ON IDEAL THEORY OF HOOPS

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Abstract. In this paper, we define and characterize the notions of (implicative, maximal, prime) ideals in hoops. Then we investigate the relation between them and prove that every maximal implicative ideal of a $\vee$-hoop with double negation property is a prime one. Also, we define a congruence relation on hoops by ideals and study the quotient that is made by it. This notion helps us to show that an ideal is maximal if and only if the quotient hoop is a simple MV-algebra. Also, we investigate the relationship between ideals and filters by exploiting the set of complements.

Keywords: Hoop; (implicative, maximal, prime) ideal; MV-algebra; Boolean algebra

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1. Introduction

Non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered in a frame of residuated lattices, see [9]. For example, Hájek's BL (basic logic, see [10]), Lukasiewiczs MV (many-valued logic, see [8]) and MTL (monoidal t-norm based logic, see [12]) are determined by the class of BL-algebras, MV-algebras and MTL-algebras, respectively. All of these algebras have lattices with residuation as a common support set. Thus, it is very important to investigate the properties of algebras with residuation. Hoops are naturally ordered commutative residuated integral monoids, introduced by Bosbach in [8] and [12] then studied by Büchi and Owens, a paper never published. In the last years, hoops theory was enriched with deep structure theorems (see [4], [8], [12]). Many of these results have a strong impact on fuzzy logic. Particularly, from the structure theorem of finite basic hoops ([4], Corollary 2.10) one obtains an elegant short proof of the completeness theorem for propositional basic logic (see [4], Theorem 3.8), introduced by Hájek in [10].

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The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops. The main example of BL-algebras in interval $[0,1]$ endowed with the structure induced by a t-norm. MV-algebras, product algebras and Gödel algebras are the best known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures. The filter theory plays an important role in studying these algebras. From the logic point of view, various filters have natural interpretation as various sets of provable formulas. At present, the filter theory of hoops has been widely studied and some important results are obtained. In particular, some types of filters such as (positive) implicative filters and fantastic filters (see [3]) were introduced and some of their characterizations were presented in [1], [2], [13], [11]. In MV-algebras, filters and ideals are dual notions, also we have to remark that residuated lattices and hoops are incomparable. Indeed, not all hoops are residuated lattices. It is noticeable that a hoop is a meet semi-lattice one with respect to the meet operator $a \wedge b = a \odot (a \to b)$ but it has not a lattice structure. So, in this paper we claim that the notion of ideals is missing in hoops. For this reason, in this paper, we define and characterize ideal, implicative, maximal and prime ideals notions in hoops. Then we investigate the relation between them and prove that every maximal implicative ideal of a $\vee$-hoop with double negation property is a prime one. Also, we define a congruence relation on a hoop by ideals and study the quotient that is made by it. This notion helps us to show that an ideal is maximal if and only if the quotient hoop is a simple MV-algebra. Also, we investigate the relationship between ideals and filters by exploiting the set of complements.

2. Preliminaries

First we recall the definition of a hoop. By a *hoop* we mean an algebraic structure $(A, \odot, \to, 1)$ where, for all $x, y, z \in A$:

(HP1) $(A, \odot, 1)$ is a commutative monoid;
(HP2) $x \to x = 1$;
(HP3) $(x \odot y) \to z = x \to (y \to z)$;
(HP4) $x \odot (x \to y) = y \odot (y \to x)$.

On a hoop A we define $x \leqslant y$ if and only if $x \to y = 1$. It is easy to see that $\leqslant$ is a partial order relation on A. A hoop A is *bounded* if there is an element $0 \in A$ such that $0 \leqslant x$ for all $x \in A$. Let A be a bounded hoop. We define negation "$\,'\,$" on A by $x' = x \to 0$ for all $x \in A$. If $x'' = x$ for all $x \in A$, then the bounded hoop A is said to have the *double negation property*, or (DNP), for short. Suppose A is a hoop such that for any $x, y \in A$, we have $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. If $\vee$

is a join operation on A, then the hoop A is called a $\vee$-*hoop*, which is a distributive lattice. The following proposition provides some properties of hoops.

Proposition 2.1 ([5], [6]). *Let $(A, \odot, \to, 1)$ be a hoop. Then the following conditions hold, for all $x, y, z, a \in A$:*

(i) *$(A, \leqslant)$ is a meet-semilattice, with $x \wedge y = x \odot (x \to y)$;*
(ii) *$x \odot y \leqslant z$ if and only if $x \leqslant y \to z$;*
(iii) *$x \odot y \leqslant x, y$ and $x \leqslant y \to x$;*
(iv) *$x \to x = 1$ and $1 \to x = x$;*
(v) *$x \leqslant y \to (x \odot y)$;*
(vi) *$x \to y \leqslant (y \to z) \to (x \to z)$;*
(vii) *$x \leqslant y$ implies $x \odot a \leqslant y \odot a$, $z \to x \leqslant z \to y$ and $y \to z \leqslant x \to z$;*
(viii) *if A is a bounded hoop, then $x \leqslant x''$, $x \odot x' = 0$ and $x''' = x'$;*
(ix) *if A is a $\vee$-hoop, then for any $n \in \mathbb{N}$, $(x \vee y)^n \to z = \{(a_1 \odot a_2 \odot \ldots \odot a_n) \to z\colon a_i \in \{x, y\}\}$;*
(x) *if A is a $\vee$-hoop, then $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.*

A nonempty subset F of A is a *filter* of A if (F1): $x, y \in F$ implies $x \odot y \in F$ and (F2): $x \in F$ and $x \leqslant y$ imply $y \in F$ for any $x, y \in A$. The set of all filters in a hoop A is denoted by $\mathcal{F}(A)$. F is a *proper filter* of a hoop A if F is a filter of A and $F \neq A$. If A is a hoop and $\emptyset \neq X \subseteq A$, then the intersection of all filters of A containing X is denoted by $\langle X\rangle$ and is characterized by

$$\begin{aligned}\langle X\rangle &= \{a \in A\colon x_1 \odot x_2 \odot \ldots \odot x_n \leqslant a \text{ for some } n \in \mathbb{N} \text{ and } x_1, \ldots, x_n \in X\}\\ &= \{a \in A\colon x_1 \to (x_2 \to (\ldots \to (x_n \to a) \ldots)) = 1\\ &\qquad\qquad\qquad\qquad\qquad \text{for some } n \in \mathbb{N},\ x_1, \ldots, x_n \in X\}.\end{aligned}$$

In particular, for any element $x \in A$ we have

$$\langle x\rangle = \{a \in A\colon x^n \leqslant a \text{ for some } n \in \mathbb{N}\} = \{a \in A\colon x^n \to a = 1 \text{ for some } n \in \mathbb{N}\}.$$

Let A and B be two hoops. A map $\varphi\colon A \to B$ is called a *homomorphism* if for all $x, y \in A$ we have $\varphi(1) = 1$, $\varphi(x \odot y) = \varphi(x) \odot \varphi(y)$ and $\varphi(x \to y) = \varphi(x) \to \varphi(y)$. If A and B are two bounded hoops, then $\varphi(0) = 0$ (see [3], [4], [5]).

Definition 2.2 ([3]). Let F be a nonempty subset of a hoop A. Then F is called a *positive implicative filter* of A if:

(PIF 1) $1 \in F$;
(PIF 2) $(x \odot y) \to z \in F$ and $x \to y \in F$ imply $x \to z \in F$ for any $x, y, z \in A$.

N o t a t i o n . From now on, in this paper, $(A, \odot, \rightarrow, 0, 1)$ or simply A is a bounded hoop, unless otherwise stated.

3. Ideal in hoops

In this section, we introduce the notion of an ideal in a hoop and investigate some of its properties.

Definition 3.1. Let I be a nonempty subset of A. I is called an *ideal* of A if it satisfies the following conditions:
(I1) $0 \in I$,
(I2) for any $x, y \in I$, $x' \rightarrow y \in I$,
(I3) for any $x, y \in A$, if $x \leqslant y$ and $y \in I$, then $x \in I$.

It is clear that A and $\{0\}$ are the trivial ideals of A. The set of all ideals of A is denoted by $\mathcal{ID}(A)$. I is called a *proper ideal* if I is an ideal of A and $I \neq A$. It can be easily seen that an ideal I is proper if and only if it is not containing 1.

E x a m p l e 3.2. Let $A = \{0, a, b, c, d, 1\}$. We define two operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Routine calculations show that A is a bounded hoop. It is easy to see that $I = \{0, a\} \in \mathcal{ID}(A)$.

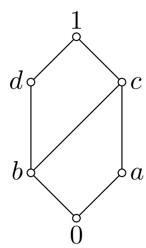

Figure 1. The Hasse diagram of A.

Let $\{I_\lambda \colon \lambda \in \Delta\}$ be a family of ideals in A. Then it is easy to see that $\bigcap\limits_{\lambda \in \Delta} I_\lambda$ is an ideal of A but $\bigcup\limits_{\lambda \in \Delta} I_\lambda$ is not an ideal of A, in general.

E x a m p l e 3.3. Let $A = \{0, a, b, 1\}$. We define two operations $\odot$ and $\rightarrow$ on A as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

By routine calculations, A with these operations is a bounded hoop. It is easy to see that $I_1 = \{0, a\}$ and $I_2 = \{0, b\}$ are two ideals of A, but $I_3 = I_1 \cup I_2 = \{0, a, b\}$ is not an ideal of A, because $a \ominus b = a' \rightarrow b = b \rightarrow b = 1 \notin I_3$.

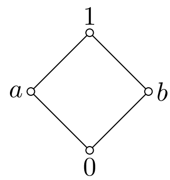

Figure 2. The Hasse diagram of A.

N o t a t i o n . For any $x, y \in A$, we define $x \ominus y = x' \rightarrow y$. Easily by an example we see that the operation $\ominus$ is not associative, but by adding (DNP) condition to hoop A, the operation $\ominus$ is associative, because since $y'' = y$, we have

$$\begin{aligned}(x \ominus y) \ominus z &= (x' \rightarrow y)' \rightarrow z = (x' \rightarrow y'')' \rightarrow z = ((x' \odot y') \rightarrow 0)' \rightarrow z \\ &= (x' \odot y')'' \rightarrow z = (x' \odot y') \rightarrow z = x' \rightarrow (y' \rightarrow z) = x \ominus (y \ominus z).\end{aligned}$$

R e m a r k 3.4. Let $I \in \mathcal{ID}(A)$. Then for any $x \in A$, $x \in I$ if and only if $x'' \in I$. By Proposition 2.1 (viii) and (I3), if $x'' \in I$, then it is easy to see that $x \in I$. Let $x \in I$, since $0 \in I$, by (I2), $x'' = x \ominus 0 \in I$.

Proposition 3.5. *Let I be a nonempty subset of A. Then, for any $x, y \in A$, the following statements are equivalent:*

(i) $I \in \mathcal{ID}(A)$,

(ii) $0 \in I$; *for any* $x, y \in I$, $x \ominus y \in I$ *and if* $x' \odot y \in I$ *and* $x \in I$, *then* $y \in I$.

(iii) $0 \in I$; *for any* $x, y \in I$, $x \ominus y \in I$ *and if* $(x' \rightarrow y')' \in I$ *and* $x \in I$, *then* $y \in I$.

P r o o f. (i) $\Rightarrow$ (ii): Let $I \in \mathcal{ID}(A)$. Then by Definition 3.1, $0 \in I$ and, for any $x, y \in I$, $x \ominus y \in I$. Now, suppose for any $x, y \in A$, $x' \odot y \in I$ and $x \in I$. Since $x' \odot y \leqslant x' \odot y$, by Proposition 2.1 (ii), $y \leqslant x' \rightarrow (x' \odot y)$. Also, since $x' \odot y \in I$ and $x \in I$, by (I2), $x' \rightarrow (x' \odot y) \in I$. Thus, by (I3), $y \in I$.

(ii) $\Rightarrow$ (iii): Suppose $x \leqslant y$ and $y \in I$. Then by Proposition 2.1 (vii), $y' \leqslant x'$, and so $(y' \rightarrow x')' = 0$. Hence, by (HP3) and Proposition 2.1 (viii), $y' \odot x \leqslant (y' \odot x)'' = 0$, and so $y'\odot x = 0 \in I$. Since $y \in I$, by (ii), $x \in I$. Now, let $(x' \to y')' \in I$ and $x \in I$ for any $x, y \in A$. By Proposition 2.1 (vii), $x' \odot y \leqslant (x' \odot y)'' = ((x' \odot y)')' = (x' \to y')'$. Then $x' \odot y \in I$. Since $x \in I$, by (ii), $y \in I$.

(iii) $\Rightarrow$ (i): It is clear that conditions (I1) and (I2) hold. Let, for any $x, y \in A$, $x \leqslant y$ and $y \in I$. By Proposition 2.1 (vii), $y' \leqslant x'$. Then $y' \to x' = 1$, and so $(y' \to x')' = 0 \in I$. Since $y \in I$, by (iii), $x \in I$. Hence, $I \in \mathcal{ID}(A)$. $\square$

In what follows, we investigate the relation between filters and ideals in any hoop A. For this, for any $\emptyset \neq X \subseteq A$, we define $X' = \{x \in A\colon\ x' \in X\}$.

Proposition 3.6. *If A has (DNP), then $I \in \mathcal{ID}(A)$ if and only if $I' = F \in \mathcal{F}(A)$.*

Proof. ($\Rightarrow$) Let $I \in \mathcal{ID}(A)$ and $F = I'$. Since $0 \in I$, we get that $1 \in F$. Suppose $x, y \in F$. Then $x', y' \in I$. Thus, $(x \odot y)' \odot x' \odot y' \leqslant y' \in I$, since $I \in \mathcal{ID}(A)$, we have $(x \odot y)' \odot x' \odot y' \in I$. Hence, $(x \odot y)' \in I$, and so $x \odot y \in F$. Now, suppose $x \leqslant y$ and $x \in F$. Then $x' \in I$ and by Proposition 2.1 (vii), $y' \leqslant x'$. Since $I \in \mathcal{ID}(A)$ and $x' \in I$, we get $y' \in I$, and so $y \in F$.

($\Leftarrow$) Let $F \in \mathcal{F}(A)$ and $I = F'$. Since $1 \in F$, we have $0 \in I$. Let $x \leqslant y$ and $y \in I$. Then $y' \leqslant x'$ and $y' \in F$. Since $F \in \mathcal{F}(A)$, we have $x' \in F$ and so $x'' \in I$. Hence, $x \in I$. Suppose $x, y \in I$. Then $x', y' \in F$. By Proposition 2.1 (viii),

$$(x' \odot y') \to (x' \to y)' = (x' \to y) \to ((x' \odot y') \to 0) = (x' \to y) \to (x' \to y'') = 1.$$

Hence, $(x' \odot y') \leqslant (x' \to y)'$. Since $F \in \mathcal{F}(A)$ and $x' \odot y' \in F$, we have $(x' \to y)' \in F$, and so $(x' \to y)'' \in I$. Then $x' \to y \in I$. $\square$

In the following example, we show that the condition (DNP) is necessary.

Example 3.7. Let $A = \{0, a, b, c, d, e, f, 1\}$. Define two operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	d	1	1	1
f	0	b	c	c	d	f	1	1
1	0	a	b	c	d	e	f	1

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

By routine calculation, we can see that A with these operations is a bounded hoop. Then it is easy to show that $I = \{0, a, b, c\} \in \mathcal{ID}(A)$, but $I' = \{1, d\} \notin \mathcal{F}(A)$ because $d \leqslant e, f$ and $e, f \notin F$.

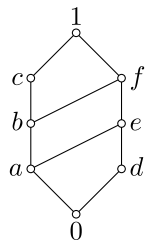

Figure 3. The Hasse diagram of A.

By Proposition 3.6 and Example 3.7, we can see that in hoops the notion of ideals is missing and filters and ideals are not dual notions, in general, except in a hoop with (DNP). So, the condition (DNP) is necessary.

Proposition 3.8. *Let A be a bounded $\vee$-hoop and $I \in \mathcal{ID}(A)$. Then for any $x, y \in A$, the following statements hold:*

(i) *$x, y \in I$ if and only if $x \vee y \in I$;*

(ii) *if $x, y \in I$, then $x \wedge y \in I$.*

Proof. Let $I \in \mathcal{ID}(A)$. By Proposition 2.1 (iii), $x \wedge y \leqslant x, y \leqslant x \vee y$. If $x, y \in I$ or $x \vee y \in I$, then by (I3) it is clear that $x \wedge y \in I$ or $x, y \in I$, respectively. Now, suppose $x, y \in I$. Then by Proposition 2.1 (x) and (viii), $x' \odot (x \vee y) = (x' \odot x) \vee (x' \odot y) = x' \odot y$. Since $x' \odot y \leqslant y$, $y \in I$ and $I \in \mathcal{ID}(A)$, we get $x' \odot y \in I$, so $x' \odot (x \vee y) \in I$. Since $x \in I$ and $I \in \mathcal{ID}(A)$, by Proposition 3.5, $x \vee y \in I$. $\square$

In the following example, we show that the converse of Proposition 3.8 (ii) may not be true, in general.

Example 3.9. According to Example 3.2, $I = \{0, a\} \in \mathcal{ID}(A)$ and $a \wedge b = a \odot (a \to b) = a \odot d = 0 \in I$ but $b \notin I$.

Proposition 3.10. *Let I be a subset of a hoop A such that $0 \in I$. Then the following conditions are equivalent:*

(i) *$I \in \mathcal{ID}(A)$;*

(ii) *$L(x, y) = \{z \in A \colon z \odot x' \leqslant y\} \subseteq I$ for any $x, y \in I$;*

(iii) *if $(z \odot x') \odot y' = 0$, then $z \in I$ for any $z \in A$ and $x, y \in I$.*

Proof. (i) $\Rightarrow$ (ii): Let $a \in L(x, y)$. Then $a \odot x' \leqslant y$. Since $y \in I$ and $I \in \mathcal{ID}(A)$, $a \odot x' \in I$. Now, by Proposition 3.5, since $x \in I$ and $I \in \mathcal{ID}(A)$, we get $a \in I$.

(ii) $\Rightarrow$ (iii): Let $x, y \in I$ and $(z \odot x') \odot y' = 0$. Since $0, y \in I$ and $(z \odot x') \odot y' \leqslant 0$, by (ii), we get $z \odot x' \in L(0, y) \subseteq I$, and so $z \odot x' \in I$. Moreover, since $z \odot x' \leqslant z \odot x'$ and $z \odot x', x \in I$, by (ii), $z \in L(z \odot x', x) \subseteq I$. Hence, $z \in I$.

(iii) $\Rightarrow$ (i): By assumption, $0 \in I$. Let $x, y \in I$. Then by Proposition 2.1 (ii), (vii) and (viii), $(x' \to y) \odot x' \odot y' \leqslant y \odot y' = 0$, and so by (iii), $x' \to y \in I$. Now, suppose $x' \odot y \in I$ and $x \in I$. Then by Proposition 2.1 (viii), $(y \odot x') \odot (x' \odot y)' = 0$, and so by (iii), $y \in I$. Hence, $I \in \mathcal{ID}(A)$. $\square$

Proposition 3.11. *Let I be a nonempty subset of A. Then for any $x, y, z \in A$, the following conditions are equivalent:*

(i) *if $I \in \mathcal{ID}(A)$ and $x \odot (y \odot x')' \in I$, then $x \in I$;*

(ii) *if $0 \in I$, $(x \odot (y \odot x')') \odot z' \in I$ and $z \in I$, then $x \in I$.*

Proof. (i) $\Rightarrow$ (ii): Since $I \in \mathcal{ID}(A)$, it is clear that $0 \in I$ and if $(x \odot (y \odot x')') \odot z' \in I$ and $z \in I$, then by (I2), $x \odot (y \odot x')' \in I$. Thus, by (i), $x \in I$.

(ii) $\Rightarrow$ (i): First we prove that $I \in \mathcal{ID}(A)$. For this, suppose $x \odot y' \in I$ and $y \in I$. Then $(x \odot (0 \odot x')') \odot y' = x \odot y' \in I$, and so by (ii), $x \in I$. Now, let $x, y \in I$. Then

$$((x' \to y) \odot (0 \odot (x' \to y)')') \odot y' \odot x' \leqslant x'' \odot x' = 0 \in I.$$

Since $x \in I$, by (ii), $(x' \to y) \odot y' \in I$. Also, from $y \in I$, we have $x' \to y \in I$. Hence, $I \in \mathcal{ID}(A)$. Now, suppose $x \odot (y \odot x')' \in I$. By considering $z = 0$ in (ii), we get $x \in I$. $\square$

Definition 3.12. Let $\emptyset \neq X \subseteq A$. We recall that the smallest ideal containing X in A is called the *ideal generated by X in A* and is denoted by $(X]$. It is also the intersection of all ideals of A containing X.

Theorem 3.13. *Let $\emptyset \neq X \subseteq A$. Then*

$$(X] = \{a \in A \colon \exists n \in \mathbb{N} \colon a \leqslant x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots) \quad \text{for } x_1, x_2, \ldots, x_n \in X\}.$$

Proof. Let $B = \{a \in A \colon \exists n \in \mathbb{N} \colon a \leqslant x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots)$ for $x_1, x_2, \ldots, x_n \in X\}$. It is enough to prove that B is the smallest ideal containing X. For this, first we show that B is an ideal of A. Since, for any $x_1, x_2 \in X$, we have $0 \leqslant x_1 \ominus x_2$, $0 \in B$, and so (I1) holds. Now, let $a, b \in A$ such that $a \leqslant b$ and $b \in B$. Since $b \in B$, there exists $n \in \mathbb{N}$ such that for $x_1, x_2, \ldots, x_n \in X$, $b \leqslant x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots)$. From $a \leqslant b$, we get $a \leqslant b \leqslant x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots)$, thus, $a \leqslant x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots)$, and so $a \in B$. Hence, (I3) holds. Now, suppose $a, b \in B$. Then there exist $n, m \in \mathbb{N}$ such that $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in X$, $a \leqslant x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots)$ and $b \leqslant y_1 \ominus (y_2 \ominus \ldots \ominus (y_{m-1} \ominus y_m) \ldots)$. By Proposition 2.1 (vii), $(x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots))' \leqslant a'$, and so $a' \to b \leqslant$

$(x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots))' \to b$. Since $b \leqslant y_1 \ominus (y_2 \ominus \ldots \ominus (y_{m-1} \ominus y_m) \ldots)$, by Proposition 2.1 (vii),

$$(x_1 \ominus (\ldots \ominus (x_{n-1} \ominus x_n) \ldots))' \to b \leqslant (x_1 \ominus (\ldots \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots))' \\ \to (y_1 \ominus \ldots \ominus \ldots \ominus (y_{m-1} \ominus y_m) \ldots)).$$

Then $a' \to b \leqslant (x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots))' \to (y_1 \ominus (y_2 \ominus \ldots \ominus (y_{m-1} \ominus y_m) \ldots))$, and so $a \ominus b \leqslant x_1 \ominus (x_2 \ominus \ldots \ominus x_n \ominus y_1 \ominus y_2 \ominus \ldots (y_{m-1} \ominus y_m) \ldots)$. Thus, $a \ominus b \in B$. Hence, B is an ideal of A. It is clear that $X \subseteq B$, because, for any $x, y \in X$, by Proposition 2.1 (iii), $x \leqslant y' \to x = y \ominus x$. Hence, $x \in B$. Now, let there exist $C \in \mathcal{ID}(A)$ such that $X \subseteq C$. It is enough to prove that $B \subseteq C$. Let $a \in B$. Then there exists $n \in \mathbb{N}$ such that for $x_1, x_2, \ldots, x_n \in X$, we have $a \leqslant x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots)$. Since $X \subseteq C$ and $C \in \mathcal{ID}(A)$, by (I2), $x_1 \ominus (x_2 \ominus \ldots \ominus (x_{n-1} \ominus x_n) \ldots) \in C$, and so by (I3), $a \in C$. Hence, B is the smallest ideal of A containing X. Therefore, $B = (X]$. □

N o t a t i o n . Consider $a \ominus (a \ominus \ldots \ominus (a \ominus a) \ldots) = na = (a')^{n-1} \to a$. If A has (DNP), then $x \ominus y = y \ominus x$ and $na = ((a')^n)'$.

Proposition 3.14. *Let $I \in \mathcal{ID}(A)$ and $a \in A$. Then the following statements hold:*

(i) $(a] = \{x \in A \colon \exists n \in \mathbb{N} \colon x \leqslant na\}$;
(ii) *if A is a hoop with (DNP), then* $(I \cup \{a\}] = \{x \in A \colon \exists n \in \mathbb{N} \colon x \odot (na)' \in I\}$;
(iii) *if A is a $\vee$-hoop with (DNP), then* $(I \cup \{x\}] \cap (I \cup \{y\}] = (I \cup \{x \wedge y\}]$.

P r o o f. (i) Let $B = \{x \in A \colon \exists n \in \mathbb{N} \colon x \leqslant na\}$. Then we prove that B is the smallest ideal of A generated by a. For this, we show that B is an ideal of A. Since $0 \leqslant a \ominus a = 2a$, it is clear that $0 \in B$. Let $x \leqslant y$ for $x, y \in A$ and $y \in B$. Then there exists $n \in \mathbb{N}$ such that $x \leqslant y \leqslant na$. So, $x \in B$. Now, suppose $x, y \in B$. Then there exist $n, m \in \mathbb{N}$ such that $x \leqslant na$ and $y \leqslant ma$, and so $x \leqslant (a')^{n-1} \to a$ and $y \leqslant (a')^{m-1} \to a$. By Proposition 2.1 (vii), $((a')^{n-1} \to a)' \leqslant x'$ and $x' \to y \leqslant ((a')^{n-1} \to a)' \to y$. Also,

$$((a')^{n-1} \to a)' \to y \leqslant ((a')^{n-1} \to a)' \to ((a')^{m-1} \to a)$$

and so,

$$x' \to y \leqslant ((a')^{n-1} \to a)' \to y \leqslant ((a')^{n-1} \to a)' \to ((a')^{m-1} \to a).$$

Hence,

$$x' \to y \leqslant ((a')^{n-1} \to a)' \to ((a')^{m-1} \to a) = na \ominus ma = (n+m)a.$$

Thus, $x \ominus y \in B$. On the other hand, by Proposition 2.1 (viii), $a \leqslant a' \to a = a \ominus a$, and so B is an ideal of A containing a. Suppose C is an ideal of A containing a. Let $x \in B$. Then there exists $n \in \mathbb{N}$ such that $x \leqslant na$. Since $a \in C$ and $C \in \mathcal{ID}(A)$, hence $na \in C$, and so $x \in C$. Hence, $B \subseteq C$. Therefore, $(a] = \{x \in A\colon \exists n \in \mathbb{N}\colon x \leqslant na\}$.

(ii) Let $E = \{x \in A\colon \exists n \in \mathbb{N}\colon x \odot (na)' \in I\}$. Since $a \odot a' = 0 \in I$ and for any $x \in I$, $x \odot a' \leqslant x \in I$ and $I \in \mathcal{ID}(A)$, so it is clear that $I \cup \{a\} \subseteq E$. Let $x, y \in E$. Then there exist $n, m \in \mathbb{N}$ such that $x \odot (na)' \in I$ and $y \odot (ma)' \in I$. Thus, there exist $\alpha, \beta \in I$ such that $x \odot (na)' \leqslant \alpha$ and $y \odot (ma)' \leqslant \beta$. Hence, by (HP3), $x \leqslant (na)' \to \alpha$ and $y \leqslant (ma)' \to \beta$. Then by Proposition 2.1 (vii),

$$x' \to y \leqslant ((na)' \to \alpha)' \to y \leqslant ((na)' \to \alpha)' \to ((ma)' \to \beta)' = (na) \ominus \alpha \ominus (ma) \ominus \beta.$$

Since A has (DNP), we get that $(x' \to y) \odot ((n+m)a)' \leqslant \alpha \ominus \beta \in I$. The proof of the other cases is similar to (i).

(iii) By definition of $\ominus$, it is easy to see that $a \odot (nx)' = a \odot (x')^n$. Suppose $a \in (I \cup \{x \wedge y\}]$. Then there exists $n \in \mathbb{N}$ such that $a \odot (n(x \wedge y))' \in I$. Thus, by Proposition 2.1 (ii), (vii) and (DNP), we have,

$$a \odot (n(x \wedge y))' = a \odot ((x \wedge y)')^n \geqslant a \odot (x')^n, \quad a \odot (y')^n = a \odot (nx)' \quad \text{and} \quad a \odot (ny)'.$$

Then $a \in (I \cup \{x\}] \cap (I \cup \{y\}]$, and so $(I \cup \{x \wedge y\}] \subseteq (I \cup \{x\}] \cap (I \cup \{y\}]$. Conversely, let $a \in (I \cup \{x\}] \cap (I \cup \{y\}]$. Then there exist $n, m \in \mathbb{N}$ such that $a \odot (nx)'$ and $a \odot (my)' \in I$, and so $a \odot (x')^n$ and $a \odot (y')^m \in I$. Let $U = a \odot (x')^n$ and $V = a \odot (y')^m$. Then by (HP3), $U' = (a \odot (x')^n)' = (x')^n \to a'$ and $V' = (a \odot (y')^m)' = (y')^m \to a'$. By routine calculations, we can see that,

$$(x')^n \to (V' \to (U' \to a')) = 1 \quad \text{and} \quad (y')^m \to (V' \to (U' \to a')) = 1.$$

Then by Proposition 2.1 (ix), there exists $p \in \mathbb{N}$ such that

$$\begin{aligned} &[(x \wedge y)']^p \to (V' \to (U' \to a')) \\ &\quad = (x' \vee y')^p \to (V' \to (U' \to a')) \\ &\quad = \bigwedge \{(a_1 \odot \ldots \odot a_p) \to (V' \to (U' \to a'))\colon a_i \in \{x', y'\}\} = 1. \end{aligned}$$

Hence, $a \in (I \cup \{x \wedge y\}]$. □

Proposition 3.15. *Let $\varphi\colon A \to B$ be a hoop homomorphism. Then the following statements hold:*

(i) *if φ is an epimorphism and $I \in \mathcal{ID}(B)$, then $\varphi^{-1}(I) \in \mathcal{ID}(A)$;*

(ii) *if φ is an isomorphism and $I \in \mathcal{ID}(A)$, then $\varphi(I) \in \mathcal{ID}(B)$;*

(iii) *if $\ker \varphi = \{x \in A\colon \varphi(x) = 0\}$, then $\ker \varphi \in \mathcal{ID}(A)$.*

P r o o f. (i) Let $I \in \mathcal{ID}(B)$. Since $\varphi(0) = 0 \in I$, we have $0 \in \varphi^{-1}(I)$. Suppose $x \leqslant y$ and $y \in \varphi^{-1}(I)$. Then $\varphi(y) \in I$. Since $x \to y = 1$ and φ is a homomorphism, we obtain that $\varphi(x) \leqslant \varphi(y)$, thus, $\varphi(x) \in I$, and so $x \in \varphi^{-1}(I)$. Let $x, y \in \varphi^{-1}(I)$. Then $\varphi(x), \varphi(y) \in I$. Since $I \in \mathcal{ID}(B)$, $\varphi(x' \to y) = \varphi'(x) \to \varphi(y) \in I$. Thus, $x' \to y \in \varphi^{-1}(I)$ and so $\varphi^{-1}(I) \in \mathcal{ID}(A)$.

(ii) Let $I \in \mathcal{ID}(A)$. It is clear that $0 \in \varphi(I)$. Let $x \leqslant y$ and $y \in \varphi(I)$. Since $y \in \varphi(I)$, there exists $a \in I$ such that $\varphi(a) = y$. Since $x = \varphi(b) \leqslant \varphi(a) = y$, we have $1 = \varphi(1) = \varphi(b \to a) = \varphi(b) \to \varphi(a)$. From the fact that φ is an isomorphism, we get $b \leqslant a$. Moreover, since $I \in \mathcal{ID}(A)$ and $a \in I$, we obtain $b \in I$, and so $x = \varphi(b) \in \varphi(I)$. Now, suppose $x, y \in \varphi(I)$. Then there exist $a, b \in I$ such that $\varphi(a) = x$ and $\varphi(b) = y$. Since $I \in \mathcal{ID}(A)$, $a' \to b \in I$, and so $x' \to y = \varphi'(a) \to \varphi(b) \in \varphi(I)$. Hence, $\varphi(I) \in \mathcal{ID}(B)$.

(iii) Let $\ker\varphi = \{x \in A\colon \varphi(x) = 0\}$. Since $\varphi(0) = 0$, we have $0 \in \ker\varphi$, and so $\ker\varphi \neq \emptyset$. Suppose $x, y \in \ker\varphi$. Then $\varphi(x) = \varphi(y) = 0$, and so $\varphi(x' \to y) = \varphi'(x) \to \varphi(y) = 0' \to 0 = 0$. Hence, $x' \to y \in \ker\varphi$. Let $x \leqslant y$ and $y \in \ker\varphi$. Since φ is monotone, $\varphi(x) \leqslant \varphi(y) = 0$. Then $\varphi(x) = 0$ and so $x \in \ker\varphi$. Hence, $\ker\varphi \in \mathcal{ID}(A)$. □

Let A be a hoop with (DNP) and I an ideal of A. Define the relation $\sim_I$ on A by $x \sim_I y$ if and only if $x' \odot y \in I$ and $y' \odot x \in I$ for any $x, y \in A$. Similarly to the proof of [7], Theorem 4.2 and Proposition 4.3, we can see that $\sim_I$ is a congruence relation on A. For any $x \in A$, we denote by x/I the equivalence class of x, that is, $x/I = \{y \in A\colon x \sim_I y\}$. Let $A/I = \{x/I\colon x \in A\}$ and we define on the set A/I the operations

$$x/I \otimes y/I = (x \odot y)/I, \quad x/I \rightsquigarrow y/I = (x \to y)/I, \quad 0/I = I,$$
$$\text{and} \quad 1/I = \{x' \in I\colon x \in A\}.$$

Also, we define a partial order on A/I by $x/I \leqslant y/I$ if and only if $x' \odot y \in I$. Then by routine calculation we can prove that $(A/I, \otimes, \rightsquigarrow, 0/I, 1/I)$ is a hoop.

N o t a t i o n. The quotient hoop via any ideal is always an MV-algebra, because by Proposition 2.1 (viii), $x \odot (x'')' = x \odot x' = 0 \in I$ and $x'' \odot x' = 0$. Then $x \sim x''$, and so $x/I = x''/I$.

4. Implicative ideal in hoops

In this section, we introduce the notion of an implicative ideal in hoops and investigate some of its properties. Then we study the quotient structures that are made by an implicative ideal.

Definition 4.1. Let $\emptyset \neq I \subseteq A$. Then I is called an *implicative ideal* of A if for any $x, y, z \in A$ it satisfies the following conditions:

(IM1) $0 \in I$;
(IM2) if $x, y \in I$, then $x \ominus y \in I$;
(IM3) if $x \odot y' \odot z' \in I$ and $y \odot z' \in I$, then $x \odot z' \in I$.

Example 4.2. According to Example 3.7, it is easy to check that $I = \{0, a, b, c\}$ is an implicative ideal of A.

Proposition 4.3. *If A has (DNP), then I is an implicative ideal if and only if $I' = F$ is a positive implicative filter of A.*

Proof. Proof is similar to the proof of Proposition 3.6. □

Theorem 4.4. *Every implicative ideal of A is an ideal of A.*

Proof. Suppose I is an implicative ideal of A and $x \leqslant y$ such that $y \in I$. Then $x \to y = 1$. By Proposition 2.1 (vii) and (viii), $x \to y \leqslant x \to y'' = (x \odot y')'$. Thus, $(x \odot y')' = 1$, and so $(x \odot y')'' = 0 \in I$. Since $x \odot y' \leqslant (x \odot y')''$, we get that $x \odot y' = 0 \in I$. Let $z = 0$. Then $x \odot y' \odot z' = x \odot y' = 0 \in I$ and $y \odot z' = y \in I$. Since I is an implicative ideal of A, $x \odot z' = x \in I$. Hence, $I \in \mathcal{ID}(A)$. □

By the following example we show that the converse of the above theorem may not be true, in general.

Example 4.5. In Example 3.2, $I = \{0, a\}$ is an ideal of A. But it is not an implicative ideal of A. Because if we let $x = 1$, $y = c$ and $z = b$, then $x \odot y' \odot z' = b \odot c = 0 \in I$. Also, $y \odot z' = c \odot c = a \in I$, but $x \odot z' = c \notin I$.

Proposition 4.6. *Let $I \in \mathcal{ID}(A)$. Then for any $x, y \in A$, the following statements are equivalent:*

(i) *I is an implicative ideal;*
(ii) *if $x \odot y'' \odot y'' \in I$, then $x \odot y \in I$;*
(iii) *if $x^2 \in I$, then $x \in I$;*
(iv) *$\{x \in A\colon\ x^2 = 0\} \subseteq I$.*

Proof. (i) ⇒ (ii): Let $x \odot y'' \odot y'' \in I$. Since I is an implicative ideal of A and by Proposition 2.1 (vii), $y' \odot y'' = 0 \in I$, we get $x \odot y'' \in I$. From $x \odot y \leqslant x \odot y''$ and $I \in \mathcal{ID}(A)$, we get $x \odot y \in I$.

(ii) ⇒ (i): Let $x \odot y' \odot z' \in I$ and $y \odot z' \in I$ for any $x, y, z \in A$. Then by Proposition 2.1 (vii), $(x \odot z' \odot z') \odot (y \odot z')' = (x \odot z' \odot z') \odot (z' \to y') \leqslant x \odot z' \odot y'$. Since $I \in \mathcal{ID}(A)$ and $x \odot y' \odot z' \in I$, we get that $(x \odot z' \odot z') \odot (y \odot z')' \in I$. Then by Propositions 3.5 and 2.1 (viii), $x \odot (z')'' \odot (z')'' = x \odot z' \odot z' \in I$. Thus, by (ii), $x \odot z' \in I$. Hence, I is an implicative ideal of A.

(ii) $\Rightarrow$ (iii): Let $x^2 \in I$. Since $I \in \mathcal{ID}(A)$, by Remark 3.4, $(x^2)'' \in I$. By Proposition 2.1 (viii), we have

$$\begin{aligned}(x'' \odot x'') \to (x^2)'' &= (x'' \odot x'') \to ((x^2)' \to 0)\\ &= x'' \to (x'' \to ((x^2)' \to 0))\\ &= x'' \to ((x^2)' \to x')\\ &= (x^2)' \to (x'' \to x')\\ &= (x^2)' \to (x \to x')\\ &= x \to ((x^2)' \to x')\\ &= x \to (x \to (x^2)'')\\ &= x^2 \to (x^2)'' = 1.\end{aligned}$$

So, $x'' \odot x'' \leqslant (x^2)''$. Since $I \in \mathcal{ID}(A)$ and $(x^2)'' \in I$, we get $1 \odot x'' \odot x'' = x'' \odot x'' \in I$. Then by (ii), $1 \odot x = x \in I$.

(iii) $\Rightarrow$ (ii): Let $x \odot y'' \odot y'' \in I$ for any $x, y \in A$. By Proposition 2.1 (viii) and (vii), $y \leqslant y''$, then $y \odot y \leqslant y'' \odot y''$. Since $x^2 \leqslant x$, we have $x^2 \odot y^2 \leqslant x \odot y'' \odot y''$. Since $x \odot y'' \odot y'' \in I$ and $I \in \mathcal{ID}(A)$, we obtain that $(x \odot y)^2 \in I$. Then by (iii), $x \odot y \in I$.

(iii) $\Rightarrow$ (iv): Let $a \in \{x \in A\colon\ x^2 = 0\}$. Then $a^2 = 0 \in I$, by (iii), $a \in I$, and so $\{x \in A\colon\ x^2 = 0\} \subseteq I$.

(iv) $\Rightarrow$ (iii): Suppose $x^2 \in I$ for any $x \in A$. Then $x^2/I = 0/I$. Since $\{0/I\} \in \mathcal{ID}(A/I)$, by (iv), $x/I \in 0/I$. Hence, $x \in I$. □

Corollary 4.7. *Let I be an implicative ideal of A and $J \in \mathcal{ID}(A)$ such that $I \subseteq J$. Then J is an implicative ideal of A.*

Proof. Since $I \subseteq J$ and I is an implicative ideal, then by Proposition 4.6, $\{x \in A\colon\ x^2 = 0\} \subseteq I \subseteq J$. As $J \in \mathcal{ID}(A)$, hence by Proposition 4.6, J is an implicative ideal of A. □

Corollary 4.8. *Let $I \in \mathcal{ID}(A)$. If I is an implicative ideal of A, then $x' \wedge x \in I$ for any $x \in A$.*

Proof. Let I be an implicative ideal of A. Since $x \wedge x' \leqslant x, x'$, we obtain that $(x \wedge x')^2 \leqslant x \odot x' = 0$. Then $(x \wedge x')^2 = 0$. Thus, by Proposition 4.6, $\{x \in A\colon\ x^2 = 0\} \subseteq I$, and so $x \wedge x' \in I$. □

In the following example, we show that the converse of Corollary 4.8 may be not true, in general.

E x a m p l e 4.9. According to Example 3.2, $I = \{0, a\} \in \mathcal{ID}(A)$. But, $d \wedge d' = d \odot (d \to d') = d \odot a = 0 \in I$ provided that, in Example 4.5, we show that I is not an implicative ideal of A.

Proposition 4.10. *Let $I \in \mathcal{ID}(A)$. If I is an implicative ideal of A such that $x'' \odot y \odot z' \in I$ and $x \odot y' \in I$, then $x \odot z' \in I$.*

P r o o f. Let for any $x, y, z \in A$, $x'' \odot y \odot z' \in I$. Then by Proposition 2.1 (vii) and (viii) we have

$$\begin{aligned}(x'' \odot y \odot z')' \odot (x'' \odot y'' \odot z') &= ((y \odot z') \to x') \odot (x'' \odot y'' \odot z') \\ &= z' \odot (z' \to (y \to x')) \odot x'' \odot y'' \\ &\leqslant (y \to x') \odot x'' \odot y'' \\ &= (y \to x') \odot (x' \to 0) \odot y'' \\ &\leqslant y' \odot y'' = 0 \in I.\end{aligned}$$

Since $I \in \mathcal{ID}(A)$, $(x'' \odot y \odot z')' \odot (x'' \odot y'' \odot z') \in I$, and so $x'' \odot y'' \odot z' \in I$. Also, from $x \odot y' \in I$, we have

$$(x \odot y')' \odot (y' \odot x'') = y' \odot (y' \to x') \odot x'' \leqslant x' \odot x'' = 0 \in I.$$

Since $I \in \mathcal{ID}(A)$, $(x \odot y')' \odot (y' \odot x'') \in I$, and so $y' \odot x'' \in I$. Since I is an implicative ideal of A, we get that $z' \odot x'' \in I$. Moreover, by Proposition 2.1 (viii), $z' \odot x \leqslant z' \odot x''$, and so $x \odot z' \in I$. □

By the next example, we can show that the converse of Proposition 4.10 may be not true, in general.

E x a m p l e 4.11. According to Example 3.2, $I = \{0, a\} \in \mathcal{ID}(A)$. Let $y \odot x'' \odot z' = b \odot c'' \odot d' = b \odot c \odot a = 0 \in I$ and $x \odot y' = c \odot b' = c \odot c = a \in I$. By assumption, $x \odot z' = c \odot d' = c \odot a = a \in I$. But by Example 4.5, we show that I is not an implicative ideal of A.

Proposition 4.12. *Let I be a nonempty subset of A. Then for any $x, y, z \in A$, the following conditions are equivalent:*
(i) *I is an implicative ideal of A;*
(ii) *$I \in \mathcal{ID}(A)$ and if $(x \odot y') \odot y' \in I$, then $x \odot y' \in I$;*
(iii) *$I \in \mathcal{ID}(A)$ and if $(x \odot y') \odot z' \in I$, then $(x \odot z') \odot (y \odot z')' \in I$;*
(iv) *$0 \in I$ and if $((x \odot y') \odot y') \odot z' \in I$ and $z \in I$, then $x \odot y' \in I$.*

P r o o f. (i) $\Rightarrow$ (ii): By Theorem 4.4, I is an ideal of A. Let $(x \odot y') \odot y' \in I$. Since I is an implicative ideal of A and by Proposition 2.1 (viii), $y \odot y' = 0 \in I$, we have $x \odot y' \in I$.

(ii) $\Rightarrow$ (iii): Let $(x \odot y') \odot z' \in I$. Then by Proposition 2.1 (vi),

$$x \odot (y \odot z')' \odot z' \odot z' = x \odot z' \odot z' \odot (z' \to y') \leqslant (x \odot y') \odot z' \in I$$

and $I \in \mathcal{ID}(A)$; thus, $x \odot (y \odot z')' \odot z' \odot z' \in I$. By (ii), $(x \odot z') \odot (y \odot z')' = x \odot (y \odot z')' \odot z' \in I$.

(iii) $\Rightarrow$ (i): Suppose $(x \odot y') \odot z' \in I$ and $y \odot z' \in I$. Then by (iii), $(x \odot z') \odot (y \odot z')' \in I$. Since $y \odot z' \in I$ and $I \in \mathcal{ID}(A)$, $x \odot z' \in I$. Hence, I is an implicative ideal of A.

(ii) $\Rightarrow$ (iv): Since $I \in \mathcal{ID}(A)$, $0 \in I$. Let $((x \odot y') \odot y') \odot z' \in I$ and $z \in I$. Since $I \in \mathcal{ID}(A)$, we have $(x \odot y') \odot y' \in I$. By (ii), $x \odot y' \in I$.

(iv) $\Rightarrow$ (ii): First we prove that $I \in \mathcal{ID}(A)$. For this, suppose $x \odot y' \in I$ and $y \in I$. We have $x \odot y' = x \odot 0' \odot 0' \odot y' \in I$. Then by (iv), $x \in I$. Now, let $x, y \in I$. By Proposition 2.1 (viii), $(x' \to y) \odot x' \odot x' \odot y' \leqslant x'' \odot x' \odot x' = 0 \in I$. By (I2), $(x' \to y) \odot x' \odot x' \odot y' \in I$. Since $y \in I$, by (iv), $(x' \to y) \odot x' \in I$. Also, from $x \in I$, we obtain that $x' \to y \in I$. Hence, $I \in \mathcal{ID}(A)$. Now, suppose $(x \odot y') \odot y' \in I$. Let $z = 0$ in (iv). Then by (iv), $x \odot y' \in I$. $\square$

N o t a t i o n. As we know, a *Boolean algebra* is a structure $(B, +, \cdot, -, 0, 1)$, with two binary operations "$+$" and "$\cdot$", a unary operation "$-$" and two distinguished elements 0 and 1 such that B with these operations makes a complemented distributive commutative algebra.

Theorem 4.13. *If A is a bounded $\vee$-hoop with (DNP), then I is an implicative ideal if and only if A/I is a Boolean algebra.*

P r o o f. ($\Rightarrow$) Let I be an implicative ideal of A. Then by Proposition 4.6, for any $a \in A$, $a \wedge a' \in I$. Since $a/I \wedge a'/I = (a \wedge a')/I$, we get $a/I \wedge a'/I = 0/I$. Also, since $(a/I \vee a'/I)' = (a \vee a')'/I = (a' \wedge a'')/I = 0/I$, we have $(a/I \vee a'/I)'' = 1/I$. Since A has (DNP), $a/I \vee a'/I = 1/I$. Hence, A/I is a Boolean algebra.

($\Leftarrow$) Suppose A/I is a Boolean algebra, $x \odot y' \odot z' \in I$ and $y \odot z' \in I$ for any $x, y, z \in A$. Since $x \odot y' \odot z' \in I$ and $y \odot z' \in I$, we have $(x \odot y' \odot z')/I = 0/I$ and $(y \odot z')/I = 0/I$. Moreover, since A/I is a Boolean algebra, $y/I \vee y'/I = 1/I$. Then by Proposition 2.1 (x),

$$\begin{aligned}(x \odot z')/I &= x/I \odot z'/I \odot 1/I = x/I \odot z'/I \odot (y/I \vee y'/I) \\ &= (x/I \odot z'/I \odot y/I) \vee (x/I \odot z'/I \odot y'/I) = 0/I.\end{aligned}$$

Hence, $x \odot z' \in I$. Therefore, I is an implicative ideal of A. $\square$

Proposition 4.14.

(i) *If $\varphi\colon A \to B$ is a hoop homomorphism and I is an implicative ideal of B, then $\varphi^{-1}(I)$ is an implicative ideal of A.*

(ii) *If $\varphi\colon A \to B$ is surjective and I is an implicative ideal of A, then $\varphi(I)$ is an implicative ideal of B.*

P r o o f. Similar to the proof of Proposition 3.15. $\square$

Proposition 4.15. *Let $I \in \mathcal{ID}(A)$. Then I is an implicative ideal of A if and only if, for any $a \in A$, the set $I_a = \{x \in A\colon x \odot a' \in I\}$ is the least ideal of A containing I and $\{a\}$.*

P r o o f. ($\Rightarrow$) Let $a \in A$. Since $0 \in A$ and $0 \odot a' = 0 \in I$, we have $0 \in I_a \neq \emptyset$. Suppose $x' \odot y \in I_a$ and $x \in I_a$. Then $(x' \odot y) \odot a' \in I$ and $x \odot a' \in I$. Since I is an implicative ideal, we get that $y \odot a' \in I$, and so $y \in I_a$. Now, let $x, y \in I_a$. Then $x \odot a' \in I$ and $y \odot a' \in I$. By Proposition 2.1 (vi), we have, $(x' \to y) \odot x' \odot a' = x' \odot (x' \to y) \odot a' \leqslant y \odot a' \in I$. Since $I \in \mathcal{ID}(A)$ and $y \odot a' \in I$, we have $(x' \to y) \odot x' \odot a' \in I$. Moreover, from $x \odot a' \in I$ and the fact that I is an implicative ideal, we obtain that $(x' \to y) \odot a' \in I$. Then $x' \to y \in I_a$. Hence, I_a is an ideal of A. Also, since $a \odot a' = 0 \in I$, we get $a \in I_a$. Let $x \in I$. By Proposition 2.1 (iii), $x \odot a' \leqslant x$. Since $I \in \mathcal{ID}(A)$ and $x \in I$, we have $x \odot a' \in I$, and so $x \in I_a$. Hence, $I \subseteq I_a$. Now, suppose there exists $J \in \mathcal{ID}(A)$ such that $I \cup \{a\} \subseteq J$. Let $x \in I_a$. Then $x \odot a' \in I \subseteq J$, and so $x \odot a' \in J$. Since $J \in \mathcal{ID}(A)$ and $a \in J$, we have $x \in J$. Hence, $I_a \subseteq J$. Therefore, I_a is the least ideal of A containing I and $\{a\}$.

($\Leftarrow$) Let $I \in \mathcal{ID}(A)$ and for any $x, y, z \in A$, let $x \odot y' \odot z' \in I$ and $y \odot z' \in I$. According to the definition of I_a, it is clear that $x \odot y' \in I_z$ and $y \in I_z$. Since $I_z \in \mathcal{ID}(A)$, we get that $x \in I_z$, and so $x \odot z' \in I$. Hence, I is an implicative ideal of A. $\square$

Proposition 4.16. *Let I, J be two ideals of A. Then the following statements hold:*

(i) *$I_a = I$ if and only if $a \in I$;*

(ii) *if $a \leqslant b$, then $I_a \subseteq I_b$;*

(iii) *if $I \subseteq J$, then $I_a \subseteq J_a$;*

(iv) *$(I \cap J)_a = I_a \cap J_a$ and $(I \cup J)_a = I_a \cup J_a$;*

(v) *$I_{a \ominus b} \subseteq (I_a)_b$.*

P r o o f. (i) By Proposition 4.15, since $I \cup \{a\} \subseteq I_a$ and $I_a = I$, we have $a \in I$. Now, if $a \in I$, since I_a is the least ideal of A containing I and $\{a\}$, it is clear that $I_a = I$.

(ii) Let $a \leqslant b$ and $x \in I_a$. Then $x \odot a' \in I$. Since $a \leqslant b$, by Proposition 2.1 (vii) and (viii), $b' \leqslant a'$, and so $x \odot b' \leqslant x \odot a'$. Since $I \in \mathcal{ID}(A)$ and $x \odot a' \in I$, we have $x \odot b' \in I$. Then $x \in I_b$.

(iii) Let $I, J \in \mathcal{ID}(A)$ and $I \subseteq J$. If $x \in I_a$, then $x \odot a' \in I$, and so $x \odot a' \in J$. Hence, $x \in J_a$.

(iv) Since $I \cap J \subseteq I, J$, by (iii), $(I \cap J)_a \subseteq I_a \cap J_a$. Let $x \in I_a \cap J_a$. Then $x \odot a' \in I$ and $x \odot a' \in J$, thus, $x \odot a' \in I \cap J$. Hence, $x \in (I \cap J)_a$. The proof of the other case is similar.

(v) Let $x \in I_{a \ominus b}$. Then $x \odot (a \ominus b)' = x \odot (a' \to b)' \in I$. By (HP3), we have,

$$(a' \odot b') \to (a' \to b)' = (a' \to b) \to ((a' \odot b') \to 0) = (a' \to b) \to (a' \to b'') = 1.$$

Thus, $(a' \odot b') \leqslant (a' \to b)'$, and so $x \odot (a' \odot b') \leqslant x \odot (a' \to b)' \in I$. Since $I \in \mathcal{ID}(A)$, $x \odot (a' \odot b') \in I$. Hence, $x \in (I_a)_b$. □

Proposition 4.17. *Let A be a bounded hoop with (DNP). Then the following statements are equivalent:*

(i) *any ideal I of A is an implicative ideal;*

(ii) *$\{0\}$ is an implicative ideal of A;*

(iii) *for any $a \in A$, the set $A(a) = \{x \in A \colon\ x \odot a' = 0\}$ is an ideal of A.*

Proof. (i) $\Rightarrow$ (ii): Since $\{0\}$ is a trivial ideal of A, by (i), the proof is clear.

(ii) $\Rightarrow$ (iii): Since $0 \in A$ and $0 \odot a' = 0$, we have $0 \in A(a) \neq \emptyset$. Suppose $x, y \in A(a)$. Then $x \odot a' = y \odot a' = 0$. Since A has (DNP), $x = x'' \in A(a)$. Then by Proposition 2.1 (vi), $(x' \to y) \odot y' \odot a' = (x' \to y) \odot (y \to 0) \odot a' \leqslant x'' \odot a' = x \odot a'$. Since $x \odot a' = 0$, we get that $(x' \to y) \odot y' \odot a' = 0 \in \{0\}$. Also, $y \odot a' = 0 \in \{0\}$. Since $\{0\}$ is an implicative ideal of A, $(x' \to y) \odot a' = 0$, so $x' \to y \in A(a)$. Now, suppose $x \leqslant y$ and $y \in A(a)$. Then by Proposition 2.1 (vii), $x \odot a' \leqslant y \odot a' = 0$, thus, $x \odot a' = 0$. Hence, $x \in A(a)$. Therefore, $A(a)$ is an ideal of A.

(iii) $\Rightarrow$ (i): Let $I \in \mathcal{ID}(A)$ such that $x \odot y' \odot z' \in I$ and $y \odot z' \in I$ for any $x, y, z \in A$. Since $I \in \mathcal{ID}(A)$, A/I is a hoop. Then by (iii), for any $a/I \in A/I$ we have $A/I(a/I) \in \mathcal{ID}(A/I)$. Then $(x \odot y')/I \odot z'/I = 0$ and $y/I \odot z'/I = 0$. Thus, $(x \odot y')/I \in A/I(z/I)$ and $y/I \in A/I(z/I)$. Since $A/I(z/I) \in \mathcal{ID}(A/I)$, we have $x/I \in A/I(z/I)$. Then $x/I \odot z'/I = 0$. Hence, $x \odot z' \in I$. Therefore, I is an implicative ideal of A. □

5. Prime, maximal and Boolean ideals in hoops

In this section, we introduce prime, maximal and Boolean ideals in a hoop and investigate the relation between these ideals and implicative one. Also, we study the quotients that are made by them.

Definition 5.1. Let P be a proper ideal of A. P is called a *prime ideal* of A if $x \wedge y \in P$ implies $x \in P$ or $y \in P$ for any $x, y \in A$. The set of all prime ideals of A is denoted by $\mathrm{Spec}(A)$.

Example 5.2. According to Example 3.3, we can easily see that both ideals I_1 and I_2 are prime ideals of A.

Proposition 5.3. *If A is a $\vee$-hoop with (DNP), then I is a prime ideal if and only if $I' = F$ is a prime filter of A.*

Proof. Proof is similar to the proof of Proposition 3.6. □

Proposition 5.4. *Let A be a $\vee$-hoop with (DNP) and let P be a proper ideal of A. Then P is a prime ideal if and only if, for any $I, J \in \mathcal{ID}(A)$ such that $I \cap J \subseteq P$, we get $I \subseteq P$ or $J \subseteq P$.*

Proof. ($\Rightarrow$) Suppose $I, J \in \mathcal{ID}(A)$ such that $I \cap J \subseteq P$, but $I \nsubseteq P$ and $J \nsubseteq P$. Then there exist $x \in I - P$ and $y \in J - P$. Since $x \wedge y \leqslant x, y$ and $I, J \in \mathcal{ID}(A)$, $x \wedge y \in I \cap J \subseteq P$, and so $x \wedge y \in P$. Since $P \in \mathrm{Spec}(A)$, we get that $x \in P$ or $y \in P$, which is a contradiction. Hence, $I \subseteq P$ or $J \subseteq P$.

($\Leftarrow$) Let $P \in \mathcal{ID}(A)$ such that for any $x, y \in A$, $x \wedge y \in P$. If $x, y \notin P$, then by Corollary 3.14, $(P \cup \{x\}] \cap (P \cup \{y\}] = (P \cup \{x \wedge y\}] = P$. Thus, by assumption, $(P \cup \{x\}] \subseteq P$ or $(P \cup \{y\}] \subseteq P$, and so $x \in P$ or $y \in P$, which is a contradiction. Hence, $P \in \mathrm{Spec}(A)$. □

Proposition 5.5. *Let $\varphi : A \to B$ be a hoop homomorphism. Then the following statements hold:*

(i) *if φ is an epimorphism and $P \in \mathrm{Spec}(B)$, then $\varphi^{-1}(P) \in \mathrm{Spec}(A)$;*

(ii) *if φ is surjective and $P \in \mathrm{Spec}(A)$ such that $P \neq B$, then $\varphi(P) \in \mathrm{Spec}(B)$.*

Proof. By Propositions 3.15 and 5.4, the proof is clear. □

Theorem 5.6. *Let A be a $\vee$-hoop with (DNP), let I be a proper ideal of A and $\emptyset \neq S \subseteq A$ such that $I \cap S = \emptyset$. If S is $\wedge$-closed, then there exists $P \in \mathrm{Spec}(A)$ such that $I \subseteq P$ and $P \cap S = \emptyset$.*

P r o o f. Let $\Sigma = \{J \in \mathcal{ID}(A)\colon\ I \subseteq J \text{ and } J \cap S = \emptyset\}$. Since $I \in \Sigma$, hence $\Sigma \neq \emptyset$. If $\{J_\lambda\}_{\lambda\in\Delta}$ is a family of ideals of A that are in Σ, then by Zorn's lemma we can see that $P = \bigcup\limits_{\lambda\in\Delta} J_\lambda$ is a maximal element of Σ. So, it is enough to prove that P is a prime ideal. Since $P \cap S = \emptyset$, it is clear that P is proper. Now, suppose $x \wedge y \in P$ for $x, y \in A$ such that $x, y \notin P$. Then by Corollary 3.14, $(P \cup \{x\}] \cap (P \cup \{y\}] = (P \cup \{x \wedge y\}] = P$. Since $P \subseteq (P \cup \{x\}] \cap (P \cup \{y\}]$ and P is a maximal element of Σ, we get $(P \cup \{x\}] \notin \Sigma$ and $(P \cup \{y\}] \notin \Sigma$, so $(P \cup \{x\}] \cap S \neq \emptyset$ and $(P \cup \{y\}] \cap S \neq \emptyset$. Then there exist $a \in (P \cup \{x\}] \cap S$ and $b \in (P \cup \{y\}] \cap S$. Since S is $\wedge$-closed, we have

$$a \wedge b \in [(P \cup \{x\}] \cap (P \cup \{y\}]] \cap S = P \cap S$$

So, we consequence that $P \cap S \neq \emptyset$, which is a contradiction. Then, $P \in \mathrm{Spec}(A)$. Therefore, there exists $P \in \mathrm{Spec}(A)$ such that $I \subseteq P$ and $P \cap S = \emptyset$. □

Corollary 5.7. *Let A be a $\vee$-hoop with (DNP). Then for any proper ideal I of A there exists $P \in \mathrm{Spec}(A)$ such that $I \subseteq P$.*

P r o o f. Since I is a proper ideal of A, there exists $x \in A - I$. Let $S = \{x\}$. Then by Theorem 5.6, the proof is clear. □

Definition 5.8. Let M be a proper ideal of A. Then M is called a *maximal ideal* of A if no proper ideal of A strictly contains M. It means that if there exists an ideal J of A such that $M \subseteq J \subseteq A$, then $M = J$ or $J = A$. The set of all maximal ideals of A is denoted by $\mathrm{Max}(A)$.

E x a m p l e 5.9. According to Example 3.3, we can easily see that both the ideals I_1 and I_2 are maximal ideals of A.

Proposition 5.10. *Let A be a $\vee$-hoop with (DNP). Then every maximal ideal of A is a prime one.*

P r o o f. Let $M \in \mathrm{Max}(A)$ and $x \wedge y \in M$ for any $x, y \in A$. If $x \notin M$, then $M \subseteq (M \cup \{x\}]$. Since $M \in \mathrm{Max}(A)$, we get $(M \cup \{x\}] = A$. In a similar way, if $y \notin M$, then $(M \cup \{y\}] = A$. By Corollary 3.14, $A = (M \cup \{x\}] \cap (M \cup \{y\}] = (M \cup \{x \wedge y\}] = M$, which is a contradiction. Hence, $M \in \mathrm{Spec}(A)$. □

Proposition 5.11. *Let $\varphi\colon\ A \to B$ be a hoop homomorphism. Then the following statements hold:*

(i) *if φ is an epimorphism and $M \in \mathrm{Max}(B)$, then $\varphi^{-1}(M) \in \mathrm{Max}(A)$;*

(ii) *if φ is surjective and $M \in \mathrm{Max}(A)$ such that $\varphi(M) \neq B$, then $\varphi(M) \in \mathrm{Max}(B)$.*

P r o o f. By Propositions 3.15 and 5.4, the proof is clear. □

Theorem 5.12. *Let A be a hoop and M a proper ideal of A. Then the following statements are equivalent:*
(i) *M is a maximal ideal of A;*
(ii) *A/M is a simple hoop;*
(iii) $|\mathcal{ID}(A/M)| = 2$.

Proof. (i) $\Rightarrow$ (ii): Let $M \in \mathrm{Max}(A)$. Then for any $J \in \mathcal{ID}(A)$ such that $M \subsetneq J$, $J/M \in \mathcal{ID}(A/M)$. Since $M \in \mathrm{Max}(A)$ and $M \subsetneq J$, we get that $J = A$. So, A/M has just trivial ideals. Hence, A/M is a simple hoop.

(ii) $\Rightarrow$ (iii): It is clear.

(iii) $\Rightarrow$ (i): Let $|\mathcal{ID}(A/M)| = 2$. Suppose $M, J \in \mathcal{ID}(A)$ such that $M \subsetneq J$. If $J \neq A$, then $\{0\} = M \subsetneq J/M \subsetneq A/M$. Thus, $|\mathcal{ID}(A/M)| > 2$, which is a contradiction. Hence, M is a maximal ideal of A. □

Definition 5.13. An ideal I of A is called a *Boolean ideal* if $x \wedge x' \in I$ for any $x, y \in A$.

Example 5.14. According to Example 3.3, we can easily see that both the ideals I_1 and I_2 are Boolean ideals of A.

According to Corollary 4.8, every implicative ideal is a Boolean ideal, but by Example 4.9, converse may be not true, in general.

Theorem 5.15. *Let A be a bounded $\vee$-hoop with (DNP) and let I be a proper ideal of A. Then the following statements are equivalent:*
(i) *I is a prime implicative ideal of A;*
(ii) *I is a maximal implicative ideal of A;*
(iii) *if $x, y \notin I$, then $x \odot y' \in I$ and $x' \odot y \in I$;*
(iv) *if $x \notin I$, then there exists $n \in \mathbb{N}$ such that $x^n_\ominus = x' \ominus x' \ominus \ldots \ominus x' \in I$;*
(v) *$x \in I$ or $x' \in I$.*

Proof. (i) $\Rightarrow$ (ii): Let I be a proper ideal of A such that $I \notin \mathrm{Max}(A)$. Then there exists $J \in \mathcal{ID}(A)$ such that $I \subsetneq J \subsetneq A$. Thus, there is an element $x \in J - I$. Since $(x' \wedge x) \odot x' = x \odot (x \to x') \odot x' = 0 \in I \subseteq J$, we get that $x \wedge x' \in (I \cup \{x\}] \subseteq J$. Since $x \wedge x' \in (I \cup \{x\}]$, if $x \wedge x' \in I$, from $I \in \mathrm{Spec}(A)$, then $x' \in I$, and so $x' \in J$. Since $J \in \mathcal{ID}(A)$, $x \ominus x' = 1 \in J$, which is a contradiction. If $x \wedge x' = x$, then $x \leqslant x'$, and so $x^2 = 0$. Since I is an implicative ideal of A, by Proposition 4.6, $x \in I$, which is a contradiction. Hence, $I \in \mathrm{Max}(A)$.

(ii) $\Rightarrow$ (i): By Proposition 5.10, the proof is clear.

(ii) $\Rightarrow$ (iii): Suppose $x, y \notin I$. Since $I \in \mathrm{Max}(A)$, we have $(I \cup \{x\}] = (I \cup \{y\}] = A$. Then $y \in (I \cup \{x\}]$ and $x \in (I \cup \{y\}]$. Thus, $x \odot y' \in I$ and $x' \odot y \in I$.

(iii) $\Rightarrow$ (iv): If $x \notin I$, since $1 \notin I$, then by (ii), $1 \odot x' \in I$ and $x \odot 1' = 0 \in I$. Thus, $x' \in I$. So, for $n = 1$, the proof is clear.

(iv) $\Rightarrow$ (v): Let $x \notin I$. Then there exists $n \in \mathbb{N}$ such that $x_{\ominus}^{n} = x' \ominus x' \ominus \ldots \ominus x' \in I$. Since $x' \leqslant x_{\ominus}^{n}$ and $I \in \mathcal{ID}(A)$, we have $x' \in I$.

(v) $\Rightarrow$ (ii): Suppose $I \notin \mathrm{Max}(A)$. Then there exists $J \in \mathcal{ID}(A)$ such that $I \subsetneq J \subsetneq A$. Let $x \in J - I$. Then by (iii), $x' \in I$, and so $x' \in J$. Since $x, x' \in J$ and $J \in \mathcal{ID}(A)$, $1 = x' \to x' \in J$. Thus, $J = A$, which is a contradiction. Hence, $I \in \mathrm{Max}(A)$. Now, suppose $x^2 \in I$ for any $x \in A$. By Proposition 4.6, it is enough to prove that $x \in I$. Let $x \notin I$. Then by (iii), $x' \in I$. Since $I \in \mathcal{ID}(A)$ and $x^2, x' \in I$, we have $(x^2)' \to x' \in I$. Then $(x \to x') \to x' = ((x \odot x) \to 0) \to x' = (x^2)' \to x' \in I$. Since $x \leqslant (x \to x') \to x'$ and $I \in \mathcal{ID}(A)$, we have $x \in I$. Therefore, I is an implicative ideal of A. $\square$

Corollary 5.16. *Let A be a bounded $\vee$-hoop with (DNP). If every proper ideal of A is an implicative ideal, then* $\mathrm{Spec}(A) = \mathrm{Max}(A)$.

P r o o f. By Theorem 5.15, the proof is clear. $\square$

6. Conclusions

In this paper we define and characterize the notions of (implicative, maximal, prime) ideals in hoops. Then we investigate the relation between them and prove that every maximal implicative ideal of a $\vee$-hoop with (DNP) is a prime one. Also, we define a congruence relation on hoops by ideals and study the quotient that is made by it. This notion helps us to show that an ideal is maximal if and only if the quotient hoop is a simple MV-algebra. Also, we investigate the relationship between ideals and filters by exploiting the set of complements.

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