

and note

$$\overline{\lambda}_{r,g} = \inf_{u\in H_0^1\setminus\{0\}} \frac{\|u\|}{\|u\|_{r,g}}.$$

Lemma 1.3 ([2])**.** *The first eigenvalue $\overline{\lambda}_{r,g}$ is positive and is achieved for some positive function $\overline{\psi}_{r,g} \in H_0^1(0,\infty)\setminus\{0\}$, i.e.*

$$\overline{\lambda}_{r,g} := \inf_{u\in H_0^1\setminus\{0\}} \frac{\|u\|}{\|u\|_{r,g}} = \frac{\|\overline{\psi}_{r,g}\|}{\|\overline{\psi}_{r,g}\|_{r,g}}.$$

Theorem 1.1 ([4], Weak Ekeland variational principle)**.** *Let (E,d) be a complete metric space and let $J\colon E \to \mathbb{R}$ be a functional that is lower semi-continuous and bounded from below. Then for each $\varepsilon > 0$ there exists $u_\varepsilon \in E$ with*

$$J(u_\varepsilon) \leqslant \inf_E J + \varepsilon,$$

and whenever $w \in E$ with $w \neq u_\varepsilon$, then

$$J(u_\varepsilon) < J(w) + \varepsilon d(u_\varepsilon, w).$$

Definition 1.1 ([6]). Let E be a Banach space and $J\colon E \to \mathbb{R}$ a C^1-functional and $c \in \mathbb{R}$. The functional J is said to satisfy the (local) Palais-Smale condition at the level c, denoted by $(\mathrm{P.S})_c$, if any sequence (u_n) in E such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0, \tag{3}$$

admits a convergent subsequence.

Lemma 1.4 (Mountain Pass Theorem). *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ with $J(0) = 0$. Suppose $J(u)$ satisfies $(\mathrm{P.S})_c$ condition and*

(a) *there are $\varrho, \alpha > 0$ such that $J(u) \geqslant \alpha$ when $\|u\|_E = \varrho$,*
(b) *there is a $e \in E$, $\|e\|_E > \varrho$ such that $J(e) < 0$.*

Define

$$\Gamma = \{\gamma \in C^1([0,1], E)\colon\ \gamma(0) = 0,\ \gamma(1) = e\}. \tag{4}$$

Then

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} J(\gamma(t)) \geqslant \alpha \tag{5}$$

is a critical value of $J(u)$.

2. Main existence results

Now we define the Euler-Lagrange functional associated to problem (1). Let $J_\lambda\colon H_0^1(0,\infty) \to \mathbb{R}$ be defined by

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta}\int_0^\infty h(x)|u|^\beta(x)\,\mathrm{d}x - \int_0^\infty q(x)F(u)\,\mathrm{d}x. \tag{6}$$

Proposition 2.1. *Suppose that the conditions (H_0)–(H_3) hold. Then the functional J_λ is continuously differentiable. The Fréchet derivative of J_λ has the form*

$$\begin{aligned}\langle J_\lambda'(u), v\rangle = &\int_0^\infty u'(x)v'(x)\,\mathrm{d}x + \int_0^\infty u(x)v(x)\,\mathrm{d}x \\ &- \lambda\int_0^\infty h(x)|u|^{\beta-2}(x)u(x)v(x)\,\mathrm{d}x - \int_0^\infty q(x)f(u)v(x)\,\mathrm{d}x\end{aligned} \tag{7}$$

for all $v \in H_0^1(0,\infty)$.

P r o o f. The proof of the proposition will be done consecutively.

Claim 2.1. *J_λ is Gâteaux-differentiable.*

For all $v \in H_0^1(0,\infty)$ and for any $t > 0$ we have

$$\begin{aligned}
&J_\lambda(u+tv) - J_\lambda(u) \\
&\quad= \frac{1}{2}\int_0^\infty |(u+tv)'|^2\,\mathrm{d}x + \frac{1}{2}\int_0^\infty |u+tv|^2\,\mathrm{d}x - \frac{\lambda}{\beta}\int_0^\infty h(x)|u+tv|^\beta\,\mathrm{d}x \\
&\qquad - \int_0^\infty q(x)F(u+tv)\,\mathrm{d}x - \frac{1}{2}\int_0^\infty |u'|^2\,\mathrm{d}x - \frac{1}{2}\int_0^\infty |u|^2\,\mathrm{d}x \\
&\qquad + \frac{\lambda}{\beta}\int_0^\infty h(x)|u|^\beta\,\mathrm{d}x + \int_0^\infty q(x)F(u)\,\mathrm{d}x \\
&\quad= \frac{t^2}{2}\int_0^\infty |v'|^2\,\mathrm{d}x + t\int_0^\infty u'v'\,\mathrm{d}x + \frac{t^2}{2}\int_0^\infty |v|^2\,\mathrm{d}x + t\int_0^\infty uv\,\mathrm{d}x \\
&\qquad - \frac{\lambda}{\beta}\int_0^\infty h(x)(|u+tv|^\beta - |u|^\beta)\,\mathrm{d}x - \int_0^\infty q(x)(F(u+tv)-F(u))\,\mathrm{d}x \\
&\quad= \frac{t^2}{2}\int_0^\infty |v'|^2\,\mathrm{d}x + t\int_0^\infty u'v'\,\mathrm{d}x + \frac{t^2}{2}\int_0^\infty |v|^2\,\mathrm{d}x + t\int_0^\infty uv\,\mathrm{d}x \\
&\qquad - t\lambda\int_0^\infty h(x)|u+t\theta v|^{\beta-2}(u+t\theta v)v\,\mathrm{d}x - t\int_0^\infty q(x)f(u+t\theta v)v\,\mathrm{d}x,
\end{aligned}$$

where $0 < \theta < 1$, and then

$$\begin{aligned}
\frac{J_\lambda(u+tv) - J_\lambda(u)}{t} &= \frac{t}{2}\int_0^\infty |v'|^2\,\mathrm{d}x + \int_0^\infty u'v'\,\mathrm{d}x + \frac{t}{2}\int_0^\infty |v|^2\,\mathrm{d}x \\
&\quad + \int_0^\infty uv\,\mathrm{d}x - \lambda\int_0^\infty h(x)|u+t\theta v|^{\beta-2}(u+t\theta v)v\,\mathrm{d}x \\
&\quad - \int_0^\infty q(x)f(u+t\theta v)v\,\mathrm{d}x.
\end{aligned}$$

Let $t \to 0$ and we have

$$\langle J_\lambda'(u), v\rangle = \int_0^\infty u'v'\,\mathrm{d}x + \int_0^\infty uv\,\mathrm{d}x - \lambda\int_0^\infty h(x)|u|^{\beta-2}uv\,\mathrm{d}x - \int_0^\infty q(x)f(u)v\,\mathrm{d}x$$

for all $v \in H_0^1(0,\infty)$.

Claim 2.2. *J_λ' is continuous.*

Let $(u_n) \subset H_0^1(0,\infty)$ with $u_n \to u$ when $n \to \infty$, so there exists $R > 0$ such that $\|u_n\| \leqslant R$ for all $n \in \mathbb{N}$.

From (H_3), given ε small enough, there exists $\delta_2 > \delta_1 > 0$ such that

$$(A-\varepsilon)|s| < f(s) < (A+\varepsilon)|s| \quad \forall\, 0 < s < \delta_1 \tag{8}$$

and

$$(B-\varepsilon)|s|^\beta < f(s) < (B+\varepsilon)|s|^\beta \quad \forall\, s > \delta_2, \tag{9}$$

so from (8) and (9) and since $f(u)$ is continuous on $[\delta_1, \delta_2]$, there exists $D_1 > 0$ such that

$$-D_1 + (A-\varepsilon)|s| + (B-\varepsilon)|s|^\beta < f(s) < D_1 + (A+\varepsilon)|s| + (B+\varepsilon)|s|^\beta \tag{10}$$

for all $s \in (0,\infty)$. This yields

$$F(s) \leqslant D_2 + \frac{A+\varepsilon}{2}s^2 + \frac{B+\varepsilon}{\beta}|s|^{\beta+1} \quad \forall\, s \in (0,\infty) \tag{11}$$

and

$$F(s) \geqslant -D_2 + \frac{A-\varepsilon}{2}s^2 + \frac{B-\varepsilon}{\beta}|s|^{\beta+1} \quad \forall\, s \in (0,\infty), \tag{12}$$

where $D_2 = D_1(\delta_2 - \delta_1)$. Furthermore, from Lemma 1.1, (H_0)–(H_1) and (10) we obtain

$$\begin{aligned} q(x)|f(u_n(x))| &\leqslant (A+\varepsilon)q(x)|u_n(x)| + (B+\varepsilon)q(x)|u_n(x)|^\beta + D_1 q(x) \\ &\leqslant (A+\varepsilon) \sup_{x\in[0,\infty)} |(pu_n)(x)| \frac{q(x)}{p(x)} \\ &\quad + (B+\varepsilon) \sup_{x\in[0,\infty)} |(pu_n)(x)|^\beta \frac{q(x)}{p^\beta(x)} + D_1 q(x) \\ &= (A+\varepsilon)\|u_n\|_{\infty,p} \frac{q(x)}{p(x)} + (B+\varepsilon)\|u_n\|_{\infty,p}^\beta \frac{q(x)}{p^\beta(x)} + D_1 q(x) \\ &\leqslant (A+\varepsilon)\sqrt{2}MR\frac{q(x)}{p(x)} + (B+\varepsilon)(\sqrt{2}MR)^\beta \frac{q(x)}{p^\beta(x)} + D_1 q(x) \in L^1(0,\infty) \end{aligned}$$

and

$$\begin{aligned} h(x)|u_n(x)|^{\beta-2}|u_n(x)| \leqslant h(x)|u_n(x)|^{\beta-1} &= p^{\beta-1}(x)|u_n(x)|^{\beta-1}\frac{h(x)}{p^{\beta-1}(x)} \\ &\leqslant \sup_{x\in[0,\infty)} |(pu_n)(x)|^{\beta-1}\frac{h(x)}{p^{\beta-1}(x)} = \|u_n\|_{\infty,p}^{\beta-1}\frac{h(x)}{p^{\beta-1}(x)} \\ &\leqslant (\sqrt{2}MR)^{\beta-1}\frac{h(x)}{p^{\beta-1}(x)} \in L^1(0,\infty). \end{aligned}$$

Then from the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\to\infty}\int_0^\infty q(x)f(u_n(x))\,\mathrm{d}x=\int_0^\infty q(x)f(u(x))\,\mathrm{d}x \tag{13}$$

and also

$$\lim_{n\to\infty}\int_0^\infty h(x)|u_n|^{\beta-2}(x)u_n(x)\,\mathrm{d}x=\int_0^\infty h(x)|u|^{\beta-2}(x)u(x)\,\mathrm{d}x. \tag{14}$$

Thus we have

$$\begin{aligned}\langle J'_\lambda(u_n)-J'_\lambda(u),v\rangle&=\int_0^\infty u'_nv'\,\mathrm{d}x+\int_0^\infty u_nv\,\mathrm{d}x-\lambda\int_0^\infty h(x)|u_n|^{\beta-2}u_nv\,\mathrm{d}x\\&\quad-\int_0^\infty q(x)f(u_n)v\,\mathrm{d}x-\int_0^\infty u'v'\,\mathrm{d}x-\int_0^\infty uv\,\mathrm{d}x\\&\quad+\lambda\int_0^\infty h(x)|u|^{\beta-2}uv\,\mathrm{d}x+\int_0^\infty q(x)f(u)v\,\mathrm{d}x\\&=\int_0^\infty(u'_n-u')v'\,\mathrm{d}x+\int_0^\infty(u_n-u)v\,\mathrm{d}x\\&\quad-\lambda\int_0^\infty h(x)(|u_n|^{\beta-2}u_n-|u|^{\beta-2}u)v\,\mathrm{d}x\\&\quad-\int_0^\infty q(x)(f(u_n)-f(u))v\,\mathrm{d}x,\end{aligned} \tag{15}$$

and from (13), (14) and the continuity of f, passing to the limit in $\langle J'_\lambda(u_n)-J'_\lambda(u),v\rangle$ when $n\to\infty$, we obtain that $J'_\lambda(u_n)\to J'_\lambda(u)$ as $n\to\infty$. □

Definition 2.1. We say that $u\in H_0^1(0,\infty)$ is a weak solution of problem (1) if for any $v\in H_0^1(0,\infty)$ we have

$$\begin{aligned}\langle J'_\lambda(u),v\rangle=\int_0^\infty u'v'\,\mathrm{d}x+\int_0^\infty uv\,\mathrm{d}x-\lambda\int_0^\infty h(x)|u|^{\beta-2}uv\,\mathrm{d}x\\-\int_0^\infty q(x)f(u)v\,\mathrm{d}x=0.\end{aligned}$$

Remark 2.1. Since the nonlinear term f is continuous, then a weak solution of problem (1) is a classical solution.

In our next two sections we will prove the main result of this paper.

Theorem 2.1. *Suppose that* (H_0)–(H_4) *hold. Then there exists* $\xi>0$ *such that for* $0<\lambda<\xi$*, problem* (1) *has at least two positive solutions.*

2.1. Existence of a first solution.

Lemma 2.1. *Suppose that the hypotheses* (H_0)–(H_4) *hold. Then there exists* $\xi_1 > 0$ *such that for* $0 < \lambda \leqslant \xi_1$, *the functional* J_λ *satisfies the geometric conditions* (a) *and* (b) *in Lemma* 1.4, *i.e.*

(a) *there are* $\varrho, \alpha > 0$ *such that* $J_\lambda(u) \geqslant \alpha$ *when* $\|u\| = \varrho$,

(b) *there is* $e \in H_0^1(0,\infty)$, $\|e\| > \varrho$ *such that* $J_\lambda(e) < 0$.

Proof. (a) From (H_0)–(H_3), (11) and using Corollary 1.1, we have

$$
\begin{aligned}
J_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta}\int_0^\infty h(x)|u|^\beta(x)\,\mathrm{d}x - \int_0^\infty q(x)F(u)\,\mathrm{d}x \qquad (16)\\
&\geqslant \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta}\int_0^\infty h(x)|u|^\beta(x)\,\mathrm{d}x - D_2\int_0^\infty q(x)\,\mathrm{d}x\\
&\quad - \frac{A+\varepsilon}{2}\int_0^\infty q(x)|u|^2\,\mathrm{d}x - \frac{B+\varepsilon}{\beta+1}\int_0^\infty q(x)|u|^{\beta+1}\,\mathrm{d}x\\
&\geqslant \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta}M_{\beta,h}^\beta\|u\|^\beta - \frac{A+\varepsilon}{2}M_{2,q}^2\|u\|^2\\
&\quad - \frac{B+\varepsilon}{\beta+1}M_{\beta+1,h}^{\beta+1}\|u\|^{\beta+1} - D_2\|q\|_{L^1}\\
&\geqslant \Big(\frac{1}{2} - \frac{A+\varepsilon}{2}M_{2,q}^2\Big)\|u\|^2 - \frac{\lambda}{\beta}M_{\beta,h}^\beta\|u\|^\beta\\
&\quad - \frac{B+\varepsilon}{\beta+1}M_{\beta+1,h}^{\beta+1}\|u\|^{\beta+1} - D_2\|q\|_{L^1}\\
&\geqslant \|u\|^2\Big(\frac{1}{2}(1-(A+\varepsilon)M_{2,q}^2) - \frac{\lambda}{\beta}M_{\beta,h}^\beta\|u\|^{\beta-1} - \frac{B+\varepsilon}{\beta+1}M_{\beta+1,h}^{\beta+1}\|u\|^\beta\Big)\\
&\quad - D_2\|q\|_{L^1}\\
&\geqslant \|u\|^2\Big(\frac{1}{2}(1-(A+\varepsilon)M_{2,q}^2) - \lambda K_1\|u\|^{\beta-1} - K_2\|u\|^\beta\Big) - K_3,
\end{aligned}
$$

where $K_1 = \beta^{-1}M_{\beta,h}^\beta$, $K_2 = ((B+\varepsilon)/(\beta+1))M_{\beta+1,h}^{\beta+1}$ and $K_3 = D_2\|q\|_{L^1}$; here ε and D_2 are given in the proof of Proposition 2.1. Let

$$g(t) = \lambda K_1 t^{\beta-2} + K_2 t^{\beta-1} \quad \text{for } t \geqslant 0.$$

Clearly,

$$g'(t) = \lambda K_1(\beta-2)t^{\beta-3} + K_2(\beta-1)t^{\beta-2} \quad \text{for } t \geqslant 0.$$

From $g'(t_0) = 0$ we have

$$t_0 = \frac{\lambda K_1(2-\beta)}{K_2(\beta-1)}.$$

Then

$$g(t_0) = \frac{2\lambda^{\beta-1}K_1^{\beta-1}}{(\beta-1)K_2^{\beta-2}}.$$

Thus, there exists

$$0 < \xi_1 < \Big(\frac{(\beta-1)K_2}{4K_1}(1-(A+\varepsilon)M_{2,q}^2)\Big)^{1/\beta-1}$$

such that

$$g(t_0) < \frac{1}{2}(1-(A+\varepsilon)M_{2,q}^2) \quad \forall\, 0<\lambda \leqslant \xi_1.$$

Consequently, taking $\varrho = t_0$ and choosing $\lambda \in (0, \xi_1)$ such that

$$m_0 = \varrho^2\Big(\frac{1}{2}(1-(A+\varepsilon)M_{2,q}^2) - \lambda K_1 \varrho^{\beta-2} - K_2\varrho^{\beta-1}\Big) > K_3,$$

from (16) we have

$$J_\lambda(u) \geqslant \alpha > 0 \ \text{ when } \|u\| = \varrho, \tag{17}$$

where $\alpha = m_0 - K_3$. Thus (a) is proved.

(b) For $t > 0$ large enough, from (12) and Lemma 1.3 we have

$$\begin{aligned}
&J_\lambda(t\overline{\psi}_{\beta+1,q})\\
&= \frac{1}{2}t^2\|\overline{\psi}_{\beta+1,q}\|^2 - \frac{\lambda}{\beta}t^\beta \int_0^\infty h(x)|\overline{\psi}_{\beta+1,q}|^\beta \,\mathrm{d}x - \int_0^\infty q(x)F(t\overline{\psi}_{\beta+1,q})\,\mathrm{d}x\\
&\leqslant \frac{1}{2}t^2\|\overline{\psi}_{\beta+1,q}\|^2 - \frac{\lambda}{\beta}t^\beta \int_0^\infty h(x)|\overline{\psi}_{\beta+1,q}|^\beta \,\mathrm{d}x - \frac{A-\varepsilon}{2}t^2\int_0^\infty q(x)|\overline{\psi}_{\beta+1,q}|^2\,\mathrm{d}x\\
&\quad - \frac{B-\varepsilon}{\beta+1}t^{\beta+1}\int_0^\infty q(x)|\overline{\psi}_{\beta+1,q}|^{\beta+1}\,\mathrm{d}x + D_2\int_0^\infty q(x)\,\mathrm{d}x\\
&\leqslant \frac{1}{2}t^2\|\overline{\psi}_{\beta+1,q}\|^2 - \frac{\lambda}{\beta}t^\beta\|\overline{\psi}_{\beta+1,q}\|_{\beta,h}^\beta - \frac{A-\varepsilon}{2}t^2\|\overline{\psi}_{\beta+1,q}\|_{2,q}^2\\
&\quad - \frac{B-\varepsilon}{\beta+1}t^{\beta+1}\|\overline{\psi}_{\beta+1,q}\|_{\beta+1,q}^{\beta+1} + D_2\|q\|_{L^1}\\
&\leqslant \frac{1}{2}(\|\overline{\psi}_{\beta+1,q}\|^2 - (A-\varepsilon)\|\overline{\psi}_{\beta+1,q}\|_{2,q}^2)t^2 - \frac{\lambda}{\beta}\|\overline{\psi}_{\beta+1,q}\|_{\beta,h}^\beta t^\beta\\
&\quad - \frac{B-\varepsilon}{\beta+1}\|\overline{\psi}_{\beta+1,q}\|_{\beta+1,q}^{\beta+1}t^{\beta+1} + K_3.
\end{aligned}$$

Therefore $J_\lambda(t\overline{\psi}_{\beta+1,q}) \to -\infty$ as $t \to \infty$. Choose $t_1 > 0$ large enough and $e = t_1\overline{\psi}_{\beta+1,q}$. Hence, we conclude that

$$J_\lambda(e) < 0 \ \text{ when } \|e\| > \varrho.$$

Thus (b) is proved. □

From a version of the Mountain Pass Theorem without the Palais-Smale condition (see [7]), there exists a $(\mathrm{P.S})_c$ sequence $(u_n) \subset H_0^1(0,\infty)$ for J_λ which satisfies (3), i.e.

$$J_\lambda(u_n) \to c \quad \text{and} \quad J'_\lambda(u_n) \to 0,$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t))$$

with

$$\Gamma = \{\gamma \in C([0,1], H_0^1(0,\infty))\colon\ \gamma(0) = 0,\ \gamma(1) = e\},$$

where e is given in Lemma 2.1.

Remark 2.2. Since the sequence (u_n^+) also satisfies (3) (see [1], Lemma 1), we assume, without of loss generality, that $u_n \geqslant 0$ for all $n \in \mathbb{N}$.

Lemma 2.2. *Suppose that the hypotheses* (H_0)–(H_4) *hold. Then the mountain level c satisfies the following inequality:*

$$c < \left(\frac{\overline{\lambda}_{\beta+1,q}^{\beta+1}}{B-\varepsilon}\right)^{2/(\beta-1)} \left(\frac{1}{2} - \frac{1}{\mu}\right) + K_3;$$

here K_3 is given in the proof of Lemma 2.1.

Proof. From the proof of Lemma 2.1 we can consider $\gamma(t) = tt_1\overline{\psi}_{\beta+1,q}$, where $t_1 > 0$ is sufficiently large such that $e = t_1\overline{\psi}_{\beta+1,q}$. Thus, from the definition of c,

$$c \leqslant \max_{t\geqslant 0} J_\lambda(t\overline{\psi}_{\beta+1,q}),$$

that is,

$$c \leqslant \max_{t\geqslant 0}\left\{\frac{1}{2}t^2\|\overline{\psi}_{\beta+1,q}\|^2 - \frac{\lambda}{\beta}t^\beta\|\overline{\psi}_{\beta+1,q}\|_{\beta,h}^\beta - \int_0^\infty q(x)F(t\overline{\psi}_{\beta+1,q})\,\mathrm{d}x\right\}.$$

From (12),

$$\begin{aligned} c \leqslant\ &\max_{t\geqslant 0}\left\{\frac{1}{2}t^2\|\overline{\psi}_{\beta+1,q}\|^2 - \frac{\lambda}{\beta}t^\beta\|\overline{\psi}_{\beta+1,q}\|_{\beta,h}^\beta - \frac{A-\varepsilon}{2}t^2\|\overline{\psi}_{\beta+1,q}\|_{2,q}^2 \right.\\ &\left. - \frac{B-\varepsilon}{\beta+1}t^{\beta+1}\|\overline{\psi}_{\beta+1,q}\|_{\beta+1,q}^{\beta+1} + K_3\right\}\\ \leqslant\ &\max_{t\geqslant 0}\left\{\frac{1}{2}t^2\|\overline{\psi}_{\beta+1,q}\|^2 - \frac{B-\varepsilon}{\beta+1}t^{\beta+1}\|\overline{\psi}_{\beta+1,q}\|_{\beta+1,q}^{\beta+1}\right\} + K_3, \end{aligned}$$

and then

$$\frac{c}{\|\overline{\psi}_{\beta+1,q}\|^2_{\beta+1,q}} \leqslant \max_{t\geqslant 0}\left\{\frac{\overline{\lambda}^2_{\beta+1,q}}{2}t^2 - \frac{B-\varepsilon}{\beta+1}\|\overline{\psi}_{\beta+1,q}\|^{\beta-1}_{\beta+1,q}t^{\beta+1}\right\} + \frac{K_3}{\|\overline{\psi}_{\beta+1,q}\|^2_{\beta+1,q}}.$$

Let

$$Z(t) = \frac{\overline{\lambda}^2_{\beta+1,q}}{2}t^2 - \frac{B-\varepsilon}{\beta+1}\|\overline{\psi}_{\beta+1,q}\|^{\beta-1}_{\beta+1,q}t^{\beta+1}.$$

Clearly,

$$Z'(t) = \overline{\lambda}^2_{\beta+1,q}t - (B-\varepsilon)\|\overline{\psi}_{\beta+1,q}\|^{\beta-1}_{\beta+1,q}t^{\beta}.$$

Since the function Z attains its maximum at

$$t = \left(\frac{\overline{\lambda}^2_{\beta+1,q}}{(B-\varepsilon)\|\overline{\psi}_{\beta+1,q}\|^{\beta-1}_{\beta+1,q}}\right)^{1/(\beta-1)},$$

it follows that

$$c < \left(\frac{\overline{\lambda}^{\beta+1}_{\beta+1,q}}{B-\varepsilon}\right)^{2/(\beta-1)}\left(\frac{1}{2} - \frac{1}{\beta+1}\right) + K_3,$$

and therefore we have

$$c < \left(\frac{\overline{\lambda}^{\beta+1}_{\beta+1,q}}{B-\varepsilon}\right)^{2/(\beta-1)}\left(\frac{1}{2} - \frac{1}{\mu}\right) + K_3.$$

□

Lemma 2.3. *There exists $\xi_2 > 0$ such that for $0 < \lambda < \xi_2$, the Palais-Smale sequence (u_n) associated with the functional J_λ satisfies*

$$\limsup_{n\to\infty}\|u_n\|^2 < 2\left(\frac{\overline{\lambda}^{\beta+1}_{\beta+1,q}}{B-\varepsilon}\right)^{2/(\beta-1)} + M^{2\beta/2-\beta}_{\beta,h} + \frac{4K_3\mu}{\mu-2}.$$

Proof. First, observe that (u_n) is bounded in $H^1_0(0,\infty)$. In fact, from (3)

$$J_\lambda(u_n)\to c \quad \text{and} \quad \langle J'_\lambda(u_n), u_n\rangle \to 0 \quad \text{as } n\to\infty.$$

Notice that from (7) we have

$$\int_0^\infty q(x)f(u_n)u_n\,\mathrm{d}x = \|u_n\|^2 - \lambda\|u_n\|^{\beta}_{\beta,h} - \langle J'_\lambda(u_n), u_n\rangle.$$

Using Corollary 1.1 and (H_4), it follows from (3) that

$$\begin{aligned}(18)\quad c+\varepsilon > J_\lambda(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{\beta}\|u_n\|_{\beta,h}^{\beta} - \int_0^\infty q(x)F(u_n)\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{\beta}\|u_n\|_{\beta,h}^{\beta} - \frac{1}{\mu}\int_0^\infty q(x)f(u_n)u_n\,\mathrm{d}x \\ &\geqslant \Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|u_n\|^2 - \lambda\Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\|u_n\|_{\beta,h}^{\beta} + \frac{1}{\mu}\langle J_\lambda'(u_n),u_n\rangle \\ &\geqslant \Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|u_n\|^2 - \lambda M_{\beta,h}^{\beta}\Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\|u_n\|^{\beta} + \frac{1}{\mu}\langle J_\lambda'(u_n),u_n\rangle \\ &\geqslant \Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|u_n\|^2 - \lambda M_{\beta,h}^{\beta}\Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\|u_n\|^{\beta} - \frac{1}{\mu}\|J_\lambda'(u_n)\|\|u_n\|.\end{aligned}$$

Since $J_\lambda'(u_n)\to 0$, there exists $N_0\in\mathbb{N}$ large enough such that

$$(19)\qquad c+\varepsilon > \Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|u_n\|^2 - \lambda M_{\beta,h}^{\beta}\Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\|u_n\|^{\beta} - o_n(1)\|u_n\| \quad \forall\, n>N_0.$$

This implies that $(u_n)\subset H_0^1(0,\infty)$ is bounded.

Now we can write (19) as

$$(20)\qquad \Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|u_n\|^2 \leqslant \lambda M_{\beta,h}^{\beta}\Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\|u_n\|^{\beta} + o_n(1)\|u_n\| + c+\varepsilon.$$

Using Young's inequality in (20), we get

$$\begin{aligned}&\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|u_n\|^2 \\ &\quad\leqslant \Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\Big(\frac{2-\beta}{2}M_{\beta,h}^{2\beta/(2-\beta)} + \frac{\beta}{2}\lambda^{2/\beta}\|u_n\|^2\Big) + o_n(1)\|u_n\| + c+\varepsilon \\ &\quad\leqslant \Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\frac{2-\beta}{2}M_{\beta,h}^{2\beta/(2-\beta)} + \frac{\beta}{2}\lambda^{2/\beta}\Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\|u_n\|^2 + o_n(1)\|u_n\| + c+\varepsilon,\end{aligned}$$

and then we have

$$\Big(\Big(\frac{1}{2}-\frac{1}{\mu}\Big) - \frac{\beta}{2}\lambda^{2/\beta}\Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\Big)\|u_n\|^2 \leqslant \Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\frac{2-\beta}{2}M_{\beta,h}^{2\beta/(2-\beta)} + o_n(1)\|u_n\| + c+\varepsilon.$$

Choosing

$$0<\lambda\leqslant \xi_2 = \Big(\frac{\frac{1}{2}-\mu^{-1}}{\beta(\beta^{-1}-\mu^{-1})}\Big)^{\beta/2},$$

then using Lemma 2.2, we conclude that

$$\begin{aligned}\|u_n\|^2 \leqslant \Big(\frac{1}{2}\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\Big)^{-1}\Big(&\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\Big(\frac{\overline{\lambda}_{\beta+1,q}^{\beta+1}}{B-\varepsilon}\Big)^{2/(\beta-1)} \\ &+ K_3 + \Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\frac{2-\beta}{2}M_{\beta,h}^{2\beta/(2-\beta)} + o_n(1)\|u_n\| + \varepsilon\Big).\end{aligned}$$

Thus

$$\limsup_{n\to\infty}\|u_n\|^2 \leqslant \Big(\frac{1}{2}\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\Big)^{-1}\Big(\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\Big(\frac{\overline{\lambda}_{\beta+1,q}^{\beta+1}}{B-\varepsilon}\Big)^{2/(\beta-1)} + K_3 + \Big(\frac{1}{\beta}-\frac{1}{\mu}\Big)\frac{2-\beta}{2}M_{\beta,h}^{2\beta/(2-\beta)}\Big)$$
$$< 2\Big(\frac{\overline{\lambda}_{\beta+1,q}^{\beta+1}}{B-\varepsilon}\Big)^{2/(\beta-1)} + M_{\beta,h}^{2\beta/2-\beta} + \frac{4K_3\mu}{\mu-2}.$$

□

Since (u_n) satisfying (3) is bounded in $H_0^1(0,\infty)$ (see Lemma 2.3), there exists $u_1 \in H_0^1(0,\infty)$ such that for a subsequence we have

(21) $$u_n \rightharpoonup u_1 \quad \text{in } H_0^1(0,\infty),$$

(22) $$u_n \to u_1 \quad \text{in } L_g^r(0,\infty)$$

for all $r \in \{\beta, 2, \beta+1\}$ and all $g \in \{h, q\}$ and

(23) $$u_n(x) \to u_1(x) \quad \text{a.e. in } (0,\infty).$$

In the next lemma we obtain some convergences results involving the sequence (u_n) and its weak limit u_1.

Lemma 2.4. *The following limits are satisfied:*

(c) $\int_0^\infty q(x)|f(u_n)-f(u_1)|\,|u_n-u_1|\,\mathrm{d}x = o_n(1)$,
(d) $\int_0^\infty h(x)||u_n|^{\beta-2}u_n - |u_1|^{\beta-2}u_1|\,|u_n-u_1|\,\mathrm{d}x = o_n(1)$.

Proof. (c) From (10) and using Corollary 1.1 and Lemmas 2.2 and 2.3, we obtain

$$\int_0^\infty q(x)|f(u_n)-f(u_1)||u_n-u_1|\,\mathrm{d}x$$
$$\leqslant \int_0^\infty q(x)|f(u_n||u_n-u_1|\,\mathrm{d}x + \int_0^\infty q(x)|f(u_1)||u_n-u_1|\,\mathrm{d}x$$
$$\leqslant 2D_1\int_0^\infty q(x)|u_n-u_1|\,\mathrm{d}x$$
$$+ (A+\varepsilon)\int_0^\infty q(x)|u_n||u_n-u_1|\,\mathrm{d}x + (A+\varepsilon)\int_0^\infty q(x)|u_1||u_n-u_1|\,\mathrm{d}x$$
$$+ (B+\varepsilon)\int_0^\infty q(x)|u_n|^\beta|u_n-u_1|\,\mathrm{d}x + (B+\varepsilon)\int_0^\infty q(x)|u_1|^\beta|u_n-u_1|\,\mathrm{d}x$$

and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_0^\infty q(x)|f(u_n)-f(u_1)||u_n-u_1|\,\mathrm{d}x\\
&\qquad\leqslant 2D_1\Big(\int_0^\infty q(x)\,\mathrm{d}x\Big)^{1/2}\Big(\int_0^\infty q(x)|u_n-u_1|^2\,\mathrm{d}x\Big)^{1/2}\\
&\qquad\quad+(A+\varepsilon)\Big(\int_0^\infty q(x)|u_n|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_0^\infty q(x)|u_n-u_1|^2\,\mathrm{d}x\Big)^{1/2}\\
&\qquad\quad+(A+\varepsilon)\Big(\int_0^\infty q(x)|u_1|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_0^\infty q(x)|u_n-u_1|^2\,\mathrm{d}x\Big)^{1/2}\\
&\qquad\quad+(B+\varepsilon)\Big(\int_0^\infty q(x)|u_n|^\beta\,\mathrm{d}x\Big)^{(\beta-1)/\beta}\Big(\int_0^\infty q(x)|u_n-u_1|^\beta\,\mathrm{d}x\Big)^{1/\beta}\\
&\qquad\quad+(B+\varepsilon)\Big(\int_0^\infty q(x)|u_1|^\beta\,\mathrm{d}x\Big)^{(\beta-1)/\beta}\Big(\int_0^\infty q(x)|u_n-u_1|^\beta\,\mathrm{d}x\Big)^{1/\beta}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^\infty q(x)|f(u_n)-f(u_1)||u_n-u_1|\,\mathrm{d}x\\
&\qquad\leqslant 2D_1\|q\|_{L^1}\|u_n-u_1\|_{2,q}+(A+\varepsilon)\|u_n\|_{2,q}\|u_n-u_1\|_{2,q}\\
&\qquad\quad+(A+\varepsilon)\|u_1\|_{2,q}\|u_n-u_1\|_{2,q}+(B+\varepsilon)\|u_n\|_{\beta,q}^{\beta-1}\|u_n-u_1\|_{\beta,q}\\
&\qquad\quad+(B+\varepsilon)\|u_1\|_{\beta,q}^{\beta-1}\|u_n-u_1\|_{\beta,q}\\
&\qquad\leqslant C_{1,\varepsilon}\|u_n-u_1\|_{2,q}+C_{2,\varepsilon}\|u_n-u_1\|_{\beta,q},
\end{aligned}$$

where

$$C_{1,\varepsilon}=2(A+\varepsilon)M_{2,q}\overline{C}^{1/2}+2D_1\|q\|_{L^1},\quad C_{2,\varepsilon}=2(B+\varepsilon)(M_{2,h}\overline{C}^{1/2})^{\beta-1},$$
$$\overline{C}=2\Big(\frac{\overline{\lambda}_{\beta+1,q}^{\beta+1}}{B-\varepsilon}\Big)^{2/(\beta-1)}+M_{\beta,h}^{2\beta/2-\beta}+\frac{4K_3\mu}{\mu-2}.$$

Then according to (22) we have

$$\int_0^\infty (q(x)|f(u_n)-f(u_1)||u_n-u_1|)\,\mathrm{d}x=o_n(1).$$

(d) From Corollary 1.1 and Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\int_0^\infty h(x)||u_n|^{\beta-2}u_n-|u_1|^{\beta-2}u_1|\,|u_n-u_1|\,\mathrm{d}x\\
&\qquad\leqslant\int_0^\infty h(x)|u_n|^{\beta-1}|u_n-u_1|\,\mathrm{d}x+\int_0^\infty h(x)|u_1|^{\beta-1}|u_n-u_1|\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
&\leqslant \Big(\int_0^\infty h(x)|u_n|^\beta \,\mathrm{d}x\Big)^{\beta-1/\beta}\Big(\int_0^\infty h(x)|u_n-u_1|^\beta \,\mathrm{d}x\Big)^{1/\beta}\\
&\quad+\Big(\int_0^\infty h(x)|u_1|^\beta \,\mathrm{d}x\Big)^{\beta-1/\beta}\Big(\int_0^\infty h(x)|u_n-u_1|^\beta \,\mathrm{d}x\Big)^{1/\beta}\\
&\leqslant \|u_n\|_{\beta,h}^{\beta-1}\|u_n-u_1\|_{\beta,h}+\|u_1\|_{\beta,h}^{\beta-1}\|u_n-u_1\|_{\beta,h}\leqslant 2C_3\|u_n-u_1\|_{\beta,h},
\end{aligned}$$

where $C_3=(M_{\beta,h}\overline{C}^{1/2})^{\beta-1}$. Then according to (22) we have

$$\int_0^\infty h(x)||u_n|^{\beta-2}u_n-|u_1|^{\beta-2}u_1|\,|u_n-u_1|\,\mathrm{d}x=o_n(1).$$

□

Proposition 2.2. *Suppose that f is a function satisfying* (H_0)–(H_4). *Then there exists a constant $\overline{\xi}>0$ such that for $0<\lambda<\overline{\xi}$, problem* (1) *has a positive solution u_1 satisfying $J_\lambda(u_1)>0$.*

P r o o f. Let u_1 be the weak limit of the sequence (u_n) that satisfies (3). Consider $\overline{\xi}=\min\{\xi_1,\xi_2\}$, where ξ_1 and ξ_2 are given in Lemmas 2.1 and 2.3, respectively. We will prove that $u_n\to u_1$ in $H_0^1(0,\infty)$.

From (15) we have

$$\begin{aligned}
&\langle J'_\lambda(u_n)-J'_\lambda(u_1),u_n-u_1\rangle\\
&\quad=\int_0^\infty (u'_n-u'_1)(u'_n-u'_1)\,\mathrm{d}x+\int_0^\infty (u_n-u_1)(u_n-u_1)\,\mathrm{d}x\\
&\quad\quad-\lambda\int_0^\infty h(x)(|u_n|^{\beta-2}u_n-|u_1|^{\beta-2}u_1)(u_n-u_1)\,\mathrm{d}x\\
&\quad\quad-\int_0^\infty q(x)(f(u_n)-f(u_1))(u_n-u_1)\,\mathrm{d}x.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|u_n-u_1\|^2&\leqslant |\langle J'_\lambda(u_n)-J'_\lambda(u_1),u_n-u_1\rangle|\\
&\quad+\lambda\int_0^\infty h(x)||u_n|^{\beta-2}u_n-|u_1|^{\beta-2}u_1||u_n-u_1|\,\mathrm{d}x\\
&\quad+\int_0^\infty q(x)|f(u_n)-f(u_1)||u_n-u_1|\,\mathrm{d}x.
\end{aligned}$$

Therefore, from Lemma 2.4 above and taking into account that J'_λ is continuous (see Proposition 2.1), we have

$$\|u_n-u_1\|^2=\int_0^\infty |u'_n-u'_1|^2\,\mathrm{d}x+\int_0^\infty |u_n-u_1|^2\,\mathrm{d}x=o_n(1).$$

Consequently,

$$\lim_{n\to\infty}\Big(\int_0^\infty |u_n' - u_1'|^2\,\mathrm{d}x + \int_0^\infty |u_n - u_1|^2\,\mathrm{d}x\Big) = 0.$$

That is, $u_n \to u_1$ as $n \to \infty$ in $H_0^1(0,\infty)$, i.e. (u_n) satisfies the Palais-Smale condition. Now by applying the Mountain Pass Theorem, we obtain

$$J_\lambda'(u_1) = 0 \quad \text{and} \quad J_\lambda(u_1) = c > 0.$$

□

2.2. Existence of a second solution. Now we apply the Ekeland variational principle to prove the existence of a weak solution u_2 which is different from the solution u_1.

Lemma 2.5. *Suppose that* (H_0)–(H_4) *hold. Then there exists a constant* $\xi_3 > 0$ *such that for* $0 < \lambda < \xi_3$*, the functional* J_λ *satisfies* $(\mathrm{P.S})_d$ *condition with* $d < 0$.

Proof. Fix $d < 0$ and suppose that $(u_n) \subset H_0^1(0,\infty)$ satisfies

$$J_\lambda(u_n) \to d \quad \text{and} \quad J_\lambda'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{24}$$

We need to show that (u_n) admits a subsequence converging strongly in $H_0^1(0,\infty)$. Proceeding as in (20) we get

$$\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2 \leqslant \lambda M_{\beta,h}^\beta\Big(\frac{1}{\beta} - \frac{1}{\mu}\Big)\|u_n\|^\beta + o_n(1)\|u_n\| + d + \varepsilon.$$

Thus, for a subsequence we have

$$\Big(\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\limsup_{n\to\infty}\|u_n\|^{2-\beta} - \lambda M_{\beta,h}^\beta\Big(\frac{1}{\beta} - \frac{1}{\mu}\Big)\Big)\limsup_{n\to\infty}\|u_n\|^\beta \leqslant d < 0.$$

Hence,

$$\limsup_{n\to\infty}\|u_n\|^2 \leqslant \Big(\lambda\frac{M_{\beta,h}^\beta(\beta^{-1} - \mu^{-1})}{(\frac{1}{2} - \mu^{-1})}\Big)^{2/(2-\beta)} < (\lambda M_{\beta,h}^\beta)^{2/(2-\beta)}. \tag{25}$$

Choosing

$$\xi_3 = M_{\beta,h}^{-\beta}\Big(2\Big(\frac{\overline{\lambda}_{\beta+1,q}^{\beta+1}}{B - \varepsilon}\Big)^{2/(\beta-1)} + M_{\beta,h}^{2\beta/(2-\beta)} + \frac{4K_3\mu}{\mu - 2}\Big)^{(2-\beta)/2},$$

we have that for $\lambda < \xi_3$,

$$\limsup_{n\to\infty} \|u_n\|^2 < 2\biggl(\frac{\overline{\lambda}_{\beta+1,q}^{\beta+1}}{B-\varepsilon}\biggr)^{2/(\beta-1)} + M_{\beta,h}^{2\beta/(2-\beta)} + \frac{4K_3\mu}{\mu-2}. \tag{26}$$

From (26) we have that (u_n) is bounded in $H_0^1(0,\infty)$ and there exists $u\in H_0^1(0,\infty)$ such that $u_n \rightharpoonup u$ in $H_0^1(0,\infty)$. Now, we can repeat the same arguments employed in the proofs of Lemma 2.3 and Proposition 2.2 to conclude that $u_n\to u$ in $H_0^1(0,\infty)$. □

Proposition 2.3. *Suppose that f is a function satisfying* (H_0)–(H_4). *Then there exists a constant $\widehat{\xi}>0$ such that for $0<\lambda<\widehat{\xi}$, problem* (1) *has a positive solution u_2 satisfying $J_\lambda(u_2)<0$.*

Proof. Consider the complete metric space

$$\overline{B}_\varrho(0) := \{u\in H_0^1(0,\infty)\colon\ \|u\|\leqslant\varrho\}$$

with a metric given by $d(u,w)=\|u-w\|$. The functional J_λ is bounded from below on $\overline{B}_\varrho(0)$ for $\lambda<\xi_1$ (see Lemma 1.4). Note that

$$\forall\, t < \min\biggl\{\frac{\delta_1}{\|\overline{\psi}_{\beta,h}\|}, \frac{1}{\|\overline{\psi}_{\beta,h}\|_{\beta,h}}\biggl(\frac{2\lambda}{\beta\overline{\lambda}_{\beta,h}^2}\biggr)^{1/2-\beta}\biggr\}$$

(t near 0) using (H_3) in (8), we get

$$\begin{aligned} J_\lambda(t\overline{\psi}_{\beta,h}) &= \frac12 t^2\|\overline{\psi}_{\beta,h}\|^2 - \frac{\lambda}{\beta}t^\beta\|\overline{\psi}_{\beta,h}\|_{\beta,h}^\beta - \int_0^\infty q(x)F(t\overline{\psi}_{\beta,h})\,\mathrm{d}x \\ &\leqslant \frac12 t^2\|\overline{\psi}_{\beta,h}\|^2 - \frac{\lambda}{\beta}t^\beta\|\overline{\psi}_{\beta,h}\|_{\beta,h}^\beta - \frac{A-\varepsilon}{2}t^2\|\overline{\psi}_{\beta,h}\|_{2,q}^2 \\ &= \frac12 t^2\|\overline{\psi}_{\beta,h}\|^2\biggl(1-\frac{2\lambda t^{\beta-2}}{\beta\overline{\lambda}_{\beta,h}^2}\|\overline{\psi}_{\beta,h}\|_{\beta,h}^{\beta-2}\biggr) - \frac{A-\varepsilon}{2}t^2\|\overline{\psi}_{\beta,h}\|_{2,q}^2 < 0, \end{aligned} \tag{27}$$

by (17). Then, in view of (27), we see that

$$\inf_{u\in\overline{B}_\varrho(0)} J(u) < 0 < \inf_{u\in\partial\overline{B}_\varrho(0)} J(u). \tag{28}$$

Consequently, by applying Ekeland's variational principle in $\overline{B}_\varrho(0)$, there is a minimizing sequence $(u_n)_{n\geqslant1}\subset\overline{B}_\varrho(0)$ such that

$$J_\lambda(u_n)\to d := \inf\{J_\lambda(u)\colon\ u\in\overline{B}_\varrho(0)\}, \tag{29}$$

i.e.

$$J(u_n) \leqslant \inf_{u \in \overline{B}_\varrho(0)} J(u) + \frac{1}{n} \quad \forall\, n \geqslant 1,$$

and for every $w \in \overline{B}_\varrho(0)$ with $w \neq u_n$,

$$J_\lambda(w) - J_\lambda(u_n) + \frac{1}{n}\|u_n - w\| > 0. \tag{30}$$

Let $v \in H_0^1(0,\infty)$. We consider the sequence $w_n := u_n + tv \subset \overline{B}_\varrho(0)$, t near 0 (small enough), and for all $n \geqslant 1$. From (30) we obtain

$$\frac{1}{t}(J_\lambda(u_n + tv) - J_\lambda(u_n)) > -\frac{1}{n}\|v\|.$$

Thus, $\langle J'_\lambda(u_n), v\rangle \geqslant -n^{-1}\|v\|$ and similarly, $\langle J'_\lambda(u_n), (-v)\rangle \geqslant -n^{-1}\|v\|$. Therefore

$$|\langle J'_\lambda(u_n), v\rangle| < \frac{1}{n}\|v\| \quad \forall\, v \in H_0^1(0,\infty).$$

Consequently,

$$\|J'_\lambda(u_n)\| \to 0 \quad \text{as } n \to \infty. \tag{31}$$

Fix $\widehat{\xi} := \min\{\xi_1, \xi_3\}$, where ξ_1 and ξ_3 are given by Lemmas 2.1 and 2.5, respectively. Then from (29) and (31) it follows that $(u_n)_{n\geqslant 1}$ is a $(\mathrm{P.S})_d$ sequence for the functional J_λ for all $0 < \lambda < \widehat{\xi}$.

Using Lemma 2.5 and Propositions 2.2, we obtain a subsequence, still denoted by $(u_n)_{n\geqslant 1}$, which converges strongly to a function $u_2 \in H_0^1(0,\infty)$. In this case

$$J'_\lambda(u_2) = 0.$$

Now we will check $J_\lambda(u_2) < 0$ to complete the proof. Note that using (H_4) and (7) we obtain

$$\begin{aligned} d + o_n(1) = J_\lambda(u_n) - \frac{1}{\mu}J'_\lambda(u_n)u_n = \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2 - \lambda\Big(\frac{1}{\beta} - \frac{1}{\mu}\Big)\int_0^\infty h(x)\|u_n\|^\beta \\ - \int_0^\infty \Big(F(u_n) - \frac{1}{\mu}f(u_n)u_n\Big) + o_n(1). \end{aligned}$$

From Fatou's lemma (see [3], Lemma 4.1) we conclude that

$$d = \liminf_{n\to\infty}\Big(J_\lambda(u_n) - \frac{1}{\mu}J'_\lambda(u_n)u_n\Big) \geqslant J_\lambda(u_2) - \frac{1}{\mu}J'_\lambda(u_2)u_2.$$

Thus

$$J_\lambda(u_2) = d < 0.$$

□

R e m a r k 2.3. If u is a nontrivial solution for problem (1), by Remark 2.2, $u \geqslant 0$. Furthermore, as a consequence of (28) and $J_\lambda(0) = 0$, we have $u > 0$ in $(0, \infty)$.

P r o o f of Theorem 2.1. We take $\xi := \min\{\overline{\xi}, \widehat{\xi}\}$ and then the proof of Theorem 2.1 follows directly from Propositions 2.2, 2.3 and Remark 2.3. □

3. Example

In this section we give an example to illustrate our results.

E x a m p l e 3.1. Consider the problem

$$\begin{cases} -u'' + u = \lambda h(x)|u|^{\beta-2}u + q(x)f(u), & x \in [0, \infty), \\ u(0) = u(\infty) = 0, \end{cases} \tag{32}$$

where

$$f(u) = \begin{cases} \dfrac{1}{2M_{2,q}^2}|u| + (\overline{\lambda}_{2,q}^2 + 1)|u|^\beta & \text{if } |u| \leqslant 1, \\ (\overline{\lambda}_{2,q}^2 + 1)|u|^\beta + \dfrac{1}{2M_{2,q}^2} & \text{if } |u| \geqslant 1, \end{cases}$$

$q(x) = \frac{1}{4}D_2^{-1}\mathrm{e}^{-3x/2}$ and $h(x) = \mathrm{e}^{-4x/3}$. Choose $p(x) = \mathrm{e}^{-x/4}$ and we see that

$$\frac{q}{p}(x) = \frac{1}{4D_2}\mathrm{e}^{-5x/4}, \quad \frac{q}{p^2}(x) = \frac{1}{4D_2}\mathrm{e}^{-x}, \quad \frac{h}{p^{\beta-1}}(x) = \mathrm{e}^{(3\beta-19)x/12},$$
$$\frac{q}{p^\beta}(x) = \frac{1}{4D_2}\mathrm{e}^{(\beta-6)x/4}, \quad \frac{h}{p^\beta}(x) = \mathrm{e}^{x(\beta/4-4/3)} \quad \text{and} \quad \frac{q}{p^{\beta+1}}(x) = \frac{1}{4D_2}\mathrm{e}^{(\beta-5)x/4}$$

are in $L^1[0, \infty)$ for all $\beta \in (1, 2)$. Note that $\overline{\lambda}_{2,q} > M_{2,q}^{-1}$, and we also obtain that

$$M_{2,q} = \frac{\sqrt{2}}{\sqrt{15D_2}}, \quad A := \lim_{u\to 0^+} \frac{f(u)}{|u|} = \frac{1}{2M_{2,q}^2} \quad \text{and} \quad B := \lim_{u\to\infty} \frac{f(u)}{|u|^\beta} = \overline{\lambda}_{2,q}^2 + 1.$$

It is easy to see that conditions (H_0)–(H_4) hold. Thus from Theorem 2.1, (32) has at least two positive solutions for each $\lambda \in (0, \xi)$.

References

[1] *C. O. Alves*: Multiple positive solutions for equations involving critical Sobolev exponent in $\mathbb{R}^N$. Electron. J. Differ. Equ. *13* (1997), Paper No. 13, 10 pages. zbl MR

[2] *D. Bouafia, T. Moussaoui, D. O'Regan*: Existence of solutions for a second order problem on the half-line via Ekeland's variational principle. Discuss. Math. Differ. Incl. Control Optim. *36* (2016), 131–140. MR doi

[3] *H. Brezis*: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext. Springer, New York, 2010. zbl MR doi

[4] *I. Ekeland*: On the variational principle. J. Math. Anal. Appl. *47* (1974), 324–353. zbl MR doi

[5] *O. Frites, T. Moussaoui, D. O'Regan*: Existence of solutions via variational methods for a problem with nonlinear boundary conditions on the half-line. Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal. *22* (2015), 395–407. zbl MR

[6] *Y. Jabri*: The Mountain Pass Theorem. Variants, Generalizations and Some Applications. Encyclopedia of Mathematics and Its Applications 95. Cambridge University Press, Cambridge, 2003. zbl MR doi

[7] *M. Willem*: Minimax Theorems. Progress in Nonlinear Differential Equations and Their Applications 24. Birkhäuser, Boston, 1996. zbl MR doi

Authors' addresses: *Dahmane Bouafia*, *Toufik Moussaoui*, Laboratory of Fixed Point Theory and Applications, Department of Mathematics, École Normale Supérieure de Kouba, B.P. 92, Vieux-Kouba, 16308, Algiers, Algeria, e-mail: `alidahmane10@yahoo.fr`, `moussaoui@ens-kouba.dz`; *Donal O'Regan*, School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, University Road, H91 TK33 Galway, Ireland, e-mail: `donal.oregan@nuigalway.ie`.