FINITENESS OF MEROMORPHIC FUNCTIONS ON AN ANNULUS SHARING FOUR VALUES REGARDLESS OF MULTIPLICITY

Duc Quang Si, An Hai Tran, Hanoi

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Abstract. This paper deals with the finiteness problem of meromorphic funtions on an annulus sharing four values regardless of multiplicity. We prove that if three admissible meromorphic functions f_1, f_2, f_3 on an annulus $\mathbb{A}(R_0)$ share four distinct values regardless of multiplicity and have the *complete identity set* of positive counting function, then $f_1=f_2$ or $f_2=f_3$ or $f_3=f_1$. This result deduces that there are at most two admissible meromorphic functions on an annulus sharing a value with multiplicity truncated to level 2 and sharing other three values regardless of multiplicity. This result also implies that there are at most three admissible meromorphic functions on an annulus sharing four values regardless of multiplicities. These results are a generalization and improvement of the previous results on finiteness problem of meromorphic functions on $\mathbb{C}$ sharing four values.

Keywords: meromorphic function; Nevanlinna theory; annulus

MSC 2010: 30D35, 32H30

1. Introduction

Let D be a domain in $\mathbb{C}$ and let f, g be two meromorphic functions on D. Let a be a value in $\mathbb{C}\cup\{\infty\}$ and k be a positive integer or ∞. We say that f and g share the value a with multiplicities counted to level k if

$$\min\{\nu^0_{f-a},k\}=\min\{\nu^0_{g-a},k\}\quad \text{on } D,$$

where ν^0_{φ} denotes the divisor of zeros of the meromorphic function φ and $\nu^0_{\varphi-\infty}$ is regarded as $\nu^0_{1/\varphi}$. We will say that f and g share a regardless of multiplicities (or share a counted with multiplicities) if $k=1$ (or $k=\infty$).

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In 1926, Nevanlinna in [12] showed that two nonconstant distinct meromorphic functions f and g on $\mathbb{C}$ cannot have the same inverse images of five distinct values and that g is a Möbius transformation of f if they share four distinct values counted with multiplicity. These results are called Nevalinna's five and four values theorems. After that, Fujimoto in [5] improved the four values theorem of Nevanlinna by proving that there are at most two meromorphic functions on $\mathbb{C}$ which share four distinct values with multiplicities truncated by level 2. This kind of results are called finiteness theorems for meromorphic function sharing values. For the case of meromorphic functions on $\mathbb{C}$, there are many extensions of the four values theorem by many authors (we refer the reader to [1], [2], [6], [7], [10] and [13], [14]). However, as far as we know, there is still no finiteness theorems for the case of meromorphic functions on doubly connected domain sharing four values, for instance on annuli $\mathbb{A}(R_0) = \{z\colon\ 1/R_0 < |z| < R_0\}$, $R_0 \in (1, \infty]$.

Recently, Khrystiyanyn and Kondratyuk (see [8], [9]) proposed the Nevanlinna theory for meromorphic functions on annuli. By using the second main theorem for meromorphic functions on annuli, Cao, Yi and Xu in [4] proved a uniqueness theory of meromorphic functions on annuli sharing values. The result of Cao, Yi and Xu may be considered as a generalization of almost all uniqueness theorems for meromorphic functions sharing finite values in the complex plane to the case of functions on annuli. However, in their result the functions are assumed to share at least five values. The purpose of this paper is to study the case where the functions on annuli share only four values regardless of multiplicity. Firstly, we give the following definition.

Let $f_1, \ldots, f_k$ be meromorphic functions on an annulus $\mathbb{A}(R_0)$. We define the "*complete identity set*" of $f_1, \ldots, f_k$, denoted by $C(f_1, \ldots, f_k)$, as the set of all points z_0 satisfying one of the following two conditions:

(i) z_0 is a common zero with the same multiplicities of $f - f(z_0)$ and $g - g(z_0)$,

(ii) z_0 is a common pole with the same multiplicities of f and g. The funtions $f_1, \ldots, f_k$ are said to have the "*complete identity set of positive counting function*" if the quantity $N(r, C(f_1, \ldots, f_k))$ is not small with respect to some f_i, $1 \leqslant i \leqslant k$, i.e.

$$N(r, C(f_1, \ldots, f_k)) \neq S_{f_1}(r) + \ldots + S_{f_k}(r).$$

Here, the counting function $N(r, C(f_1, \ldots, f_k))$ and the quantities $S_{f_i}(r)$ are defined in Section 2. Our main result will be stated as follows.

Theorem 1.1. *Let f_1, f_2, f_3 be three meromorphic functions on an annulus $\mathbb{A}(R_0)$, $1 < R_0 \leqslant \infty$ and let a_1, a_2, a_3, a_4 be four distinct values in $\mathbb{C} \cup \{\infty\}$. Assume that f_1, f_2, f_3 share a_1, a_2, a_3, a_4 regardless of multiplicities. If f_1 is admissible and*

f_1, f_2, f_3 have the identity complete set of positive counting function, then $f_1 = f_2$ or $f_2 = f_3$ or $f_3 = f_1$.

From our result above, we will show that there are at most two meromorphic functions sharing a value with multiplicities truncated by level 2 and sharing three other values regardless of multiplicities. For details, we have the following corollary.

Corollary 1.2. *Let f_1, f_2, f_3 be three meromorphic functions on an annulus $\mathbb{A}(R_0)$, $1 < R_0 \leqslant \infty$ and let a_1, a_2, a_3, a_4 be four distinct values in $\mathbb{C} \cup \{\infty\}$. Assume that f_1, f_2, f_3 share a_1 with multiplicities counted to level 2 and share a_2, a_3, a_4 regardless of multiplicities. If f_1 is admissible, then $f_1 = f_2$ or $f_2 = f_3$ or $f_3 = f_1$.*

With weaker assumption that the meromorphic functions share all four values regardless of multiplicities, our main result also implies the following corollary.

Corollary 1.3. *Let f_1, f_2, f_3, f_4 be four meromorphic functions on an annulus $\mathbb{A}(R_0)$, $1 < R_0 \leqslant \infty$ and a_1, a_2, a_3, a_4 be four distinct values in $\mathbb{C} \cup \{\infty\}$. Assume that f_1, f_2, f_3, f_4 share all a_1, a_2, a_3, a_4 regardless of multiplicities. If f_1 is admissible, then there are two functions among $\{f_1, f_2, f_3, f_4\}$ identical to each other.*

2. Some definitions and results from Nevanlinna theory on annuli

In this section, we will recall some important basic notions of Nevanlinna theory for meromorphic functions on annuli from [11] (see also [3], [8] and [9]).

For a divisor ν on $\mathbb{A}(R_0)$, which we may regard as a function on $\mathbb{A}(R_0)$ with values in $\mathbb{Z}$ whose support is a discrete subset of $\mathbb{A}(R_0)$, and for a positive integer M (maybe $M = \infty$), we define the counting function of ν as

$$n_0^{[M]}(t) = \begin{cases} \sum\limits_{1 \leqslant |z| \leqslant t} \min\{M, \nu(z)\} & \text{if } 1 \leqslant t < R_0, \\ \sum\limits_{t \leqslant |z| < 1} \min\{M, \nu(z)\} & \text{if } \dfrac{1}{R_0} < t < 1, \end{cases}$$

$$N_0^{[M]}(r, \nu) = \int_{1/r}^{1} \frac{n_0^{[M]}(t)}{t}\,\mathrm{d}t + \int_{1}^{r} \frac{n_0^{[M]}(t)}{t}\,\mathrm{d}t, \quad 1 < r < \infty.$$

For brevity we will omit the character $^{[M]}$ if $M = \infty$.

For a divisor ν and a positive integer k (maybe $k = \infty$), we define

$$\nu_{\leqslant k}(z) = \begin{cases} \nu(z) & \text{if } \nu(z) \leqslant k, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{>k}(z) = \begin{cases} \nu(z) & \text{if } \nu(z) > k, \\ 0 & \text{otherwise.} \end{cases}$$

For a meromorphic function φ we define

$\triangleright$ ν^0_φ (or ν^∞_φ) the divisor of zeros (or divisor of poles) of φ,

$\triangleright$ $\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi$,

$\triangleright$ $\nu^0_{\varphi,\leqslant k} = (\nu^0_\varphi)_{\leqslant k}$, $\nu^0_{\varphi,>k} = (\nu^0_\varphi)_{>k}$.

Similarly, we define $\nu^\infty_{\varphi,\leqslant k}$, $\nu^\infty_{\varphi,>k}$, $\nu_{\varphi,\leqslant k}$, $\nu_{\varphi,>k}$ and their counting functions.

For a discrete subset $S \subset \mathbb{A}(R_0)$ we consider it as a reduced divisor (denoted again by S) whose support is S, and denote by $N_0(r, S)$ its counting function. We also set $\chi_S(z) = 0$ if $z \notin S$ and $\chi_S(z) = 1$ if $z \in S$.

Let f be a nonconstant meromorphic function on $\mathbb{A}(R)$. We define

$$m_0(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\Bigl|f\Bigl(\frac{1}{r}\mathrm{e}^{\mathrm{i}\theta}\Bigr)\Bigr|\,\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(r\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta - \frac{1}{\pi}\int_0^{2\pi}\log^+|f(\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta$$

and

$$T_0(r,f) = m_0(r,f) + N_0(r,\nu^\infty_f).$$

Throughout this paper, we denote by $S_f(r)$ the quantities satisfying

(i) in the case $R_0 = \infty$,

$$S_f(r) = O(\log(rT_0(r,f)))$$

for $r \in (1,\infty)$ except for the set Δ_R such that $\int_{\Delta_R} r^{\lambda-1}\,\mathrm{d}r < \infty$, $\lambda > 0$,

(ii) in the case $R_0 < \infty$,

$$S_f(r) = O\Bigl(\log\frac{T_0(r,f)}{R_0-r}\Bigr) \quad \text{as } r \to R_0$$

for $r \in (1,R_0)$ except for the set Δ'_R such that $\int_{\Delta'_R}(R_0-r)^{1-\lambda}\,\mathrm{d}r < \infty$, $\lambda > 0$.

The function f is said to be admissible if it satisfies

$$\limsup_{r\to\infty}\frac{T_0(r,f)}{\log r} = \infty \quad \text{in the case } R_0 = \infty$$

or

$$\limsup_{r\to R_0}\frac{T_0(r,f)}{-\log(R_0-r)} = \infty \quad \text{in the case } 1 < R_0 < \infty.$$

Thus, for an admissible meromorphic function f on the annulus $\mathbb{A}(R_0)$ we have $S_f(r) = o(T_0(r,f))$ as $r \to R_0$ for all $1 \leqslant r < R_0$ except for the set Δ_R or the set Δ'_R mentioned above, respectively.

Lemma 2.1 (Lemma on logarithmic derivatives [3], [8], [9], [11]). *Let f be a nonzero meromorphic function on $\mathbb{A}(R_0)$. Then for each $k \in \mathbb{N}$ we have*

$$m_0\Bigl(r, \frac{f^{(k)}}{f}\Bigr) = S_f(r), \quad 1 \leqslant r < R_0.$$

Theorem 2.2 (First main theorem [3], [8], [9], [11]). *Let f be a meromorphic function on $\mathbb{A}(R_0)$. Then for each $a \in \mathbb{C}$ we have*

$$T_0(r,f) = T_0\Big(r, \frac{1}{f-a}\Big) + S_f(r), \quad 1 \leqslant r < R_0.$$

Theorem 2.3 (Second main theorem [3], [8], [9], [11]). *Let f be a nonconstant meromorphic function on $\mathbb{A}(R_0)$. Let $a_1, \ldots, a_q$, $q \geqslant 3$ be q distinct values in $\mathbb{C} \cup \{\infty\}$. We have*

$$(q-2)T_0(r,f) \leqslant \sum_{i=1}^{q} N_0^{[1]}(r, \nu^0_{f-a_i}) + S_f(r), \quad 1 \leqslant r < R_0.$$

Lemma 2.4. *Let f be an admissible meromorphic function on $\mathbb{A}(R_0)$, $1 < R_0 \leqslant \infty$ and let a_1, a_2, a_3 be three distinct values in $\mathbb{C} \cup \{\infty\}$. Let g be a meromorphic function on $\mathbb{A}(R_0)$ such that f and g share all a_1, a_2, a_3 regardless of multiplicities. Then we have*

$$T_0(r,f) = O(T_0(r,g)) + S_f(r) \quad \text{and} \quad T_0(r,g) = O(T_0(r,f)) + S_g(r) \quad \text{as } r \to R_0.$$

In particular, g is admissible.

Proof. By Theorem 2.3 we have

$$T_0(r,f) \leqslant \sum_{i=1}^{3} N_0^{[1]}(r, \nu^0_{f-a_i}) + S_f(r) = \sum_{i=1}^{3} N_0^{[1]}(r, \nu^0_{g-a_i}) + S_f(r) \leqslant 3T_0(r,g) + S_f(r).$$

Similarly, we have $T_0(r,g) \leqslant 3T_0(r,f) + S_g(r)$. The lemma is proved. $\square$

3. Some preparations

Throughout this section, let f_1, f_2, f_3 be three meromorphic functions on $\mathbb{A}(R_0)$ and let a_1, a_2, a_3, a_4 be four distinct values in $\mathbb{C} \setminus \{0\}$ satisfying the following two conditions:

(1) f_1, f_2, f_3 share four values $a_1, \ldots, a_4$ regardless of multiplicities,
(2) f_1 is an admissible meromorphic function.

By Lemma 2.4, we see that $T_0(r, f_k) = O(T_0(r, f_l)) + S_{f_k}(r)$, $1 \leqslant k, l \leqslant 3$ as $r \to R_0$. In particular, f_s is admissible for every $s = 1, 2, 3$. Therefore the quantities $S_{f_1}(r)$,

$S_{f_2}(r)$, $S_{f_3}(r)$ are equivalent, and hence we denote them by the same notation $S(r)$. We set

$$T_0(r) = T_0(r, f_1) + T_0(r, f_2) + T_0(r, f_3).$$

For $i \in \{1, \dots, 4\}$ we put $F_i^k = (f_k - a_i)/f_k$. Then

$$T_0(r, F_i^k) = T_0(r, f_k) + S(r).$$

We define

- ▷ $\nu_i = \{z \colon \nu^0_{f_1 - a_i}(z) > 0\}$,
- ▷ $\nu_{i,s}$, $0 \leqslant s \leqslant 3$: the set of all points $z \in \nu_i$ satisfying that there are exactly s values among $\{\nu^0_{f_s - a_i}(z)\}_{s=1}^3$ bigger than 1.
- ▷ $C' = C(f_1, f_2, f_3) \setminus \bigcup\limits_{1 \leqslant i \leqslant 4} \nu_i$, where $C(f_1, f_2, f_3)$ is the complete identity set of f_1, f_2, f_3.

Lemma 3.1. *If f_1, f_2, f_3 are distinct, then the following assertions hold:*

(1) $2T_0(r, f_k) = \sum\limits_{i=1}^4 N_0(r, \nu_i) + S(r)$, $1 \leqslant k \leqslant 3$,

(2) $N_0(r, C') = S(r)$,

(3) $N_0(r, \nu_{i,s}) = S(r)$ *for all* $1 \leqslant i \leqslant 4$, $2 \leqslant s \leqslant 3$.

P r o o f. Suppose that each f_k has a reduced representation $f_k = (f_{k0} : f_{k1})$, where f_{k0}, f_{k1} are holomorphic functions without common zero. For $k, l \in \{1, 2, 3\}$, $k \neq l$, we have

$$\begin{aligned}
N_0(r, C') &+ \sum_{i=1}^4 N_0(r, \min\{\nu^0_{f_k - a_i}, \nu^0_{f_l - a_i}\}) \\
&\leqslant N_0(r, f_{k0}f_{l1} - f_{k1}f_{l0}) = \frac{1}{2\pi}\int_0^{2\pi} \log|f_{k0}f_{l1} - f_{k1}f_{l0}| \, \mathrm{d}\theta \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log(|f_{k0}|^2 + |f_{k1}|^2)^{1/2} \, \mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} \log(|f_{l0}|^2 + |f_{l1}|^2)^{1/2} \, \mathrm{d}\theta \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log\Bigl(\frac{|f_{k0}|^2}{|f_{k1}|^2} + 1\Bigr)^{1/2} \, \mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|f_{k1}| \, \mathrm{d}\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \log\Bigl(\frac{|f_{l0}|^2}{|f_{l1}|^2} + 1\Bigr)^{1/2} \, \mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|f_{l1}| \, \mathrm{d}\theta \\
&= m_0(r, f_k) + N_0(r, \nu^\infty_{f_k}) + m_0(r, f_l) + N_0(r, \nu^\infty_{f_l}) = T_0(r, f_k) + T_0(r, f_l).
\end{aligned}$$

Therefore

$$N_0(r, C') + \sum_{i=1}^4 \sum_{1 \leqslant k < l \leqslant 3} N_0(r, \min\{\nu^0_{f_k - a_i}, \nu^0_{f_l - a_i}\}) \leqslant 2T_0(r). \tag{3.1}$$

It is easy to see that for every $z \in \nu_i = \bigcup_{s=0}^{3} \nu_{i,s}$ we have

$$\sum_{1 \leqslant k < l \leqslant 3} \min\{\nu^0_{f_k - a_i}(z), \nu^0_{f_l - a_i}(z)\}$$
$$\geqslant \begin{cases} \min\{1,1\} + \min\{1,1\} + \min\{1,1\} = 3 & \text{if } z \in \nu_{i,0}, \\ \min\{1,1\} + \min\{1,2\} + \min\{1,2\} = 3 & \text{if } z \in \nu_{i,1}, \\ \min\{1,2\} + \min\{1,2\} + \min\{2,2\} = 4 & \text{if } z \in \nu_{i,2}, \\ \min\{2,2\} + \min\{2,2\} + \min\{2,2\} = 6 & \text{if } z \in \nu_{i,3}. \end{cases}$$

Then we have

$$\begin{aligned} \sum_{1 \leqslant k < l \leqslant 3} \min\{\nu^0_{f_k - a_i}(z), \nu^0_{f_l - a_i}(z)\} &\geqslant 3\chi_{\nu_{i,0}} + 3\chi_{\nu_{i,1}} + 4\chi_{\nu_{i,2}} + 6\chi_{\nu_{i,3}} \\ &= 3\chi_{\nu_i} + \chi_{\nu_{i,2}} + 3\chi_{\nu_{i,3}}. \end{aligned}$$

This yields that

$$\sum_{1 \leqslant k < l \leqslant 3} N_0(r, \min\{\nu^0_{f_k - a_i}, \nu^0_{f_l - a_i}\}) \geqslant 3N_0(r, \nu_i) + N_0(r, \nu_{i,2}) + 3N_0(r, \nu_{i,3}).$$

From (3.1) we have

$$2T_0(r) \geqslant \sum_{i=1}^{4} (3N_0(r, \nu_i) + N_0(r, \nu_{i,2}) + 3N_0(r, \nu_{i,3})). \tag{3.2}$$

On the other hand, by the second main theorem we have

$$2T_0(r, f_k) \leqslant \sum_{i=1}^{4} N_0^{[1]}(r, \nu_i) + S(r), \quad 1 \leqslant k \leqslant 3. \tag{3.3}$$

Summing-up both sides of (3.3) over all $1 \leqslant k \leqslant 3$, we obtain

$$2T_0(r) \leqslant 3 \sum_{i=1}^{4} N_0^{[1]}(r, \nu_i) + S(r). \tag{3.4}$$

Then, combining (3.2), (3.3) and (3.4), we easily see that

$$\begin{aligned} &T_0(r, f_k) \leqslant \sum_{i=1}^{4} N_0^{[1]}(r, \nu_i) + S(r), \quad 1 \leqslant k \leqslant 3, \\ &N_0(r, \nu_{i,2}) + 3N_0(r, \nu_{i,3}) = S(r), \quad 1 \leqslant i \leqslant 4, \\ &N_0(r, C') = S(r). \end{aligned}$$

This obviously implies the conclusions of the lemma. □

Now we recall the Cartan's auxiliary function (see [5], Definition 3.1). Let F, G, H be three nonzero meromorphic functions, we define Cartan's auxiliary function by

$$\Phi(F,G,H) := F\cdot G\cdot H\cdot\begin{vmatrix} 1 & 1 & 1\\ 1/F & 1/G & 1/H\\ (1/F)' & (1/G)' & (1/H)'\end{vmatrix} \tag{3.5}$$
$$= F\Big(\frac{(1/H)'}{1/H}-\frac{(1/G)'}{1/G}\Big)+G\Big(\frac{(1/F)'}{1/F}-\frac{(1/H)'}{1/H}\Big)$$
$$+H\Big(\frac{(1/G)'}{1/G}-\frac{(1/F)'}{1/F}\Big).$$

It is easy to see that for every meromorphic function h we have the property

$$\Phi(hF,hG,hH)=h\cdot\Phi(F,G,H).$$

Lemma 3.2. *If f_1, f_2, f_3 are distinct, then $\Phi(F_i^1,F_i^2,F_i^3)\not\equiv 0$ for every $1\leqslant i\leqslant 4$.*

Proof. Suppose contrarily that $\Phi(F_i^1,F_i^2,F_i^3)=0$ for an index $i\in\{1,\dots,4\}$. We have

$$0=\begin{vmatrix} 1 & 1 & 1\\ \dfrac{1}{F_i^1} & \dfrac{1}{F_i^2} & \dfrac{1}{F_i^3}\\ \Big(\dfrac{1}{F_i^1}\Big)' & \Big(\dfrac{1}{F_i^2}\Big)' & \Big(\dfrac{1}{F_i^3}\Big)'\end{vmatrix}=\begin{vmatrix} \dfrac{1}{F_i^2}-\dfrac{1}{F_i^1} & \dfrac{1}{F_i^3}-\dfrac{1}{F_i^1}\\ \Big(\dfrac{1}{F_i^2}-\dfrac{1}{F_i^1}\Big)' & \Big(\dfrac{1}{F_i^3}-\dfrac{1}{F_i^1}\Big)'\end{vmatrix}$$
$$=\Big(\frac{1}{F_i^2}-\frac{1}{F_i^1}\Big)\Big(\frac{1}{F_i^3}-\frac{1}{F_i^1}\Big)'-\Big(\frac{1}{F_i^3}-\frac{1}{F_i^1}\Big)\Big(\frac{1}{F_i^2}-\frac{1}{F_i^1}\Big)'.$$

It follows that

$$\frac{1/F_i^3-1/F_i^1}{1/F_i^2-1/F_i^1}=\lambda\in\mathbb{C},$$

i.e.

$$\frac{1/(f_3-a_i)-1/(f_1-a_i)}{1/(f_2-a_i)-1/(f_1-a_1)}=\lambda.$$

Since f_1, f_2, f_3 are supposed to be distinct, $\lambda\notin\{0,1\}$ and

$$(1-\lambda)\frac{1}{f_1-a_i}+\lambda\frac{1}{f_2-a_i}=\frac{1}{f_3-a_i}.$$

Then for every $z\in\mathbb{A}(R_0)$, one has $\nu^0_{f_s-a_i}(z)=\nu^0_{f^t-a_i}(z)\geqslant\nu^0_{f^l-a_i}(z)$ with a permutation (s,t,l) of $(1,2,3)$. We consider the meromorphic function $\varphi=(f_2-a_i)/(f_1-a_i)$.

Suppose that $\varphi = $ constant, i.e. $(f_2 - a_i)/(f_1 - a_i) = a \in \mathbb{C} \setminus \{0,1\}$. Then $\bigcup\limits_{j\neq i} \nu_j = \emptyset$. Therefore

$$T_0(r, f_1) \leqslant \sum_{j\neq i} N_0(r, \nu^0_{f_1 - a_j}) + S(r) = S(r).$$

This contradicts the fact that f_1 is admissible.

Then φ is not constant. We see that if z is a zero of some functions among $\{\varphi, 1/\varphi, \varphi + \lambda/(1-\lambda)\}$, then z is zero of only one function among them and $\nu^0_{f_s - a_i}(z) = \nu^0_{f^t - a_i}(z) > \nu^0_{f^l - a_i}(z)$, and hence $z \in \nu_{i,2} \cup \nu_{i,3}$. Then by the second main theorem and by Lemma 3.2 we have

$$\begin{aligned} T_0(r, \varphi) &\leqslant N_0^{[1]}(r, \nu^0_\varphi) + N_0^{[1]}(r, \nu^0_{1/\varphi}) + N_0^{[1]}(r, \nu^0_{\varphi + \lambda/(1-\lambda)}) + S(r) \\ &\leqslant N_0(r, \nu_{i,2}) + N_0(r, \nu_{i,3}) + S(r) = S(r). \end{aligned}$$

On the other hand, again by the second main theorem we have

$$T_0(r, \varphi) \geqslant N_0(r, \nu^0_{\varphi - 1}) + S(r) \geqslant \sum_{j=1,\, j\neq i}^{4} N_0^{[1]}(r, \nu^0_{f_1 - a_j}) + S(r) \geqslant T_0(r, f_1) + S(r).$$

Therefore we have $T_0(r, f_1) = S(r)$. This contradicts the fact that f_1 is admissible.

Then the supposition is untrue and the lemma is proved. □

Lemma 3.3. *Let i be an index in $\{1, \ldots, 4\}$ and let $\Phi := \Phi(F_i^1, F_i^2, F_i^3)$. If f_1, f_2, f_3 are distinct, then*

$$N_0(r, \nu_{i,0}) + 2 \sum_{j\neq i,\, j=1}^{4} N_0^{[1]}(r, \nu_j) \leqslant N_0(r, \nu^0_\Phi) \leqslant T_0(r) + S(r).$$

Proof. Without loss of generality we may assume that $i = 1$. From Lemma 3.2 we see that $\Phi \not\equiv 0$.

a) We prove the first inequality of the lemma. Let $S = \nu_{1,0} \cup \bigcup\limits_{j=2}^{4} \nu_j$. For a fixed point $z_0 \in S$, we consider the following two cases.

Case 1: Suppose that $z_0 \in \nu_{1,0}$. Then there exists a neighborhood U of z_0 such that all $F_1^k/(z - z_0)$, $1 \leqslant k \leqslant 3$ are nowhere zero holomorphic functions on U. We rewrite the function Φ on U as

$$\Phi = (z - z_0)\Phi\Big(\frac{F_1^1}{z - z_0}, \frac{F_1^2}{z - z_0}, \frac{F_1^3}{z - z_0}\Big).$$

Then, it yields that

$$\nu_\Phi^0(z_0) \geqslant \nu^0_{(z-z_0)}(z_0) = 1 = \chi_{\nu_{1,0}}(z_0) + \sum_{j=2}^{4} \chi_{\nu_j}(z_0).$$

Case 2: Suppose that $z_0 \in \nu_t$ with $t > 1$. We rewrite the function Φ as

$$\begin{aligned}
\Phi &= F_1^1 \cdot F_1^2 \cdot F_1^3 \begin{vmatrix} \dfrac{1}{F_1^2} - \dfrac{1}{F_1^1} & \dfrac{1}{F_1^3} - \dfrac{1}{F_1^1} \\ \Big(\dfrac{1}{F_1^2} - \dfrac{1}{F_1^1}\Big)' & \Big(\dfrac{1}{F_1^3} - \dfrac{1}{F_1^1}\Big)' \end{vmatrix} \\
&= F_1^1 \cdot F_1^2 \cdot F_1^3 \begin{vmatrix} \dfrac{a_1(f_2 - f_1)}{(f_2 - a_1)(f_1 - a_1)} & \dfrac{a_1(f_3 - f_1)}{(f_3 - a_1)(f_1 - a_1)} \\ \Big(\dfrac{a_1(f_2 - f_1)}{(f_2 - a_1)(f_1 - a_1)}\Big)' & \Big(\dfrac{a_1(f_3 - f_1)}{(f_3 - a_1)(f_1 - a_1)}\Big)' \end{vmatrix} \\
&= (z - z_0)^2 F_1^1 \cdot F_1^2 \cdot F_1^3 \\
&\quad \times \begin{vmatrix} \dfrac{a_1(f_2 - f_1)}{(z - z_0)(f_2 - a_1)(f_1 - a_1)} & \dfrac{a_1(f_3 - f_1)}{(z - z_0)(f_3 - a_1)(f_1 - a_1)} \\ \Big(\dfrac{a_1(f_2 - f_1)}{(z - z_0)(f_2 - a_1)(f_1 - a_1)}\Big)' & \Big(\dfrac{a_1(f_3 - f_1)}{(z - z_0)(f_3 - a_1)(f_1 - a_1)}\Big)' \end{vmatrix}.
\end{aligned}$$

We note that all functions $a_1(f_k - f_1)/((z - z_0)(f_k - a_1)(f_1 - a_1))$, $k = 2, 3$ are holomorphic on a neighborhood of z_0. Therefore it follows that

$$\nu_\Phi^0(z_0) \geqslant 2\nu^0_{(z-z_0)}(z_0) = 2 = 2\chi_{\nu_{1,0}}(z_0) + 2\sum_{j=2}^{4} \chi_{\nu_j}(z_0).$$

From the above two cases, we have

$$\nu_\Phi^0(z) \geqslant \chi_{\nu_{i,0}} + 2\sum_{j=1}^{4} \chi_{\nu_j}$$

for all $z \in S$. This implies that

$$N_0(r, \nu_\Phi^0) \geqslant N_0(r, \nu_{1,0}) + 2\sum_{j=2}^{4} N_0(r, \nu_j).$$

Then we have the desired inequality.

b) We prove the second inequality of the lemma. We have

$$\begin{aligned}
N_0(r,\nu^0_\Phi) &\leqslant T_0(r,\Phi) = m_0(r,\Phi) + N_0(r,\nu^\infty_\Phi)\\
&\leqslant \sum_{s=1}^{3} m_0(r,F_i^s) + \sum_{s=1}^{3} N_0(r,\nu^\infty_{F_i^s}) + N_0(r,\nu^\infty_\Phi) - \sum_{s=1}^{3} N_0(r,\nu^\infty_{F_i^s}) + S(r)\\
&= T_0(r) + N_0(r,\nu^\infty_\Phi) - \sum_{s=1}^{3} N_0(r,\nu^\infty_{F_1^s}) + S(r).
\end{aligned}$$

Then it suffices for us to prove that

$$N_0(r,\nu^\infty_\Phi) \leqslant \sum_{s=1}^{3} N_0(r,\nu^\infty_{F_i^s}).$$

In order to prove the above inequality, it is enought to show that the inequality

$$\nu^\infty_\Phi \leqslant \sum_{s=1}^{3} \nu^\infty_{F_i^s} \tag{3.6}$$

holds for every z outside an analytic subset of counting function equal to $S(r)$.

For fixed point z_0, we consider the following two cases:

Case 1: Suppose that $z_0 \in \nu_{1,0} \cup \nu_{1,1}$. Similarly as in Case 1 of the above part, we see that Φ is holomorphic on a neighborhood of z_0.

Case 2: Suppose that $z_0 \notin \nu_1$. Then $1/F_1^s$ is holomorphic at z_0 for all s. Hence, we have

$$\nu^\infty_\Phi(z_0) \leqslant \sum_{k=1}^{3} \nu^\infty_{F_1^k}(z_0).$$

Then from the above two cases we have

$$\nu^\infty_\Phi \leqslant \sum_{s=1}^{3} \nu^\infty_{F_1^s}$$

for all z outside the set $\nu_{1,2} \cup \nu_{1,2}$, which has the counting function equal to $S(r)$. Then we have the desired inequality. □

4. Proofs of results

By using Möbius transformation if necessary, we may assume that all values a_1, a_2, a_3, a_4 belong to $\mathbb{C}$. We will use the same notations given in Section 3 for the proofs of Theorem 1.1 and Corollary 1.2 below.

P r o o f of Theorem 1.1. Let f_1, f_2, f_3 be distinct. Then $\Phi(F_i^1, F_i^2, F_i^3) \not\equiv 0$ for every $i = 1, \ldots, 4$. Lemma 3.3 yields that

$$N_0(r, \nu_{i,0}) + 2 \sum_{j=1,\, j\neq i}^{4} N_0^{[1]}(r, \nu_j) \leqslant N_0(r, \nu_\Phi^0) \leqslant T_0(r) + S(r), \quad 1 \leqslant i \leqslant 4.$$

Summing-up both sides of these inequalities, we get

$$\sum_{i=1}^{4} N_0(r, \nu_{i,0}) + 6 \sum_{i=1}^{4} N_0^{[1]}(r, \nu_j) \leqslant 4T_0(r) + S(r).$$

On the other hand, by the second main theorem we have

$$T_0(r) = \sum_{k=1}^{3} T_0(r, f_k) \leqslant \frac{3}{2} \sum_{i=1}^{4} N_0^{[1]}(r, \nu_j) + S(r).$$

The above two inequalities imply that

$$\sum_{i=1}^{4} N_0(r, \nu_{i,0}) = S(r).$$

Therefore, by Lemma 3.1 we have

$$\begin{aligned} N_0(r, C(f_1, f_2, f_3)) &= N_0(r, C') + \sum_{\substack{0\leqslant s\leqslant 3\\ 1\leqslant i\leqslant 4}} N_0(r, C(f_1, f_2, f_3) \cap \nu_{i,s}) \\ &= N_0(r, C') + \sum_{i=1}^{4} (N_0(r, \nu_{i,0}) + N_0(r, \nu_{i,3})) = S(r) \end{aligned}$$

(here we note that $C(f_1, f_2, f_3) \cap \nu_{i,s} = \emptyset$ for all $1 \leqslant s \leqslant 2$). This is a contradiction.

Then the supposition is impossible. Hence, we must have $f_1 = f_2$ or $f_2 = f_3$ or $f_3 = f_2$. The theorem is proved. $\square$

P r o o f of Corollary 1.2. Suppose that f_1, f_2, f_3 are distinct. By the assumption, we see that if z is a simple zero of some functions $(f_i - a_1)$, then it will be a common simple zero of all funtions $(f_j - a_1)$, $1 \leqslant j \leqslant 4$. This implies that $\nu_{1,1} = \nu_{1,2} = \emptyset$.

Also by Lemma 3.1 (3), the set $\nu_{1,3}$ is of counting function equal to $S(r)$. On the other hand, $\nu_{1,0} \subset C(f_1, f_2, f_3)$ and from Theorem 1.1 we have

$$N_0(r, \nu_{1,0}) \leqslant N_0(r, C(f_1, f_2, f_3)) = S(r).$$

These facts imply that

$$N_0(r, \nu_1) = \sum_{s=0}^{3} N_0(r, \nu_{1,s}) = N_0(r, \nu_{1,0}) + N_0(r, \nu_{1,3}) = S(r).$$

Now, from Lemma 3.3 and the second main theorem we have

$$\begin{aligned} 2T_0(r) = 2\sum_{k=1}^{3} T_0(r, f_k) &\leqslant 3\sum_{i=1}^{4} N_0^{[1]}(r, \nu_i) + S(r) \\ &= 3\sum_{i=2}^{4} N_0^{[1]}(r, \nu_i) + S(r) \leqslant \frac{3}{2}T_0(r) + S(r). \end{aligned}$$

Letting $r \to R_0$, we get $2 \leqslant \frac{3}{2}$. This is a contradiction.

Then the supposition is impossible. Hence, we must have $f_1 = f_2$ or $f_2 = f_3$ or $f_3 = f_2$. The corollary is proved. $\square$

P r o o f of Corollary 1.3. Suppose that f_1, f_2, f_3, f_4 are distinct. Similarly as in Section 2, we set

$$T_0(r) = \sum_{k=1}^{4} T_0(r, f_k)$$

and denote by $S(r)$ the quantities $S_{f_k}(r)$, $1 \leqslant k \leqslant 4$ (these quantities are equivalent). Denote by $\nu_{i,s}$ the set of all points z which are common zeros of $\{f_k - a_i\colon\ 1 \leqslant k \leqslant 4\}$ such that there are exactly s values in $\{\nu^0_{f_k - a_i}(z)\colon\ 1 \leqslant k \leqslant 4\}$ exceeding 2. From Lemma 3.1 (3) we see that $\nu_{i,s}$ consists of counting functions equal to $S(r)$ for all $s \geqslant 2$. On the other hand, for each $i \in \{1, \ldots, 4\}$ we see that if $z \in \nu_{i,0} \cup \nu_{i,1}$, then there are at least three distinct indices $s, k, t \in \{1, \ldots, 4\}$ such that z is a common simple zero of $\{f_s - a_i, f_t - a_i, f_k - a_i\}$, and hence z belongs to $C(f_s, f_t, f_k)$. This yields that

$$\bigcup_{i=1}^{4} (\nu_{i,0} \cup \nu_{i,1}) \subset C(f_s, f_t, f_k).$$

Therefore, from Theorem 1.1 we have

$$\begin{aligned} \sum_{i=1}^{4} N_0(r, \nu_i) &= \sum_{i=1}^{4} (N_0(r, \nu_{i,0}) + N_0(r, \nu_{i,1})) + S(r) \\ &\leqslant \sum_{1 \leqslant s < t < k \leqslant 4} N_0(r, C(f_s, f_t, f_k)) + S(r) = S(r). \end{aligned}$$

Then by the second main theorem, we have

$$2T_0(r, f_1) \leqslant \sum_{i=1}^{4} N_0(r, \nu_i) + S(r) = S(r).$$

This contradicts the fact that f_1 is admissible. Then the supposition is impossible. Hence, there are two funtions among $\{f_1, f_2, f_3, f_4\}$ identical to each other. The corollary is proved. $\square$

References

[1] *A. Banerjee*: Weighted sharing of a small function by a meromorphic function and its derivative. Comput. Math. Appl. *53* (2007), 1750–1761. zbl MR doi

[2] *S. S. Bhoosnurmath, R. S. Dyavanal*: Uniqueness and value-sharing of meromorphic functions. Comput. Math. Appl. *53* (2007), 1191–1205. zbl MR doi

[3] *T.-B. Cao, Z.-S. Deng*: On the uniqueness of meromorphic functions that share three or two finite sets on annuli. Proc. Indian Acad. Sci., Math. Sci. *122* (2012), 203–220. zbl MR doi

[4] *T.-B. Cao, H.-X. Yi, H.-Y. Xu*: On the multiple values and uniqueness of meromorphic functions on annuli. Comput. Math. Appl. *58* (2009), 1457–1465. zbl MR doi

[5] *H. Fujimoto*: Uniqueness problem with truncated multiplicities in value distribution theory. Nagoya Math. J. *152* (1998), 131–152. zbl MR doi

[6] *G. G. Gundersen*: Meromorphic functions that share four values. Trans. Am. Math. Soc. *277* (1983), 545–567. zbl MR doi

[7] *K. Ishizaki, N. Toda*: Unicity theorems for meromorphic functions sharing four small functions. Kodai Math. J. *21* (1998), 350–371. zbl MR doi

[8] *A. Y. Khrystiyanyn, A. A. Kondratyuk*: On the Nevanlinna theory for meromorphic functions on annuli. I. Mat. Stud. *23* (2005), 19–30. zbl MR

[9] *A. Y. Khrystiyanyn, A. A. Kondratyuk*: On the Nevanlinna theory for meromorphic functions on annuli. II. Mat. Stud. *24* (2005), 57–68. zbl MR

[10] *X. Li, H. Yi, H. Hu*: Uniqueness results of meromorphic functions whose derivatives share four small functions. Acta Math. Sci., Ser. B, Engl. Ed. *32* (2012), 1593–1606. zbl MR doi

[11] *M. Lund, Z. Ye*: Nevanlinna theory of meromorphic functions on annuli. Sci. China, Math. *53* (2010), 547–554. zbl MR doi

[12] *R. Nevanlinna*: Einige Eindeutigkeitssätze in der Theorie der meromorphen Funktionen. Acta Math. *48* (1926), 367–391. (In German.) zbl MR doi

[13] *S. D. Quang*: Unicity of meromorphic functions sharing some small functions regardless of multiplicities. Int. J. Math. *23* (2012), Article ID 1250088, 18 pages. zbl MR doi

[14] *S. D. Quang*: Finiteness problem of meromorphic functions sharing four small functions regardless of multiplicities. Int. J. Math. *25* (2014), Article ID 1450102, 20 pages. zbl MR doi

Authors' addresses: *Duc Quang Si*, Department of Mathematics, Hanoi National University of Education, 136-Xuan Thuy, Cau Giay, 123106 Hanoi, Vietnam, and Thang Long Institute of Mathematics and Applied Sciences, Nghiem Xuan Yem, Hoang Mai, 100000 Hanoi, Vietnam, e-mail: `quangsd@hnue.edu.vn`; *An Hai Tran*, Division of Mathematics, Banking Academy, 12-Chua Boc, Dong Da, 100000 Hanoi, Vietnam, e-mail: `trananhai@wru.vn`.