ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR LINEAR EVOLUTIONARY BOUNDARY VALUE PROBLEM OF VISCOELASTIC DAMPED WAVE EQUATION

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Abstract. We study the existence of global in time and uniform decay of weak solutions to the initial-boundary value problem related to the dynamic behavior of evolution equation accounting for rotational inertial forces along with a linear nonlocal frictional damping arises in viscoelastic materials. By constructing appropriate Lyapunov functional, we show the solution converges to the equilibrium state polynomially in the energy space.

Keywords: global existence; uniqueness; uniform stabilization

MSC 2010: 35B33, 47J35

1. Introduction

We study the boundedness and asymptotic properties of solutions as $t \to \infty$ of the linear wave equation with memory

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u + k * u_t = 0, & t > 0,\ x \in \Omega, \\ u = 0, & (x,t) \in \Gamma \times \mathbb{R}_+, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where the unknown $u(x,t)$ is real valued function and Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\Gamma = \partial\Omega$. The functions $u_0(x)$ and $u_1(x)$ are given initial data. The kernel relaxation function $k(t)$, often called the relaxation, will be specified later on. The term $-\Delta u_{tt}$ accounts for rotational inertia forces and the convolution term $k * u_t := \int_0^t k(t-s)u_t(s)\,\mathrm{d}s$ represents the memory effect with a real-valued function k.

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This paper deals with polynomial stability of different kinds for vibrations, modeled by the standard linear model of conducting material with memory. Problem (1.1) without fourth order term has its origin in continuum mechanics for viscoelastic materials which describes the evolution of electromagnetic field through a linear viscoelastic solid body ([6], [7], [11], [10], [19]). It is known that the memory effect exhibits natural damping, which is due to the special property of these materials to retain a memory of their past history. From the mathematical point of view, these linear viscoelastic damping effects are modeled by convolution integrals. Therefore, the dynamics of electromagnetic constitutive relations are of great importance and interest since they have immense applications in the applied and engineering sciences.

The mathematical study on stabilization of vibrating viscoelastic structures is an active area of research among others. The question of stabilization of boundary value problems for the damped wave equation with various approaches, such as abstract semigroup theory, multiplier techniques method, etc. has earlier been studied by several authors (see e.g. [15], [8], [12], [22], [1], [4], [23], [5], [9] and references therein).

Let us briefly give an overview of some related results in the literature. The seminal paper [8] by Dafermos was among earliest results on the asymptotic behavior of solutions to the equations of linear viscoelasticity at large time. Rivera et al. in [22] investigated evolutions systems of the theory of free hereditary electromagnetic field, specifically, they study a model for general conducting material with memory and another for ionospheric phenomena. In both models, the constitutive relations of the material considers the past history of the electric field. Then they use a Lyapunov functionals and semigroup approach to show the existence of solutions, lack of exponential decay and polynomial decay of the solution. A similar problem was studied by Matos and Dmitriev in [18], with the presence of a frictional damping and with effect of two memories in the constitutive relations, where the uniform exponential stability of the associated energy has been obtained via the semigroup method.

Pata and Zucchi in [25] established the theory of finite dimensional attractors for a damped hyperbolic equation with a linear memory. In an innovative work [4], Cavalcanti and Oquendo studied the energy decay rate for a partially viscoelastic nonlinear wave equation subject to a nonlinear and localized frictional damping. Recently, Messaoudi in [21] investigated the general decay rate properties in the energy norm for solutions of a wave equation with viscoelastic damping. We refer the reader to [24], [14], [13], [26] and the references therein for other related results. However, in the case of convolution terms $k * u_t$ involving singular kernels, in particular for problems of fractional time order, there seems to be few results concerning the uniform decay via Lyapunov functionals in the literature (see Chill and Fašangová [5], Zacher [28], Yassine [27]).

The purpose of this paper is to show whether the dissipation given by the memory kernel is strong enough to produce a uniform decay of the solution for problem (1.1). The main issue we encounter here comes from the non-local nature of the memory damping term, moreover, the kernel $k(t)$ may possibly be singular at $t = 0$, which prevents us from reducing such complications. Overcoming this by adopting energy multipliers techniques method consists of constructing new suitable Lyapunov functional, using ideas from Komornik and Zuazua [15], Giorgi et al. [12]. Chill and Fašangová in [5] instead of having to use of semigroup theory (cf. [17], [20], [9], [18], [2]).

The remainder of the paper is organized as follows. In Section 2, we give some lemmas which are useful for the proofs of Theorems 3.1 and 3.2. In Section 3 the main result is enunciated and the proof of well-posedness theorem is provided. Finally, in Section 4 the uniform decay of the solution is proved.

2. Preliminaries

In this section, we shall present some material needed in the proof of our results which are stated at the end of this section. We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual norms $\|\cdot\|_2$ and $\|\cdot\|_{H_0^1}$, respectively. We will write $(\cdot,\cdot)$ to denote the inner product in $L^2(\Omega)$. Let C_* be the smallest positive constant such that

$$\|u\|_2 \leqslant C_* \|\nabla u\|_2 \quad \text{for } u \in H_0^1(\Omega).$$

Throughout this paper, C and C_i are used to denote generic positive constants.

2.1. Assumptions on the memory kernel. In order to prove the result for our problem, we suppose that the function k satisfies the following assumption:

(A) The kernel k is assumed to be positive, convex and integrable on $(0,\infty)$, and there exists a constant $C > 0$ such that

$$\mathrm{d}k'(s) + Ck'(s)\,\mathrm{d}s \geqslant 0, \tag{2.1}$$

where $\mathrm{d}k'$ is the distributional derivative of k'.

Typical examples we consider for k include time fractional derivative kernel

$$k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \mathrm{e}^{-\beta t} \quad \text{with } \alpha \in (0,1] \text{ and } \beta > 0.$$

R e m a r k 2.1. If we integrate inequality (2.1), we obtain an inequality which will be used in the sequel:

$$0 \leqslant k(s)\,\mathrm{d}s \leqslant -k_0 k'(s)\,\mathrm{d}s \leqslant k_1\,\mathrm{d}k'(s) \quad \text{on } (0,\infty)$$

for some $k_0, k_1 \geqslant 0$.

As a consequence of assumption (A), the kernel k verifies the following lemma (see [3], [5]).

Lemma 2.2. *Let k satisfy* (A). *Then*

$$\lim_{s\to 0^+} sk(s) = \lim_{s\to 0^+} s^2 k'(s) = 0, \quad \lim_{s\to\infty} sk(s) = \lim_{s\to\infty} s^2 k'(s) = 0.$$

Now, following the approach of Dafermos (see [8]), let us introduce the following notation:

$$\eta(t,s) = \int_{t-s}^{t} v(r)\,\mathrm{d}r, \quad 0 \leqslant s \leqslant t.$$

The following lemma is required in the construction of the Lyapunov functional. Its proof can be found in [5].

Lemma 2.3. *Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be positive and convex. Let $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H)$. Then we have for almost every $t \geqslant 0$,*

$$\begin{aligned}(k * v(t), v(t))_H = {}& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (-k'(s))\|\eta(t,s)\|_H^2\,\mathrm{d}s + \frac{\mathrm{d}}{\mathrm{d}t}(k(t)\|\eta(t,t)\|_H^2)\\ &+ \frac{1}{2}\int_0^t \|\eta(t,s)\|_H^2\,\mathrm{d}k'(s) + \frac{1}{2}(-k'(t))\|\eta(t,t)\|_H^2.\end{aligned}$$

The next lemma is useful in showing well-posedness of the result; it was introduced in [22], and its proof follows directly by developing the term

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(s)|\varphi(t) - \varphi(t-s)|^2\,\mathrm{d}s.$$

Lemma 2.4. *For any function $k \in C^1(\mathbb{R})$ and any $\varphi \in W^{1,2}(0,T)$ we have that*

$$\begin{aligned}(k*\varphi)(t)\varphi_t(t) = {}& -\frac{1}{2}k(t)|\varphi(t)|^2 + \frac{1}{2}\int_0^t k'(s)|\varphi(t)-\varphi(t-s)|^2\,\mathrm{d}s\\ &- \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t k(s)|\varphi(t)-\varphi(t-s)|^2\,\mathrm{d}s - \int_0^t k(s)\,\mathrm{d}s\,|\varphi(t)|^2\Big).\end{aligned}$$

3. Existence and uniqueness of weak solutions

The well-posedness of system (1.1) is given by the following proposition.

Proposition 3.1. *Let us take* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$ *and let us suppose that assumption* (A) *holds. Then there exists a unique solution* u *of problem* (1.1) *satisfying*

$$u \in L^\infty(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)), \quad u_t \in L^\infty(0,\infty; H_0^1(\Omega)), \quad u_{tt} \in L^2(0,\infty; L^2(\Omega)).$$

Theorem 3.2. *Let all the conditions of Proposition* 3.1 *be satisfied. Then the energy for the global solution of system* (1.1) *decays polynomially, i.e. there exists positive constant* M *such that*

$$E(t) \leqslant \frac{M}{t+1} \quad \forall\, t \geqslant 0.$$

Proof. Let us denote by A the operator

$$Aw = -\Delta w, \quad D(A) = H_0^1(\Omega) \cap H^2(\Omega).$$

It is well known that A is a positive self-adjoint operator in the Hilbert space $L^2(\Omega)$ for which there exist sequences $\{w_m\}_{m\in\mathbb{N}}$ and $\{\lambda_m\}_{m\in\mathbb{N}}$ of eigenfunctions and eigenvalues of A, respectively, such that the set of linear combinations of $\{w_m\}_{m\in\mathbb{N}}$ is dense in $D(A)$ and $\lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_m \to \infty$ as $m \to \infty$.

Now for any integer $m \in \mathbb{N}$ we consider the finite-dimensional subspace

$$V_m = \operatorname{Span}\{w_1, \ldots, w_m\} \subset H_0^1(\Omega) \cap H^2(\Omega),$$

and for given initial data $(u_0, u_1) \in D(A) \times H_0^1(\Omega)$ we search for functions

$$u^m(t) = \sum_{j=1}^{m} y_{mj}(t) w_j$$

which satisfy the approximate problem

$$(u_{tt}^m(t), w_j) + (\nabla u_{tt}^m(t), \nabla w_j) + (\nabla u^m(t), \nabla w_j) + (k * u_t^m(t), w_j) = 0, \quad j = 1, \ldots, m, \tag{3.1}$$

with initial conditions

$$u^m(0) = u_0^m, \quad u_t^m(0) = u_1^m,$$

where

$$u_0^m \to u_0 \text{ in } D(A) \quad \text{and} \quad u_1^m \to u_1 \text{ in } H_0^1(\Omega) \quad \text{as } m \to \infty.$$

We note that the approximate problem (3.1) can be reduced to an ordinary differential equation (ODE) system and by the standard existence theory for ODEs, this problem has a local solution $u^m(t)$ in some interval $[0, T_m)$ with $0 < T_m \leqslant T$. The estimate below will allow us to extend the local solutions $u^m(t)$ to the interval $[0, T]$ for any given $T > 0$.

Now, we derive the first estimate. Multiplying (3.1) by $y'_{mj}(t)$ and summing with respect to j, we conclude from Lemma 2.3 that

$$\begin{aligned}(3.2)\qquad & \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{1}{2}(\|u_t^m\|^2 + \|\nabla u_t^m\|^2 + \|\nabla u^m\|^2) \\ &\qquad + \frac{1}{2}\int_0^t (-k'(s))\|\eta^m(t,s)\|^2 \,\mathrm{d}s + k(t)\|\eta^m(t,t)\|^2\bigg) \\ &= -\frac{1}{2}\int_0^t \|\eta^m(t,s)\|^2 \,\mathrm{d}k'(s) + \frac{1}{2}k'(t)\|\eta^m(t,t)\|^2,\end{aligned}$$

where $\eta^m(t,s) = u^m(t) - u^m(t-s)$.

Integrating (3.2) over $(0,t)$ and using assumption (A), we infer that

$$\begin{aligned}(3.3)\qquad & \frac{1}{2}(\|u_t^m\|^2 + \|\nabla u_t^m\|^2 + \|\nabla u^m\|^2) \\ &\qquad + \frac{1}{2}\int_0^t (-k'(s))\|\eta^m(t,s)\|^2 \,\mathrm{d}s + k(t)\|\eta^m(t,t)\|^2 \\ &\leqslant \frac{1}{2}(\|u_1^m\|^2 + \|\nabla u_1^m\|^2 + \|\nabla u_0^m\|^2) + \lim_{s\to 0} k(s)\|\eta^m(s,s)\|^2.\end{aligned}$$

Since $\lim\limits_{s\to 0} k(s)\|\eta^m(s,s)\|^2 = \lim\limits_{s\to 0} s^2 k(s)\|\eta^m(s,s)\|^2/s^2 = 0$, we obtain

$$\begin{aligned}(3.4)\qquad & \frac{1}{2}(\|u_t^m\|^2 + \|\nabla u_t^m\|^2 + \|\nabla u^m\|^2) \\ &\qquad + \frac{1}{2}\int_0^t (-k'(s))\|\eta^m(t,s)\|^2 \,\mathrm{d}s + k(t)\|\eta^m(t,t)\|^2 \leqslant C_1,\end{aligned}$$

where C_1 is a positive constant depending only on $\|u_0\|_{H_0^1}$ and $\|u_1\|_{H_0^1}$.

It follows from (3.4) that

$$\begin{aligned} &u^m \text{ is uniformly bounded in } L^\infty(0,T; H_0^1(\Omega)), \\ &u_t^m \text{ is uniformly bounded in } L^\infty(0,T; H_0^1(\Omega)).\end{aligned}$$

Then we derive the second estimate. Multiplying (3.1) by $y''_{mj}(t)$ and then summing with respect to j, it holds that

$$\|u_{tt}^m\|_2^2 + \|\nabla u_{tt}^m\|_{L^2}^2 = -(\nabla u^m, \nabla u_{tt}^m) - (u_{tt}^m, k * u_t^m). \tag{3.5}$$

By Young's inequality, the first term on the right-hand side of (3.5) can be estimated as

$$|-(\nabla u^m, \nabla u_{tt}^m)| \leqslant \lambda \|\nabla u_{tt}^m\|_2^2 + \frac{1}{4\lambda}\|\nabla u^m\|_2^2, \quad \lambda > 0. \tag{3.6}$$

On the other hand, in view of Lemma 2.4, we have that

$$\begin{aligned}(u_{tt}^m, k * u_t^m) = &-\frac{1}{2}k(t)\|u_t^m\|^2 + \frac{1}{2}\int_0^t k'(s)\|\eta^m(t,s)\|^2\,\mathrm{d}s \\ &-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t k(s)\|\eta^m(t,s)\|^2\,\mathrm{d}s - \int_0^t k(\tau)\,\mathrm{d}\tau\,\|u_t^m\|^2\right),\end{aligned} \tag{3.7}$$

using (3.7) in (3.5), we find

$$\begin{aligned}\|u_{tt}^m\|_2^2 + \|\nabla u_{tt}^m\|_{L^2}^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t k(s)\|\eta^m(t,s)\|^2\,\mathrm{d}s - \int_0^t k(\tau)\,\mathrm{d}\tau\,\|u_t^m\|^2\right) \\ = -(\nabla u^m, \nabla u_{tt}^m) + \frac{1}{2}k(t)\|u_t^m\|^2 - \frac{1}{2}\int_0^t k'(s)\|\eta^m(t,s)\|^2\,\mathrm{d}s.\end{aligned} \tag{3.8}$$

Integrating (3.8) over $(0,t)$ and using (3.6) yields

$$\begin{aligned}\int_0^t \|u_{tt}^m(s)\|_2^2\,\mathrm{d}s + (1-\lambda)\int_0^t \|\nabla u_{tt}^m(s)\|_{L^2}^2\,\mathrm{d}s + \frac{1}{2}\int_0^t k(\tau)\,\mathrm{d}\tau\,\|u_t^m\|^2 \\ \leqslant \frac{1}{4\lambda}\int_0^t \|\nabla u^m(s)\|_2^2\,\mathrm{d}s + \frac{1}{2}\int_0^t k(s)\|u_t^m(s)\|^2\,\mathrm{d}s \\ - \frac{1}{2}\int_0^t k'(s)\|\eta^m(t,s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t k(s)\|\eta^m(t,s)\|^2\,\mathrm{d}s \\ \leqslant \frac{1}{4\lambda}C_2 T + C_3,\end{aligned} \tag{3.9}$$

where C_2 and C_3 are positive constants depending only on $\|u_0\|_{H_0^1}$ and $\|u_1\|_{H_0^1}$ and $\int_0^\infty k(s)\,\mathrm{d}s$.

Estimate (3.9) implies that u_{tt}^m is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$.

According to (3.4) and (3.9) we can extract subsequence $\{u^\nu\} \subset \{u^m\}$ which verifies:

$$u^\nu \rightharpoonup u \quad \text{weak star in } L^\infty(0,T;H_0^1(\Omega)),$$

$$u_t^\nu \rightharpoonup u_t \quad \text{weak star in } L^\infty(0,T;H_0^1(\Omega)), \tag{3.10}$$

$$u_{tt}^\nu \rightharpoonup u_{tt} \quad \text{weak in } L^2(0,T;H_0^1(\Omega)). \tag{3.11}$$

Hence by Aubin's compactness lemma (see [16]), it follows from (3.10) and (3.11) that there exists a subsequence of $\{u^\nu\}$ still denoted by $\{u^\nu\}$ such that

$$\tag{3.12} \begin{aligned} u_t^\nu &\rightharpoonup u_t \quad \text{strongly in } L^\infty(0,T;L^2(\Omega)),\\ u_{tt}^\nu &\rightharpoonup u_{tt} \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \end{aligned}$$

which implies $u_t^\nu \rightharpoonup u_t$ almost everywhere in $(0,T)\times\Omega$.

Multiplying (3.1) by $\theta(t)\in\mathcal{D}(0,T)$ (here $\mathcal{D}(0,T)$ denotes the space of functions in C^∞ with compact support in $(0,T)$) and integrating over $(0,T)$, it follows that

$$\tag{3.13} \begin{aligned} \int_0^T (u_{tt}^m(t),w_j)\theta(t)\,\mathrm{d}t + \int_0^T (\nabla u_{tt}^m(t),\nabla w_j)\theta(t)\,\mathrm{d}t + \int_0^T (\nabla u^m(t),\nabla w_j)\theta(t)\,\mathrm{d}t \\ + \int_0^T (k*u_t^m(t),w_j)\theta(t)\,\mathrm{d}t = 0 \quad \forall\, j=1,\dots,m. \end{aligned}$$

Convergences (3.10)–(3.11) and (3.12) are sufficient to pass to the limit in (3.13) in order to obtain

$$u_{tt}-\Delta u_{tt}-\Delta u + k*u_t = 0 \quad \text{in } L^2_{\mathrm{loc}}(0,\infty;H^{-1}(\Omega)).$$

Uniqueness of solutions: We derive the uniqueness of solutions using the usual energy method.

In fact, let u and v be two solutions of problem (1.1). Then from problem (3.1), the function $z=u-v$ satisfies

$$(z_{tt},w)+(\nabla z_{tt},\nabla w)+(\nabla z,\nabla w)+(k*z_t,w)=0, \quad t>0,\ x\in\Omega$$

for all $w\in H_0^1(\Omega)$. Taking $w=z_t$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |z_t|^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla z_t|^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla z|^2\,\mathrm{d}x + (k*z_t,z_t)=0, \quad t>0,$$

then from Lemma 2.3 and taking Remark 2.1 into account, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |z_t(t)|^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla z_t(t)|^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla z(t)|^2\,\mathrm{d}x \\ + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Bigl(\int_0^t (-k'(s))\|\eta(t,s)\|^2\,\mathrm{d}s + k(t)\|\eta(t,t)\|^2\Bigr) \\ = -\frac{1}{2}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k'(s) + \frac{1}{2}k'(t)\|\eta(t,t)\|^2 \leqslant 0, \quad t>0. \end{aligned}$$

Integrating the last inequality over $(0,t)$, we deduce that

$$\begin{aligned}(3.14)\qquad &\frac{1}{2}\int_\Omega |z_t(t)|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega |\nabla z_t(t)|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega |\nabla z(t)|^2\,\mathrm{d}x\\ &\quad+\frac{1}{2}\Big(\int_0^t(-k'(s))\|\eta(t,s)\|^2\,\mathrm{d}s+k(t)\|\eta(t,t)\|^2\Big)\\ &\quad+\frac{1}{2}\int_0^t\int_0^s\|\eta(s,\tau)\|^2\,\mathrm{d}k'(\tau)\,\mathrm{d}s+\frac{1}{2}\int_0^t k'(s)\|\eta(s,s)\|^2\,\mathrm{d}s\\ &\leqslant\frac{1}{2}(\|z_1\|^2+\|\nabla z_1\|^2+\|\nabla z_0\|^2)=0,\end{aligned}$$

from (3.14) and employing Gronwall's lemma we conclude that $|z_t(t)|=|\nabla z(t)|=0$. This finishes the proof. □

4. Polynomial decay

In the following lemmas we will prove some technical inequalities which will be useful for showing the polynomial decay of the solution.

By E we denote the first-order energy associated to problem (1.1):

$$E(t):=\frac{1}{2}(\|u_t\|^2+\|\nabla u_t\|^2+\|\nabla u\|^2)+\frac{1}{2}\int_0^t(-k'(s))\|\eta(t,s)\|^2\,\mathrm{d}s+k(t)\|\eta(t,t)\|^2.$$

Using Lemma 2.3, we easily conclude that

$$(4.1)\qquad \frac{\mathrm{d}}{\mathrm{d}t}E(t)=-\frac{1}{2}\int_0^t\|\eta(t,s)\|^2\,\mathrm{d}k'(s)+\frac{1}{2}k'(t)\|\eta(t,t)\|^2\leqslant 0,$$

and the energy decreases.

In order to show the uniform decay result, we need the following lemmas.

Lemma 4.1. *Under the assumptions of Theorem* 3.2, *the functional*

$$\begin{aligned}\mathcal{E}(t)=\frac{1}{2}\Big(&\|\nabla u_t\|^2+\|\Delta u_t\|^2+\|\Delta u\|^2\\ &+\int_0^t(-k'(s))\|\nabla\eta(t,s)\|^2\,\mathrm{d}s+2k(t)\|\nabla\eta(t,t)\|^2\Big),\quad t\geqslant 0,\end{aligned}$$

satisfies along the solution of (1.1) *the equality*

$$(4.2)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)=-\frac{1}{2}\int_0^t\|\nabla\eta(t,s)\|^2\,\mathrm{d}k'(s)+\frac{1}{2}k'(t)\|\nabla\eta(t,t)\|^2.$$

P r o o f. Taking the inner product of (1.1) with $-\Delta u_t$ for $H = L^2(\Omega)$,

$$(\nabla u_t, \nabla u_{tt}) + (\Delta u_t, \Delta u_{tt}) + (\Delta u_t, \Delta u) + (\nabla u_t, k * \nabla u_t) = 0,$$

using Lemma 2.3 on the last term, we get immediately

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Bigl(\|\nabla u_t\|^2 + \|\Delta u_t\|^2 + \|\Delta u\|^2 + \int_0^t (-k'(s))\|\nabla\eta(t,s)\|^2\,\mathrm{d}s + 2k(t)\|\nabla\eta(t,t)\|^2\Bigr)$$
$$= -\frac{1}{2}\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k'(s) + \frac{1}{2}k'(t)\|\nabla\eta(t,t)\|^2.$$

□

Lemma 4.2. *The functional*

$$\Phi(t) := (u_t, u) + (\nabla u_t, \nabla u) - \Bigl(u, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Bigr), \quad t \geqslant 0,$$

satisfies along the solution of (1.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \leqslant -\frac{3}{4}\|\nabla u\|^2 + \|\nabla u_t\|^2 + \Bigl(\frac{3}{2} + \frac{3}{4}C\|k\|^2_{L^1(\mathbb{R}_+)}\Bigr)\|u_t\|^2 + \frac{3}{4}C(-k'(t))\|\eta(t,t)\|^2 + \frac{C}{2}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k'(s). \tag{4.3}$$

P r o o f. Considering the scalar product in $L^2(\Omega)$ of equation (1.1) with u, we get

$$(u_{tt}, u) - (\Delta u_{tt}, u) - (\Delta u, u) + (u, k * u_t) = 0,$$

by using Green's formula we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t, u) - \|u_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}(\nabla u_t, \nabla u) - \|\nabla u_t\|^2 + \|\nabla u\|^2 + (u, k * u_t) = 0, \tag{4.4}$$

and with the aid of

$$k * u_t = k(t)\eta(t,t) + u_t\int_0^t k(s)\,\mathrm{d}s - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, \tag{4.5}$$

from (4.4) and (4.5) we infer that

$$\begin{aligned}(u, k * u_t) &= (u, k(t)\eta(t,t)) + (u, u_t)\int_0^t k(s)\,\mathrm{d}s - \Bigl(u, \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Bigr) \\ &= (u, k(t)\eta(t,t)) + (u, u_t)\int_0^t k(s)\,\mathrm{d}s - \frac{\mathrm{d}}{\mathrm{d}t}\Bigl(u, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Bigr) \\ &\quad + \Bigl(u_t, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Bigr).\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.4), we find

$$\begin{aligned}(4.7)\qquad &\frac{\mathrm{d}}{\mathrm{d}t}(u_t,u)+\frac{\mathrm{d}}{\mathrm{d}t}(\nabla u_t,\nabla u)-\frac{\mathrm{d}}{\mathrm{d}t}\Big(u,\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big)\\ &\quad-\|u_t\|^2-\|\nabla u_t\|^2+\|\nabla u\|^2+(u,k(t)\eta(t,t))\\ &\quad+(u,u_t)\int_0^t k(s)\,\mathrm{d}s+\Big(u_t,\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big)=0.\end{aligned}$$

From (4.7), we see that

$$\Phi:=(u_t,u)+(\nabla u_t,\nabla u)-\Big(u,\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big)$$

verifies

$$\begin{aligned}(4.8)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\Phi+\|\nabla u\|^2&=\|u_t\|^2+\|\nabla u_t\|^2-(u,k(t)\eta(t,t))-\int_0^t k(s)\,\mathrm{d}s(u,u_t)\\ &\quad-\Big(u_t,\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big).\end{aligned}$$

Now, we need to estimate three terms on the right-hand side of (4.8). The Cauchy-Schwarz, Poincaré inequalities and assumption (A) imply for all $t\geqslant t_0>0$

$$\begin{aligned}(4.9)\qquad (u,k(t)\eta(t,t))&\leqslant\frac14\|\nabla u\|^2+C_*k^2(t)\|\eta(t,t)\|^2\\ &\leqslant\frac14\|\nabla u\|^2+C(-k'(t))\|\eta(t,t)\|^2,\end{aligned}$$

$$(4.10)\qquad \int_0^t k(s)\,\mathrm{d}s(u,u_t)\leqslant\frac14\|\nabla u\|^2+C\|k\|^2_{L^1(\mathbb{R}_+)}\|u_t\|^2$$

and

$$(4.11)\qquad \Big(u_t,\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big)\leqslant\frac12\|u_t\|^2+\frac{C}{2}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t\|\eta(t,s)\|^2\,\mathrm{d}k'(s).$$

A combination of (4.9), (4.10) and (4.11) with (4.8) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)\leqslant&-\frac34\|\nabla u\|^2+\|\nabla u_t\|^2+\Big(\frac32+\frac34C\|k\|^2_{L^1(\mathbb{R}_+)}\Big)\|u_t\|^2\\ &+\frac34C(-k'(t))\|\eta(t,t)\|^2+\frac{C}{2}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t\|\eta(t,s)\|^2\,\mathrm{d}k'(s),\end{aligned}$$

which proves the claim. □

Lemma 4.3. *The functional*

$$\begin{aligned}\Psi(t) := &-\Big(u_t, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big) - \frac{1}{2}\int_0^t (-k'(s))\|\eta(t,s)\|^2\,\mathrm{d}s - k(t)\|\eta(t,t)\|^2 \\ &+ \frac{1}{2}\Big\|\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big\|^2 - \Big(\nabla u_t, \int_0^t k(s)\nabla\eta(t,s)\,\mathrm{d}s\Big) \\ &- \frac{1}{2}\int_0^t (-k'(s))\|\nabla\eta(t,s)\|^2\,\mathrm{d}s - k(t)\|\nabla\eta(t,t)\|^2\end{aligned}$$

satisfies along the solution for any $\delta > 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) \leqslant\ &\delta\|u_t\|^2 + \frac{C}{4\delta}(-k'(t))\|\eta(t,t)\|^2 - \int_0^t k(s)\,\mathrm{d}s\|u_t\|^2 + \frac{1}{2}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k' \\ &+ \frac{1}{2}(-k'(t))\|\eta(t,t)\|^2 + \delta\|\nabla u_t\|^2 + \frac{C}{4\delta}(-k'(t))\|\nabla\eta(t,t)\|^2 \\ &- \int_0^t k(s)\,\mathrm{d}s\|\nabla u_t\|^2 + \frac{1}{2}\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k' + \frac{1}{2}(-k'(t))\|\nabla\eta(t,t)\|^2 \\ &+ \delta\|\nabla u\|^2 + \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k'(s) \\ &+ \frac{C}{2}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k'(s) + \frac{C}{2}(-k'(t))\|\eta(t,t)\|^2 \\ &+ \delta\|k\|^2_{L^1(\mathbb{R}_+)}\|u_t\|^2 + \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k'(s).\end{aligned} \tag{4.12}$$

P r o o f. Multiplying (1.1) by $\int_0^t k(s)\eta(t,s)\,\mathrm{d}s$, integrating on Ω along the solution and performing straightforward calculations we obtain

$$\begin{aligned}\Big(u_{tt}, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big) = &\frac{\mathrm{d}}{\mathrm{d}t}\Big(u_t, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big) \\ &- \Big(u_t, \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big).\end{aligned} \tag{4.13}$$

By making use of Leibniz's rule, the right-hand side of (4.13) can also be expressed in the form

$$\begin{aligned}\Big(u_{tt}, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(u_t, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big) - (u_t, k(t)\eta(t,t)) \\ &\quad - \Big(u_t, \int_0^t k(s)\frac{\partial}{\partial t}\eta(t,s)\,\mathrm{d}s\Big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(u_t, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\Big) - (u_t, k(t)\eta(t,t)) \\ &\quad - (u_t, u_t(t))\int_0^t k(s)\,\mathrm{d}s + (u_t, k * u_t).\end{aligned}$$

Using

$$(u_t, k * u_t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^t (-k'(s))\|\eta(t,s)\|^2\,\mathrm{d}s + k(t)\|\eta(t,t)\|^2\right) + \frac{1}{2}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k' + \frac{1}{2}(-k'(t))\|\eta(t,t)\|^2,$$

we get

$$\begin{aligned}\left(u_{tt}, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(u_t, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\right) \\ - (u_t, k(t)\eta(t,t)) - \int_0^t k(s)\,\mathrm{d}s\|u_t\|^2 \\ + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^t (-k'(s))\|\eta(t,s)\|^2\,\mathrm{d}s + k(t)\|\eta(t,t)\|^2\right) \\ + \frac{1}{2}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k' + \frac{1}{2}(-k'(t))\|\eta(t,t)\|^2.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\left(\nabla u_{tt}, \int_0^t k(s)\nabla\eta(t,s)\,\mathrm{d}s\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla u_t, \int_0^t k(s)\nabla\eta(t,s)\,\mathrm{d}s\right) \\ - (\nabla u_t, k(t)\nabla\eta(t,t)) - \int_0^t k(s)\,\mathrm{d}s\|\nabla u_t\|^2 \\ + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^t (-k'(s))\|\nabla\eta(t,s)\|^2\,\mathrm{d}s + k(t)\|\nabla\eta(t,t)\|^2\right) \\ + \frac{1}{2}\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k' + \frac{1}{2}(-k'(t))\|\nabla\eta(t,t)\|^2.\end{aligned}$$

Therefore the functional

$$\begin{aligned}\Psi(t) := & -\left(u_t, \int_0^t k(s)\eta(t,s)\,\mathrm{d}s\right) - \frac{1}{2}\int_0^t (-k'(s))\|\eta(t,s)\|^2\,\mathrm{d}s - k(t)\|\eta(t,t)\|^2 \\ & + \frac{1}{2}\left\|\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\right\|^2 - \left(\nabla u_t, \int_0^t k(s)\nabla\eta(t,s)\,\mathrm{d}s\right) \\ & - \frac{1}{2}\int_0^t (-k'(s))\|\nabla\eta(t,s)\|^2\,\mathrm{d}s - k(t)\|\nabla\eta(t,t)\|^2\end{aligned}$$

fulfills the identity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) + (u_t, k(t)\eta(t,t)\,\mathrm{d}s) + \int_0^t k(s)\,\mathrm{d}s\|u_t\|^2 - \frac{1}{2}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k' \\ - \frac{1}{2}(-k'(t))\|\eta(t,t)\|^2 + (\nabla u_t, k(t)\nabla\eta(t,t)\,\mathrm{d}s) + \int_0^t k(s)\,\mathrm{d}s\|\nabla u_t\|^2\end{aligned}$$

$$
-\frac{1}{2}\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k' - \frac{1}{2}(-k'(t))\|\nabla\eta(t,t)\|^2
$$

$$
-\left(\nabla u, \int_0^t k(s)\nabla\eta(t,s)\,\mathrm{d}s\right) - \left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, k * u_t\right) = 0.
$$

In addition, invoking once more (4.5)

$$
\begin{aligned}
&\left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, k * u_t\right)\\
&\quad = \left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, \left(k(t)\eta(t,t) + u_t\int_0^t k(s)\,\mathrm{d}s - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\right)\right)\\
&\quad = \left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, k(t)\eta(t,t)\right) + \left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, u_t\int_0^t k(s)\,\mathrm{d}s\right)\\
&\qquad - \left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\right)\\
&\quad = \left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, k(t)\eta(t,t)\right) + \left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, u_t\int_0^t k(s)\,\mathrm{d}s\right)\\
&\qquad - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\int_0^t k(s)\eta(t,s)\,\mathrm{d}s\right\|^2,
\end{aligned}
$$

we infer

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = {} & -(u_t, k(t)\eta(t,t)\,\mathrm{d}s) - \int_0^t k(s)\,\mathrm{d}s\|u_t\|^2\\
&+ \frac{1}{2}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k' + \frac{1}{2}(-k'(t))\|\eta(t,t)\|^2\\
&- (\nabla u_t, k(t)\nabla\eta(t,t)\,\mathrm{d}s) - \int_0^t k(s)\,\mathrm{d}s\|\nabla u_t\|^2\\
&+ \frac{1}{2}\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k' + \frac{1}{2}(-k'(t))\|\nabla\eta(t,t)\|^2\\
&+ \left(\nabla u, \int_0^t k(s)\nabla\eta(t,s)\,\mathrm{d}s\right) + \left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, k(t)\eta(t,t)\right)\\
&+ \left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, u_t\int_0^t k(s)\,\mathrm{d}s\right).
\end{aligned}
\tag{4.14}
$$

Let us examine in detail the four terms appearing on the right-hand side of (4.14).

Using Cauchy-Schwarz inequality, we find for any $t \geqslant t_0 > 0$

$$
\begin{aligned}
(u_t, k(t)\eta(t,t)) &\leqslant \delta\|u_t\|^2 + \frac{k^2(t)}{4\delta}\|\eta(t,t)\|^2\\
&\leqslant \delta\|u_t\|^2 + \frac{C}{4\delta}(-k'(t))\|\eta(t,t)\|^2,
\end{aligned}
$$

$$(\nabla u_t, k(t)\nabla\eta(t,t)\,\mathrm{d}s) \leqslant \delta\|\nabla u_t\|^2 + \frac{C}{4\delta}(-k'(t))\|\nabla\eta(t,t)\|^2,$$

$$\left(\nabla u, \int_0^t k(s)\nabla\eta(t,s)\,\mathrm{d}s\right) \leqslant \delta\|\nabla u\|^2 + \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k'(s),$$

$$\left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, k(t)\eta(t,t)\right) \leqslant \frac{C}{2}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k'(s) + \frac{C}{2}(-k'(t))\|\eta(t,t)\|^2$$

and

$$\left(\int_0^t k(s)\eta(t,s)\,\mathrm{d}s, u_t\int_0^t k(s)\,\mathrm{d}s\right) \leqslant \delta\|k\|^2_{L^1(\mathbb{R}_+)}\|u_t\|^2 + \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k'(s).$$

Combining the above estimates, we easily see that for every $\delta > 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) \leqslant{}& \delta\|u_t\|^2 + \frac{C}{4\delta}(-k'(t))\|\eta(t,t)\|^2 - \int_0^t k(s)\,\mathrm{d}s\|u_t\|^2 + \frac{1}{2}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k' \\ &+ \frac{1}{2}(-k'(t))\|\eta(t,t)\|^2 + \delta\|\nabla u_t\|^2 + \frac{C}{4\delta}(-k'(t))\|\nabla\eta(t,t)\|^2 \\ &- \int_0^t k(s)\,\mathrm{d}s\|\nabla u_t\|^2 + \frac{1}{2}\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k' + \frac{1}{2}(-k'(t))\|\nabla\eta(t,t)\|^2 \\ &+ \delta\|\nabla u\|^2 + \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k'(s) \\ &+ \frac{C}{2}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k'(s) + \frac{C}{2}(-k'(t))\|\eta(t,t)\|^2 + \delta\|k\|^2_{L^1(\mathbb{R}_+)}\|u_t\|^2 \\ &+ \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k'(s),\end{aligned}$$

yielding

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) \leqslant{}& \left(\delta(1+\|k\|^2_{L^1(\mathbb{R}_+)}) - \int_0^t k(s)\,\mathrm{d}s\right)\|u_t\|^2 + \left(\delta - \int_0^t k(s)\,\mathrm{d}s\right)\|\nabla u_t\|^2 \\ &+ \delta\|\nabla u\|^2 + \left(\frac{C}{4\delta} + \frac{1}{2} + \frac{C}{2}\right)(-k'(t))\|\eta(t,t)\|^2 \\ &+ \left(\frac{1}{2} + \frac{C}{2}\|k\|_{L^1(\mathbb{R}_+)} + \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\right)\int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k'(s) \\ &+ \left(\frac{C}{4\delta} + \frac{1}{2}\right)(-k'(t))\|\nabla\eta(t,t)\|^2 \\ &+ \left(\frac{1}{2} + \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\right)\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k'(s).\end{aligned}$$

Now, we are in position to prove the main result. For small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, we denote by $V(t)$ the Lyapunov functional

$$V(t) = E + \mathcal{E} + \varepsilon_1\Phi(t) + \varepsilon_2\Psi(t). \tag{4.15}$$

Exploiting Cauchy-Schwarz and Poincaré inequalities, it is easy to check that, provided ε_1 and ε_2 are small enough, there exists a constant $C > 0$ depending on ε_1 and ε_2 such that $V(t) \geqslant CE(t)$ ($V(t)$ is positive definite). Combining (4.1), (4.2), (4.3) and (4.12) with (4.15) together, we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t) \leqslant{}& \Bigl(\varepsilon_2\Bigl(\delta - \int_0^t k(s)\,\mathrm{d}s\Bigr) + \varepsilon_1\Bigl(\frac{3}{2} + \frac{3}{4}C\|k\|^2_{L^1(\mathbb{R}_+)}\Bigr)\Bigr)\|u_t\|^2 \\ &+ \Bigl(\varepsilon_2\Bigl(\delta - \int_0^t k(s)\,\mathrm{d}s\Bigr) + \varepsilon_1\Bigr)\|\nabla u_t\|^2 + \Bigl(\varepsilon_2\delta - \frac{3}{4}\varepsilon_1\Bigr)\|\nabla u\|^2 \\ &+ \Bigl(\varepsilon_1\frac{C}{2}\|k\|_{L^1(\mathbb{R}_+)} + \varepsilon_2\Bigl(\frac{1}{2} + \frac{C}{2}\|k\|_{L^1(\mathbb{R}_+)} + \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\Bigr) - \frac{1}{2}\Bigr) \\ &\times \int_0^t \|\eta(t,s)\|^2\,\mathrm{d}k'(s) \\ &+ \Bigl(-\frac{1}{2} + \varepsilon_2\Bigl(\frac{C}{4\delta} + \frac{1}{2} + \frac{C}{2}\Bigr)\Bigr)(-k'(t))\|\eta(t,t)\|^2 \\ &\times \Bigl(-\frac{1}{2} + \frac{3}{4}\varepsilon_1 C + \varepsilon_2\Bigl(\frac{C}{4\delta} + \frac{1}{2}\Bigr)\Bigr)(-k'(t))\|\nabla\eta(t,t)\|^2 \\ &+ \Bigl(-\frac{1}{2} + \varepsilon_2\Bigl(\frac{1}{2} + \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\Bigr)\Bigr)\int_0^t \|\nabla\eta(t,s)\|^2\,\mathrm{d}k'(s)\end{aligned}$$

for every $t \geqslant t_0$. Therefore, fixing δ small such that $\delta < 3k_0/8(1 + C\|k\|^2_{L^1(\mathbb{R}_+)})$, $k_0 = \int_0^{t_0} k(s)\,\mathrm{d}s$. Thus

$$\delta - \int_0^t k(s)\,\mathrm{d}s \leqslant -\frac{k_0}{2}.$$

Once $\delta > 0$ fixed, we take $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough and such that

$$\frac{4}{3}\delta\varepsilon_2 < \varepsilon_1 < \varepsilon_2\frac{k_0}{2(1 + C\|k\|^2_{L^1(\mathbb{R}_+)})}, \tag{4.16}$$

$$\varepsilon_2\Bigl(\frac{C}{4\delta} + \frac{1}{2} + \frac{C}{2}\Bigr) < \frac{1}{4}$$

and

$$\frac{3}{4}\varepsilon_1 C + \varepsilon_2\Bigl(\frac{C}{4\delta} + \frac{1}{2}\Bigr) < \frac{1}{4}, \quad \varepsilon_2\Bigl(\frac{1}{2} + \frac{C}{4\delta}\|k\|_{L^1(\mathbb{R}_+)}\Bigr) < \frac{1}{4}.$$

Therewith (4.16) imply that

$$\varepsilon_2\Bigl(\delta - \int_0^t k(s)\,\mathrm{d}s\Bigr) + \varepsilon_1\Bigl(\frac{3}{2} + \frac{3}{4}C\|k\|^2_{L^1(\mathbb{R}_+)}\Bigr) < 0, \quad \varepsilon_2\delta - \frac{3}{4}\varepsilon_1 < 0.$$

We then end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) \leqslant -CE(t), \quad t \geqslant t_0. \tag{4.17}$$

Integrating both sides of (4.17) on (t_0, t), we get

$$V(t) - V(t_0) \leqslant -C \int_{t_0}^{t} E(s)\,\mathrm{d}s,$$

which yields that

$$\int_{t_0}^{t} E(s)\,\mathrm{d}s \leqslant \frac{1}{C} V(t_0).$$

We recall that from (4.1), $E(t)$ is decreasing. Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(tE(t)) = E(t) + tE'(t) \leqslant E(t).$$

Performing an integration over (t_0, t), we deduce

$$tE(t) - t_0 E(t_0) \leqslant \int_{t_0}^{t} E(s)\,\mathrm{d}s \leqslant \frac{1}{C} V(t_0),$$

which implies that for any $t \geqslant t_0$

$$E(t) \leqslant \frac{C}{t} \tag{4.18}$$

for some positive constant C. On the other hand, we have

$$tE(t) \leqslant t_0 E(0) \quad \text{on } [0, t_0]. \tag{4.19}$$

Hence, we deduce from (4.18) and (4.19)

$$E(t) \leqslant \frac{C}{t+1} \quad \text{for any } t \geqslant 0,$$

which is exactly the desired inequality in Theorem 3.2. □

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