COVARIANTIZATION OF QUANTIZED CALCULI OVER QUANTUM GROUPS

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Abstract. We introduce a method for construction of a covariant differential calculus over a Hopf algebra A from a quantized calculus $da = [D, a]$, $a \in A$, where D is a candidate for a Dirac operator for A. We recover the method of construction of a bicovariant differential calculus given by T. Brzeziński and S. Majid created from a central element of the dual Hopf algebra A°. We apply this method to the Dirac operator for the quantum SL(2) given by S. Majid. We find that the differential calculus obtained by our method is the standard bicovariant 4D-calculus. We also apply this method to the Dirac operator for the quantum SL(2) given by P. N. Bibikov and P. P. Kulish and show that the resulted differential calculus is 8-dimensional.

Keywords: Hopf algebra; quantum group; covariant first order differential calculus; quantized calculus; Dirac operator

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1. Introduction

In Connes' noncommutative differential geometry, the quantized differential calculus over a *-algebra A is given by $d_D a = [D, a]$, built on a "Dirac operator" D, acting on a Hilbert space $\mathcal{H}$ (see [3]). On the other hand, in the theory of quantum groups one usually needs covariant differential calculi over a Hopf algebra A (see [7]). Since Connes' calculus is not covariant, it seems that these two theories do not match with each other. Our goal in this paper is to convert any differential calculus over a Hopf algebra to a covariant one.

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Our strategy to do this task is as follows. Let (Γ, d) be a left covariant first order differential calculus (l.c.FODC) over a Hopf algebra A and let $\omega\colon\ A \to {}_{\mathrm{inv}}\Gamma$ be the *fundamental map* generating the space of left invariant 1-forms, i.e.

$$\omega(a) := S(a_{(1)})da_{(2)}, \quad a \in A, \tag{1.1}$$

where S is the antipode of A (see [4]). It is known that Γ is freely generated by the set $\omega(A)$ as a left A-module and $\omega(A)$ is closed under the right adjoint action of A on Γ. Namely, we have

$$Ad_r(b)\omega(a) = \omega(\overline{a}b), \quad a, b \in A, \tag{1.2}$$

where $Ad_r(b)(\varrho) = S(b_{(1)})\varrho b_{(2)}$, $b \in A$, $\varrho \in \Gamma$. On the other hand, if (Γ, d) is a FODC (not necessarily l.c.) over the Hopf algebra A, then we can still define the map ω by (1.1). We have $\Gamma = A\omega(A) = \omega(A)A$ and ω obeys the relation (1.2), but since Γ is not freely generated by the set $\omega(A)$ as a left A-module, in general Γ is not left covariant. The simple but essential idea of this paper is to replace the not necessarily free left action of A on $\omega(A)$ by the formal free left action. Hence we convert any FODC, Γ, to a l.c.FODC, which is the smallest l.c.FODC with Γ as its quotient.

In Connes' approach, the essential idea is to define the differential by $da = [D, a]$, $a \in A$. But in our approach, the essential idea is to introduce left invariant 1-forms as operators

$$\omega(a) := S(a_{(1)})[D, a_{(2)}], \quad a \in A,$$

and then construct a covariant FODC based on these invariant forms (see [7]). We apply this method to an operator constructed from a central element of the dual Hopf algebra A° and we find that our method gives a bicovariant FODC over A which coincides with the FODC given in [2]. We also apply this method to the Dirac operator for $A = \mathrm{SL}_q(2)$ constructed by Majid in [6]. We show that the FODC obtained by this Dirac operator is bicovariant and 4-dimensional, and it is indeed the standard 4D-calculus of $\mathrm{SL}_q(2)$. Finally, we apply our method to the Dirac operator constructed by Bibikov and Kulish over $\mathrm{SL}_q(2)$ (see [1]), and show that it is 8-dimensional.

2. Preliminaries

Throughout this paper, we follow the notation of [4]. A denotes a Hopf algebra over $\mathbb{C}$ with coproduct Δ, antipode S and counit ε. We use the Sweedler's notation $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ and most often we omit the summation symbol. For $a \in A$, we use the notation $\overline{a} = a - \epsilon(a)1$. We first recall some concepts from [4].

A first order differential calculus (abbreviated a FODC) over an algebra X is a X-bimodule Γ with a linear mapping $d\colon X \to \Gamma$ such that (i) d satisfies the Leibniz rule $d(xy) = x \cdot dy + dx \cdot y$ for any $x, y \in X$, (ii) Γ is the linear span of elements $x \cdot dy \cdot z$ with $x, y, z \in X$. A left-covariant bimodule (abbreviated l.c. bimodule) over Hopf algebra A is a bimodule Γ over A which is a left comodule of A with coaction $\Delta_\Gamma\colon \Gamma \to A \otimes \Gamma$, such that $\Delta_\Gamma(a\varrho b) = \Delta(a)\Delta_\Gamma(\varrho)\Delta(b)$ for $a, b \in A$ and $\varrho \in \Gamma$. In Sweedler's notation, the last condition can be written as $\sum (a\varrho b)_{(-1)} \otimes (a\varrho b)_{(0)} = \sum a_{(1)}\varrho_{(-1)}b_{(1)} \otimes a_{(2)}\varrho_{(0)}b_{(2)}$. An element ϱ of a left-covariant bimodule Γ is called left-invariant if $\Delta_\Gamma(\varrho) = 1 \otimes \varrho$. The vector space of left-invariant elements of Γ is denoted by ${}_{\mathrm{inv}}\Gamma$. A FODC Γ over A is called left-covariant if it is left-covariant as an A-bimodule with the left coaction $\Delta_\Gamma\colon \Gamma \to A \otimes \Gamma$ and, moreover, $\Delta_\Gamma(adb) = \Delta(a)(\mathrm{id} \otimes d)\Delta(b)$ for all $a, b \in A$.

There is a well-known one-to-one correspondence between l.c. A-bimodules and right A-modules as follows (see [4], Chapter 13, pages 474–475). Let $(\Lambda, \triangleleft)$ be a right A-module. By defining

$$b(a \otimes \alpha)c := bac_{(1)} \otimes \alpha \triangleleft c_{(2)}, \tag{2.1}$$

$$\Delta_\Gamma(a \otimes \alpha) := a_{(1)} \otimes a_{(2)} \otimes \alpha \tag{2.2}$$

for all $a, b, c \in A$, $\alpha \in \Lambda$, the vector space $\Gamma := A \otimes \Lambda$ becomes a l.c. bimodule over A (see [4]). Conversely, let Γ be a l.c. bimodule over A and let Λ be the subspace of left invariant elements of Γ. For $a \in A$, $\alpha \in \Lambda$, we set

$$\alpha \triangleleft a := Ad_r(a)\alpha = S(a_{(1)})\alpha a_{(2)}. \tag{2.3}$$

This is a right A-module structure on Λ. Let $\Gamma' := A \otimes \Lambda$ denote the l.c. A-bimodule given by (2.1) and (2.2) with respect to this right A-action (2.3). It is known that Γ and Γ' are isomorphic as l.c. A-bimodules (see [4]). Now let (Γ, d) be a l.c.FODC over the Hopf algebra A. We define the *fundamental form* of Γ as the map

$$\omega(a) := S(a_{(1)})da_{(2)}, \quad a \in A, \tag{2.4}$$

the *fundamental ideal* of Γ as the following right ideal of $\ker\epsilon$,

$$R = \{a \in \ker\epsilon\colon\ \omega(a) = 0\}, \tag{2.5}$$

and the *tangent space* of Γ as the following set of linear forms on A,

$$T = \{X \in A' \colon\ X(1) = X(a) = 0 \text{ for all } a \in R\}. \tag{2.6}$$

3. Covariantization of a FODC

Definition 3.1. A differential right module (abbreviated DRM) over a Hopf algebra A is a triple $(\Lambda, \triangleleft, \omega)$, where

(i) $(\Lambda, \triangleleft)$ is a right A-module, $\triangleleft\colon\ \Lambda \otimes A \to \Lambda$, and
(ii) $\omega\colon\ A \to \Lambda$ is a surjective linear map satisfying

$$\omega(ab) = \omega(a) \triangleleft b + \epsilon(a)\omega(b), \quad a, b \in A. \tag{3.1}$$

Lemma 3.1. *There is a correspondence between the classes of all l.c.FODC's $(\Gamma, d, \Delta_\Gamma)$ and all DRM's $(\Lambda, \triangleleft, \omega)$ over a Hopf algebra A as follows:*

(i) *If $(\Gamma, d, \Delta_\Gamma)$ is a l.c.FODC over A, then Λ is defined as the space of left invariant 1-forms, $\triangleleft$ is defined by (2.3) and ω is the fundamental form of Γ.*
(ii) *Conversely, given a DRM $(\Lambda, \triangleleft, \omega)$ then $\Gamma := A \otimes \Lambda$ equipped with (2.1), (2.2) and*

$$da := a_{(1)} \otimes \omega(a_{(2)}), \quad a \in A. \tag{3.2}$$

Proof. (i) As we mentioned in the previous section, $(\Lambda, \triangleleft)$ is a right A-module. We have

$$\begin{aligned}\omega(ab) &= S((ab)_{(1)})d(ab)_{(2)} = S(b_{(1)})S(a_{(1)})a_{(2)}db_{(2)} + S(b_{(1)})S(a_{(1)})da_{(2)}b_{(2)} \\ &= \epsilon(a)S(b_{(1)})db_{(2)} + S(b_{(1)})\omega(a)b_{(2)} = \epsilon(a)\omega(b) + \omega(a) \triangleleft b,\end{aligned}$$

so $\omega(ab) = \omega(a) \triangleleft b + \epsilon(a)\omega(b)$. Now we show that ω is surjective. By the definition of a FODC, we have $\Gamma = AdA$. According to Chapter 13 of [4], first we show that for any $a \in A$, $\omega(a) = P(da)$, where $P := \cdot(S \otimes \mathrm{id}_A)\Delta_\Gamma$ has been introduced in Lemma 1 of Chapter 13 of [4] (page 473–474). Here $\cdot\colon\ A \otimes \Gamma \to \Gamma$ is the left action of A on Γ. We have

$$P(da) = \cdot((S \otimes \mathrm{id}_A)\Delta_\Gamma(da)) = \cdot((S \otimes \mathrm{id}_A)(a_{(1)} \otimes da_{(2)})) = S(a_{(1)})da_{(2)} = \omega(a).$$

Also, if $\alpha \in \Lambda$ then $P(\alpha) = \alpha$ (see [4]). Now since $\Gamma = AdA$ then for $\alpha \in \Lambda \subseteq \Gamma$ there exist some elements $x_i, y_i \in A$ such that $\alpha = \sum\limits_i x_i dy_i$. According to the formula (3) on page 473 of [4], $\alpha = P(\alpha) = \sum\limits_i \epsilon(x_i)\omega(y_i) = \omega\big(\sum \epsilon(x_i)y_i\big)$. Thus ω is surjective.

(ii) In the previous section we mentioned that (Γ, Δ_Γ) is a l.c. A-bimodule. Now we have

$$\begin{aligned} d(ab) &= (ab)_{(1)} \otimes \omega((ab)_{(2)}) = a_{(1)}b_{(1)} \otimes \omega(a_{(2)}b_{(2)}) \\ &= a_{(1)}b_{(1)} \otimes (\omega(a_{(2)}) \triangleleft b_{(2)} + \epsilon(a_{(2)})\omega(b_{(2)})) \\ &= a_{(1)}b_{(1)} \otimes \omega(a_{(2)}) \triangleleft b_{(2)} + ab_{(1)} \otimes \omega(b_{(2)}) \\ &= a_{(1)}b_{(1)} \otimes \omega(a_{(2)}) \triangleleft b_{(2)} + a(b_{(1)} \otimes \omega(b_{(2)})) \\ &= (a_{(1)} \otimes \omega(a_{(2)}))b + a(b_{(1)} \otimes \omega(b_{(2)})) = (da)b + a(db). \end{aligned}$$

So the linear map d satisfies the Leibniz rule. To show that $\Gamma = AdA$, let $\varrho = a \otimes \alpha \in \Gamma$. By the surjectivity of ω, there is an element $b \in A$ such that $\alpha = \omega(b)$. Therefore $\varrho = a \otimes \omega(b)$ and

$$\begin{aligned} \varrho &= a \otimes \omega(b) = a(1 \otimes \omega(b)) = a(\epsilon(b_{(1)}) \otimes \omega(b_{(2)})) = a(S(b_{(1)})b_{(2)} \otimes \omega(b_{(3)})) \\ &= (aS(b_{(1)}))(b_{(2)} \otimes \omega(b_{(3)})) = (aS(b_{(1)}))(db_{(2)}). \end{aligned}$$

Thus $\varrho \in AdA$ and (Γ, d) is a FODC. Finally, for all $a \in A$ we have

$$\begin{aligned} \Delta_\Gamma(da) &= \Delta_\Gamma(a_{(1)} \otimes \omega(a_{(2)})) = a_{(1)} \otimes a_{(2)} \otimes \omega(a_{(3)}) \\ &= a_{(1)} \otimes da_{(2)} = (\mathrm{id} \otimes d)(a_{(1)} \otimes a_{(2)}) = (\mathrm{id} \otimes d)(\Delta(a)). \end{aligned}$$

Thus $(\Gamma, d, \Delta_\Gamma)$ is a l.c.FODC. □

Proposition 3.1. *Let $(\Lambda, \triangleleft, \omega)$ be the DRM associated with a l.c.FODC $(\Gamma, d, \Delta_\Gamma)$ by part* (i) *of Lemma* 3.1 *and also $(\Gamma', d', \Delta_{\Gamma'})$ be the l.c.FODC constructed from this DRM $(\Lambda, \triangleleft, \omega)$ by part* (ii) *of Lemma* 3.1. *Then $(\Gamma, d, \Delta_\Gamma)$ and $(\Gamma', d', \Delta_{\Gamma'})$ are isomorphic as l.c.FODC's.*

Proof. We have

$$(\Gamma, d, \Delta_\Gamma) \xrightarrow{\text{part (i) of Lemma 3.1}} (\Lambda, \triangleleft, \omega) \xrightarrow{\text{part (ii) of Lemma 3.1}} (\Gamma', d', \Delta_{\Gamma'}) \ .$$

We define

$$\nu\colon\ \Gamma \to \Gamma', \quad \nu(\alpha) = (\mathrm{id} \otimes P) \circ \Delta_\Gamma(\alpha),$$

where the map P was introduced in the proof of Lemma 3.1. It is well-known that ν is an isomorphism of l.c. bimodules ([4], page 475). We must show that for all $a \in A$,

$$\nu(da) = d'a.$$

We have $\nu(da) = a_{(1)} \otimes P(da_{(2)}) = a_{(1)} \otimes S(a_{(2)})da_{(3)} = a_{(1)} \otimes \omega(a_{(2)}) = d'a$. □

Theorem 3.1. *Let (Γ, d) be a FODC over A. Then we obtain a DRM $(\Lambda, \triangleleft, \omega)$ over A by defining $\Lambda = \omega(A)$, where $\omega\colon A \to \Gamma$ is the fundamental form of Γ and $\alpha \triangleleft a = S(a_{(1)})\alpha a_{(2)}$. Hence, by part* (ii) *of Lemma* 3.1 *we obtain a l.c.FODC $(\Gamma', d', \Delta_{\Gamma'})$. The map $\zeta\colon \Gamma' \to \Gamma$, $a \otimes b \mapsto ab$ for $a \in A$, $b \in \Lambda = \omega(A)$, is a surjective map of FODC's such that $\zeta({}_{\rm inv}\Gamma') \subseteq \omega(A)$ and $(\Gamma', d', \Delta_{\Gamma'})$ is the smallest l.c.FODC among all l.c.FODC's for which there exists a map ζ with the above mentioned properties. Finally, if $(\Gamma, d, \Delta_\Gamma)$ is a l.c.FODC, then $(\Gamma, d, \Delta_\Gamma)$ and $(\Gamma', d', \Delta_{\Gamma'})$ are isomorphic as l.c.FODC's.*

Proof. It is clear that Λ is a vector space. We have

$$\begin{aligned}\omega(ab) &= S((ab)_{(1)})d(ab)_{(2)}\\ &= S(b_{(1)})S(a_{(1)})a_{(2)}db_{(2)} + S(b_{(1)})S(a_{(1)})da_{(2)}b_{(2)}\\ &= \epsilon(a)\omega(b) + \omega(a) \triangleleft b\end{aligned}$$

for all $a, b \in A$. Thus $\omega(a) \triangleleft b = \omega(ab - \epsilon(a)b)$. This identity shows that Λ is closed with respect to $\triangleleft$. Also it is well-known that $\triangleleft$ is a right action of A on Γ. Thus $\triangleleft$ is a well-defined right action of A on Λ. Thus $(\Lambda, \triangleleft, \omega)$ is a DRM over A. Next we have

$$\begin{aligned}\zeta(a(c\otimes e)b) &= \zeta(acb_{(1)} \otimes (e \triangleleft b_{(2)})) = \zeta(acb_{(1)} \otimes S(b_{(2)})eb_{(3)})\\ &= acb_{(1)}S(b_{(2)})eb_{(3)} = ac\epsilon(b_{(1)})eb_{(2)} = aceb = a\zeta(c\otimes e)b\end{aligned}$$

for all $a, b, c \in A$, $e \in \Lambda$. Also

$$\zeta(d'a) = \zeta(a_{(1)} \otimes \omega(a_{(2)})) = a_{(1)}\omega(a_{(2)}) = a_{(1)}S(a_{(2)})da_{(3)} = \epsilon(a_{(1)})da_{(2)} = da.$$

Thus ζ is a map of FODC's. Next, since $\Gamma = AdA$, then for $\alpha \in \Gamma$ there exist some elements $x_i, y_i \in A$ such that $\alpha = \sum\limits_i x_i dy_i$. Thus $\alpha = \sum\limits_i x_i dy_i = \sum\limits_i x_i \zeta(d'y_i) = \zeta\Bigl(\sum\limits_i x_i d'y_i\Bigr)$ and therefore ζ is surjective and $\Gamma'/\ker(\zeta) \simeq \Gamma$.

Now, for $\alpha = \sum\limits_i a_i \otimes \beta_i \in {}_{\rm inv}\Gamma'$, $a_i \in A$ and $\beta_i \in \omega(A)$ we have $\Delta_{\Gamma'}(\alpha) = 1 \otimes \alpha$, i.e. $\sum\limits_i (a_i)_{(1)} \otimes (a_i)_{(2)} \otimes \beta_i = \sum\limits_i 1 \otimes a_i \otimes \beta_i$. Thus by applying the mapping

$$(m_A \otimes {\rm id}_\Gamma)(S \otimes {\rm id}_A \otimes {\rm id}_\Gamma)$$

followed by the left action of A on Γ to both sides of the latter equation, where $m_A\colon A \otimes A \to A$ is the product of A, we get $\sum\limits_i S((a_i)_{(1)})(a_i)_{(2)}\beta_i = \sum\limits_i S(1)a_i\beta_i$, so $\sum\limits_i \epsilon(a_i)\beta_i = \sum\limits_i a_i\beta_i$, and hence $\zeta(\alpha) = \sum\limits_i a_i\beta_i = \sum\limits_i \epsilon(a_i)\beta_i \in \omega(A)$. Therefore $\zeta({}_{\rm inv}\Gamma') \subseteq \omega(A)$.

Next, we show that Γ' is the smallest l.c.FODC pre-quotient of Γ. Suppose that $(\Upsilon, \Delta_\Upsilon)$, $\Delta_\Upsilon(\alpha) = \alpha_{(-1)} \otimes \alpha_{(0)}$ is an arbitrary l.c.FODC and $\psi\colon \Upsilon \to \Gamma$ is a surjective map of FODC's such that $\psi({}_{\rm inv}\Upsilon) \subseteq \omega(A)$. We define $\overline{\psi}\colon \Upsilon \to \Gamma'$, $\overline{\psi} := (\mathrm{id} \otimes \psi)(\mathrm{id} \otimes P_\Upsilon)\Delta_\Upsilon$, where again $P_\Upsilon = \cdot(S \otimes \mathrm{id})\Delta_\Upsilon$, i.e. $P_\Upsilon(\alpha) = S(\alpha_{(-1)})\alpha_{(0)}$. It follows that for all $\alpha \in \Upsilon$

$$\begin{aligned}(\zeta \circ \overline{\psi})(\alpha) &= \zeta(\alpha_{(-2)} \otimes \psi(S(\alpha_{(-1)})\alpha_{(0)})) = \alpha_{(-2)}\psi(S(\alpha_{(-1)})\alpha_{(0)})\\ &= \psi(\alpha_{(-2)}S(\alpha_{(-1)})\alpha_{(0)}) = \psi(\alpha).\end{aligned}$$

Therefore, $\zeta \circ \overline{\psi} = \psi$.

Finally, if (Γ, d) is left-covariant, then $\Lambda = \omega(A) = {}_{\rm inv}\Gamma$. Therefore, by Proposition 3.1, (Γ, d) is isomorphic with (Γ', d'). □

Corollary 3.1. *Let V be a complex vector space and $\pi\colon A \to L(V)$ be an algebra representation of the Hopf algebra A in V, where $L(V)$ denotes the algebra of linear endomorphisms of V. Also, let D be a linear operator on V. Then the map d: $A \to L(V)$, $da := [D, \pi(a)]$ is a differential operator and the space $\Gamma := A(dA)A$ equipped with d and A-bimodule structure given by $aT := \pi(a)T$, $Ta := T\pi(a)$ for all $a \in A$ and $T \in L(V)$ is a FODC over A. Then by Theorem* 3.1 *we obtain a DRM $\Lambda = \omega_D(A)$ where $\omega_D\colon A \to L(V)$,*

$$\omega_D(a) := \pi(S(a_{(1)}))[D, \pi(a_{(2)})], \quad a \in A. \tag{3.3}$$

Here the bracket denotes the commutator of two operators.

The proof is obvious. We denote the l.c.FODC associated with this triple by Γ_D.

R e m a r k 3.1. Let (A, H, D) be a commutative spectral triple where A is the Hopf algebra of smooth functions over a Lie group. Then, since it is known that the quantized calculus $da = [D, a]$ is the classical calculus, which is automatically bicovariant (see [3]), we conclude that covariantization of this calculus by our approach using the Dirac operator D gives the classical calculus.

According to the previous Corollary, we have the following result.

Proposition 3.2. *Let $a \mapsto L_a$ for $a \in A$ denote the left regular representation of a Hopf algebra A on itself, where $L_a(b) = ab$, $b \in A$, and let φ be a linear functional on A. We define the operator D_φ on A by*

$$D_\varphi(a) := a_{(1)}\varphi(a_{(2)}), \quad a \in A. \tag{3.4}$$

(i) *The map (3.3), which we denote by* ω_φ, *takes the form*

$$(\omega_\varphi(a))(x) = x_{(1)}\varphi(\overline{a}x_{(2)}), \quad a, x \in A. \tag{3.5}$$

We denote the associated l.c.FODC by Γ_φ.

(ii) *The fundamental ideal of* Γ_φ *is*

$$R_\varphi = \{a \in \ker\epsilon\colon\ \varphi(ax) = 0 \text{ for all } x \in A\}. \tag{3.6}$$

(iii) *If the dual Hopf algebra* A° *(see [1]) separates the elements of* A *and* $\varphi \in A^\circ$ *is a central element, then the tangent space of* Γ_φ *is*

$$T_\varphi = \operatorname{span}\{X_a := \varphi_{(2)}(a)\varphi_{(1)} - \varphi(a)\epsilon\colon\ a \in A\}, \tag{3.7}$$

where $\Delta\varphi = \varphi_{(1)} \otimes \varphi_{(2)}$ *is the coproduct of Hopf algebra* A°. *Moreover,* Γ_φ *is finite-dimensional and bicovariant. Finally we have* $D_\varphi(a) := \varphi(a_{(1)})a_{(2)}$.

Proof. We have

$$\pi\colon\ A \to L(A), \quad a \mapsto L_a.$$

For $x \in A$, $\pi(ab)(x) = L_{ab}(x) = ab(x) = L_a(L_b(x)) = (\pi(a)\pi(b))(x)$. Thus π is a linear representation.

(i) According to the definition of D_φ,

$$\begin{aligned}
(\omega_\varphi(a))(x) &= (\pi(S(a_{(1)}))[D_\varphi, \pi(a_{(2)})])(x)\\
&= \pi(S(a_{(1)}))(D_\varphi\pi(a_{(2)})(x) - \pi(a_{(2)})D_\varphi(x))\\
&= \pi(S(a_{(1)}))(D_\varphi(a_{(2)}x) - \pi(a_{(2)})x_{(1)}\varphi(x_{(2)}))\\
&= \pi(S(a_{(1)}))D_\varphi(a_{(2)}x) - \pi(S(a_{(1)})a_{(2)})x_{(1)}\varphi(x_{(2)})\\
&= S(a_{(1)})a_{(2)}x_{(1)}\varphi(a_{(3)}x_{(2)}) - S(a_{(1)})a_{(2)}x_{(1)}\varphi(x_{(2)})\\
&= x_{(1)}\varphi(\epsilon(a_{(1)})a_{(2)}x_{(2)}) - \epsilon(a)x_{(1)}\varphi(x_{(2)})\\
&= x_{(1)}\varphi(ax_{(2)} - \epsilon(a)x_{(2)}) = x_{(1)}\varphi(\overline{a}x_{(2)}).
\end{aligned}$$

(ii) Let R be the fundamental ideal of Γ. We show that $R = R_\varphi$. First, we prove that $R \subseteq R_\varphi$. For $a \in R$, we have $\overline{a} = a$ and $\omega(a) = 0$, thus $\omega_\varphi(a)(x) = 0$, and so $x_{(1)}\varphi(ax_{(2)}) = 0$. We get $\epsilon(x_{(1)}\varphi(ax_{(2)})) = 0$, so $\varphi(ax) = 0$. Therefore $a \in R_\varphi$. If $a \in R_\varphi$, then for each $x \in A$, $\varphi(ax) = 0$, therefore $x_{(1)}\varphi(ax_{(2)}) = 0$, and $\omega(a) = 0$ and so $R_\varphi \subseteq R$. Hence $R = R_\varphi$.

(iii) We recall that $A^\circ = \{f \in A' \colon \Delta(f) \in A' \otimes A'\}$, where $\Delta(f)(a \otimes b) = f(ab)$ and A' is the space of all linear functionals on A. Now let $\varphi \in A^\circ$ and $\Delta(\varphi) = \varphi_{(1)} \otimes \varphi_{(2)}$. Let $R' := \{a \in \ker\epsilon \colon X_b(a) = 0 \text{ for all } b \in A\}$. We have

$$\begin{aligned} R' &= \{a \in \ker\epsilon \colon \varphi_{(1)}(a)\varphi_{(2)}(b) = 0 \text{ for all } b \in A\} \\ &= \{a \in \ker\epsilon \colon \varphi(ab) = 0 \text{ for all } b \in A\} = R_\varphi. \end{aligned}$$

Thus R' is a right ideal of $\ker\epsilon$ and we obtain a FODC Γ'. It is well-known that if there are two FODC's with the same fundamental ideal, then they are isomorphic (see [4], Chapter 14, the proof of Proposition 5 for the existence and Proposition 1, part (ii) for the uniqueness). Here, R' is equal to R_φ, so Γ' is isomorphic to Γ_φ, and thus they have identical tangent spaces. On the other hand, Γ' is a bicovariant finite-dimensional FODC over A such that its tangent space is given by

$$T' = \{X_a = \varphi_{(2)}(a)\varphi_{(1)} - \varphi(a)\epsilon \colon a \in A\}$$

(see [4], page 502, Proposition 11). Thus Γ_φ is also a bicovariant finite-dimensional FODC over A and (3.7) is proved.

To prove the last assertion, we let h be an arbitrary linear form in A°. We have

$$\begin{aligned} h(\varphi(a_{(1)})a_{(2)}) &= \varphi(a_{(1)})h(a_{(2)}) = (\varphi h)(a) = (h\varphi)(a) \\ &= h(a_{(1)})\varphi(a_{(2)}) = h(a_{(1)}\varphi(a_{(2)})) \end{aligned}$$

for all $a \in A$. But since A° separates the elements of A, we conclude that

$$\varphi(a_{(1)})a_{(2)} = a_{(1)}\varphi(a_{(2)}), \quad a \in A.$$

Thus $D_\varphi(a) = \varphi(a_{(1)})a_{(2)}$. $\square$

So, we observe that if we choose the operator D in Corollary 3.1 of the form D_φ then the covariant FODC constructed by our method coincides with the covariant FODC constructed by the method mentioned in [2]. Thus we can construct, for example, the standard 4D-calculus over $\mathrm{SL}_q(2)$ through our method of covariantization by choosing φ to be the Casimir element. In the next section, we construct examples of covariant FODC's from operators which are not of this form.

4. Example: The l.c.FODC associated with the Dirac-Majid operator of the quantum group $\mathrm{SL}_q(2)$

In this section, we use our method to answer the question whether there exists a suitably defined operator on some Hilbert space such that the FODC associated to it is the 4D-calculus on quantum SL(2). We find that the FODC associated to the Dirac operator of Majid (see [6]) is 4-dimensional and coincides with the standard 4D-calculus on quantum SL(2).

We take $\mathcal{A} = \mathrm{SL}_q(2)$ and let $\mathcal{A}^\circ$ be its dual Hopf algebra (see [4]). It is well-known that this is a coquasitriangular Hopf algebra (see [4], Chapter 10, [5], Chapter 2 and [6]). Thus it is equipped with the standard universal R-form $\mathrm{R}\colon \mathcal{A}\otimes\mathcal{A} \to \mathbb{C}$. Consider the linear form $\mathrm{Q} = \mathrm{R}_{21}\mathrm{R}\colon \mathcal{A}\otimes\mathcal{A} \to \mathbb{C}$, namely $\mathrm{Q}(a\otimes b) = R(b_{(1)}, a_{(1)})R(a_{(2)}, b_{(2)})$. We view it as a linear map $\mathrm{Q}\colon \mathcal{A} \to \mathcal{A}^\circ$ by evaluation, i.e. $\langle \mathrm{Q}(a), b\rangle = \mathrm{Q}(a\otimes b)$ for $a, b\in\mathcal{A}$. Let W be the spin $\frac{1}{2}$-corepresentation of $\mathcal{A}$ (see [4]), which we view as a two-dimensional representation of $\mathcal{A}^\circ$ with action $\alpha\colon \mathcal{A}^\circ\otimes W \to W$ or equivalently $\alpha\colon \mathcal{A}^\circ \to L(W)$ where $L(W)$ is the algebra of linear operators on W. If $t_{11} = a$, $t_{12} = b$, $t_{21} = c$, $t_{22} = d$ are the standard generators of $\mathcal{A}$ then a basis for W is $\{a, b\}$. If we identify W with $\mathbb{C}^2$ via $a\mapsto e_1$, $b\mapsto e_2$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{C}^2$, then $\alpha(x)$ is the matrix $(\alpha(x))_{ij} = \langle x, t_{ij}\rangle$, $x\in\mathcal{A}^\circ$.

Next, we represent $\mathcal{A}$ on the vector space $\mathcal{A}\oplus\mathcal{A} \simeq \mathcal{A}\otimes\mathbb{C}^2$ as

$$\theta\colon \mathcal{A} \to L(\mathcal{A}\oplus\mathcal{A}),\quad \theta(a)\begin{pmatrix} x\\ y\end{pmatrix} = \begin{pmatrix} ax\\ ay\end{pmatrix},\quad x, y\in\mathcal{A}. \tag{4.1}$$

The Dirac operator defined by Majid (see [6]) on the linear space $\mathcal{A}\oplus\mathcal{A}$ is

$$D = \Big(\partial^i_j - \sum_{k=1}^{2} \mathrm{A}^i_k(\beta(S^{-1}(t_{kj})))\Big)_{1\leqslant i,j\leqslant 2}. \tag{4.2}$$

In other words, for $a = \begin{pmatrix} a^1\\ a^2\end{pmatrix} \in \mathcal{A}\oplus\mathcal{A}$ the entries of $Da = \begin{pmatrix} (Da)^1\\ (Da)^2\end{pmatrix}$ are given by

$$(Da)^i = \sum_{j=1}^{2} \partial^i_j(a^j) - \sum_{j,k=1}^{2} \mathrm{A}^i_k(\beta(S^{-1}(t_{kj})))a^j, \tag{4.3}$$

where

$$\partial^i_j(x) = x_{(1)}\langle \bar{L}^i_j, \bar{x}_{(2)}\rangle = x_{(1)}\langle \underline{L}^i_j, x_{(2)}\rangle \quad \forall\, x\in\mathcal{A}$$

and $\bar{L}^i_j, \underline{L}^i_j \in \mathcal{A}^\circ$ are defined by $\bar{L}^i_j(a) = \mathrm{Q}(a, t_{ij})$ for all $a\in\mathcal{A}$, $\underline{L}^i_j = \bar{L}^i_j - \delta^i_j 1$, δ^i_j is the Kronecker delta, $\beta(a) = (\alpha\circ\mathrm{Q})(\overline{a})$, $\overline{a} = a - \epsilon(a)1$, and $\mathrm{A}^i_j\colon L(W) \to \mathbb{C}$ are some given linear functionals called connections. In the sequel, we need the following $L^{\pm}$ functionals on $\mathcal{A}$,

$$L_j^{+i}(a) = \mathrm{R}(a, t_{ij}), \quad L_j^{-i}(a) = \mathrm{R}(S(t_{ij}), a). \tag{4.4}$$

It is known that

$$\Delta(L_j^{+i}) = \sum_k L_k^{+i} \otimes L_j^{+k}, \quad \Delta(L_j^{-i}) = \sum_k L_k^{-i} \otimes L_j^{-k}, \tag{4.5}$$

and

$$\bar{L}_j^i = \sum_k S(L_k^{-i}) L_j^{+k}. \tag{4.6}$$

We conclude that

$$\Delta(\bar{L}_j^i) = \sum_{k,l=1}^{2} \bar{L}_l^k \otimes S(L_k^{-i}) L_j^{+l}, \tag{4.7}$$

because

$$\begin{aligned}\Delta(\bar{L}_j^i) &= \Delta\left(\sum_m S(L_m^{-i}) L_j^{+m}\right) = \sum_m \left(\sum_k S(L_m^{-k}\right) \otimes S(L_k^{-i}))\left(\sum_l L_l^{+m} \otimes L_j^{+l}\right)\\ &= \sum_{m,k,l} S(L_m^{-k}) L_l^{+m} \otimes S(L_k^{-i}) L_j^{+l} = \sum_{k,l} \bar{L}_l^k \otimes S(L_k^{-i}) L_j^{+l}.\end{aligned}$$

Lemma 4.1. *There is a faithful representation of $M_2(\mathcal{A}^\circ)$, the algebra of 2×2-matrices over $\mathcal{A}^\circ$, in the vector space $\mathcal{A}\oplus\mathcal{A}$ given by*

$$\phi\colon\ M_2(\mathcal{A}^\circ) \to L(\mathcal{A}\oplus\mathcal{A}), \quad u = (u_{ij})_{i,j=1}^2 \mapsto (D_{u_{ij}})_{i,j=1}^2.$$

Namely

$$\phi(u)\begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \begin{pmatrix} D_{u_{11}} a^1 + D_{u_{12}} a^2 \\ D_{u_{21}} a^1 + D_{u_{22}} a^2 \end{pmatrix} \quad \forall\, a^1, a^2 \in \mathcal{A},$$

where

$$D_x(a) := a_{(1)}\langle x, a_{(2)}\rangle, \quad a\in\mathcal{A},\ x\in\mathcal{A}^\circ. \tag{4.8}$$

P r o o f. It is clear that ϕ is linear and we show that ϕ is multiplicative. We first show that D is a faithful representation of $\mathcal{A}^\circ$ in the vector space $\mathcal{A}$. For each $x \in \mathcal{A}^\circ$, D_x is linear and also it is clear that D is linear. We show that D is multiplicative.

$$D_{xy}(a) = a_{(1)}\langle xy, a_{(2)}\rangle = a_{(1)}\langle x, a_{(2)}\rangle\langle y, a_{(3)}\rangle = D_x(a_{(1)}\langle y, a_{(2)}\rangle) = (D_x \circ D_y)(a).$$

To show that D is faithful, let $D_x = 0$. Thus $D_x(a) = 0$ for all $a \in \mathcal{A}$, so $a_{(1)}\langle x, a_{(2)}\rangle = 0$. By applying the counit map to the latter, we get $\langle x, a\rangle = 0$ for all $a \in \mathcal{A}$. Thus we conclude that $x = 0$.

Now for all $u, v \in M_2(\mathcal{A}^\circ)$, we have

$$\begin{aligned}\left(\phi(uv)\begin{pmatrix} a^1 \\ a^2 \end{pmatrix}\right)^i &= \sum_j D_{(uv)_{ij}} a^j = \sum_{j,k} D_{u_{ik}v_{kj}} a^j \\ &= \sum_{j,k} D_{u_{ik}} D_{v_{kj}} a^j = \left(\phi(u)\phi(v)\begin{pmatrix} a^1 \\ a^2 \end{pmatrix}\right)^i.\end{aligned}$$

Thus ϕ is a representation. The faithfulness of ϕ is obtained by the faithfulness of D. $\square$

Henceforth, we embed $M_2(\mathcal{A}^\circ)$ in $L(\mathcal{A} \oplus \mathcal{A})$ by identifying (u_{ij}), $u_{i,j} \in \mathcal{A}^\circ$, with the linear operator $(D_{u_{ij}})$ on $\mathcal{A} \oplus \mathcal{A}$.

Theorem 4.1. *By applying our method of covariantization to Majid's Dirac operator of the quantum group* $\mathrm{SL}_q(2)$, *the associated fundamental form is*

$$\omega_M(a) = \sum_{k,l=1}^{2} \langle \bar{L}_l^k, \overline{a}\rangle (S(L_k^{-i})L_j^{+l})_{i,j=1}^2, \tag{4.9}$$

the associated fundamental ideal is

$$R_M = \ker\epsilon \cap \ker\beta = \{a \in \ker\epsilon\colon\ \bar{L}_j^i(a) = 0 \text{ for all } i, j = 1, 2\}, \tag{4.10}$$

and the associated tangent space is

$$T_M = \operatorname{span}\{\bar{L}_j^i - \epsilon_U(\bar{L}_j^i)1\colon\ i, j = 1, 2\}. \tag{4.11}$$

The l.c.FODC associated to this operator denoted by Γ_M *is nothing other than the well-known* $4D$*-calculus over quantum group* $\mathrm{SL}_q(2)$ *and therefore is bicovariant.*

P r o o f. According to the representation (4.1) and Corollary 3.1, for $a \in \ker\epsilon$, $x^1, x^2 \in \mathcal{A}$ we have

$$
\begin{aligned}
\omega_M(a)\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} &= \theta(S(a_{(1)}))D\theta(a_{(2)})\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \theta(S(a_{(1)}))D\begin{pmatrix} a_{(2)}x^1 \\ a_{(2)}x^2 \end{pmatrix} \\
&= \theta(S(a_{(1)}))\begin{pmatrix} \sum\limits_{j=1}^{2}\partial^1_j(a_{(2)}x^j) - \sum\limits_{j,k=1}^{2}\mathrm{A}^1_k(\beta(S^{-1}(t_{kj})))a_{(2)}x^j \\ \sum\limits_{j=1}^{2}\partial^2_j(a_{(2)}x^j) - \sum\limits_{j,k=1}^{2}\mathrm{A}^2_k(\beta(S^{-1}(t_{kj})))a_{(2)}x^j \end{pmatrix} \\
&= \begin{pmatrix} S(a_{(1)})\sum\limits_{j=1}^{2}\partial^1_j(a_{(2)}x^j) - \sum\limits_{j,k=1}^{2}\mathrm{A}^1_k(\beta(S^{-1}(t_{kj})))S(a_{(1)})a_{(2)}x^j \\ S(a_{(1)})\sum\limits_{j=1}^{2}\partial^2_j(a_{(2)}x^j) - \sum\limits_{j,k=1}^{2}\mathrm{A}^2_k(\beta(S^{-1}(t_{kj})))S(a_{(1)})a_{(2)}x^j \end{pmatrix} \\
&= \begin{pmatrix} S(a_{(1)})\sum\limits_{j=1}^{2}\partial^1_j(a_{(2)}x^j) - \sum\limits_{j,k=1}^{2}\mathrm{A}^1_k(\beta(S^{-1}(t_{kj})))\epsilon(a)x^j \\ S(a_{(1)})\sum\limits_{j=1}^{2}\partial^2_j(a_{(2)}x^j) - \sum\limits_{j,k=1}^{2}\mathrm{A}^2_k(\beta(S^{-1}(t_{kj})))\epsilon(a)x^j \end{pmatrix} \\
&= \begin{pmatrix} S(a_{(1)})\sum\limits_{j=1}^{2}\partial^1_j(a_{(2)}x^j) \\ S(a_{(1)})\sum\limits_{j=1}^{2}\partial^2_j(a_{(2)}x^j) \end{pmatrix} = \begin{pmatrix} S(a_{(1)})\sum\limits_{j=1}^{2}a_{(2)}x^j_{(1)}\langle \underline{L}^1_j, a_{(3)}x^j_{(2)}\rangle \\ S(a_{(1)})\sum\limits_{j=1}^{2}a_{(2)}x^j_{(1)}\langle \underline{L}^2_j, a_{(3)}x^j_{(2)}\rangle \end{pmatrix} \\
&= \begin{pmatrix} \sum\limits_{j=1}^{2}x^j_{(1)}\langle \underline{L}^1_j, ax^j_{(2)}\rangle \\ \sum\limits_{j=1}^{2}x^j_{(1)}\langle \underline{L}^2_j, ax^j_{(2)}\rangle \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=1}^{2}x^j_{(1)}\langle \bar{L}^1_j - \delta^1_j 1, ax^j_{(2)}\rangle \\ \sum\limits_{j=1}^{2}x^j_{(1)}\langle \bar{L}^2_j - \delta^2_j 1, ax^j_{(2)}\rangle \end{pmatrix} \\
&= \begin{pmatrix} \sum\limits_{j=1}^{2}x^j_{(1)}\langle \bar{L}^1_j, ax^j_{(2)}\rangle \\ \sum\limits_{j=1}^{2}x^j_{(1)}\langle \bar{L}^2_j, ax^j_{(2)}\rangle \end{pmatrix} = \begin{pmatrix} \sum\limits_{j,k,l}x^j_{(1)}\langle \bar{L}^k_l, a\rangle\langle S(L^{-1}_k)L^{+l}_j, x^j_{(2)}\rangle \\ \sum\limits_{j,k,l}x^j_{(1)}\langle \bar{L}^k_l, a\rangle\langle S(L^{-2}_k)L^{+l}_j, x^j_{(2)}\rangle \end{pmatrix} \\
&= \begin{pmatrix} \sum\limits_{k,l}\langle \bar{L}^k_l, a\rangle S(L^{-1}_k)L^{+l}_1 & \sum\limits_{k,l}\langle \bar{L}^k_l, a\rangle S(L^{-1}_k)L^{+l}_2 \\ \sum\limits_{k,l}\langle \bar{L}^k_l, a\rangle S(L^{-2}_k)L^{+l}_1 & \sum\limits_{k,l}\langle \bar{L}^k_l, a\rangle S(L^{-2}_k)L^{+l}_2 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \\
&= \left(\sum_{k,l=1}^{2}\langle \bar{L}^k_l, a\rangle S(L^{-i}_k)L^{+l}_j\right)^{2}_{i,j=1}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \\
&= \sum_{k,l=1}^{2}\langle \bar{L}^k_l, a\rangle (S(L^{-i}_k)L^{+l}_j)^2_{i,j=1}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.
\end{aligned}
$$

In the above, we used the facts $\varepsilon(a) = 0$ in the first equation in line 5, $\langle 1, ax\rangle = \varepsilon(ax) = \varepsilon(a)\varepsilon(x) = 0$ at the begining of line 7 and the faithful representation (4.8), $u \mapsto D_u$, where $u = S(L^{-i}_k)L^{+l}_j$, at the begining of line 8. So, we proved (4.9) for

$a \in \ker\epsilon$ and the general case is the result of the identity $\omega_M(\overline{a}) = \omega_M(a)$. To prove (4.10), we write $\Delta(\bar{L}^i_j)$ as $\sum \bar{L}^i_{j\,(1)} \otimes \bar{L}^i_{j\,(2)}$ such that for each fixed i, j, the set of all $\bar{L}^i_{j\,(2)}$ is linearly independent. Now we rewrite the above calculation of $\omega_M(a)$ until line 7 and then continue as follows:

$$\omega_M(a)\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=1}^{2} x^j_{(1)}\langle \bar{L}^1_j, a x^j_{(2)}\rangle \\ \sum\limits_{j=1}^{2} x^j_{(1)}\langle \bar{L}^2_j, a x^j_{(2)}\rangle \end{pmatrix} = \begin{pmatrix} \sum\limits_j \sum x^j_{(1)}\langle \bar{L}^1_{j\,(1)}, a\rangle\langle \bar{L}^1_{j\,(2)}, x^j_{(2)}\rangle \\ \sum\limits_j \sum x^j_{(1)}\langle \bar{L}^2_{j\,(1)}, a\rangle\langle \bar{L}^2_{j\,(2)}, x^j_{(2)}\rangle \end{pmatrix}$$
$$= \begin{pmatrix} \sum\langle \bar{L}^1_{1\,(1)}, a\rangle \bar{L}^1_{1\,(2)} & \sum\langle \bar{L}^1_{2\,(1)}, a\rangle \bar{L}^1_{2\,(2)} \\ \sum\langle \bar{L}^2_{1\,(1)}, a\rangle \bar{L}^2_{1\,(2)} & \sum\langle \bar{L}^2_{2\,(1)}, a\rangle \bar{L}^2_{2\,(2)} \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.$$

Now using this computation and our assumption on the linear independence of functionals $\bar{L}^i_{j\,(2)}$ for each fixed i, j, and putting $x^2 = 0$ or $x^1 = 0$, we find that $R_M = \{a \in \ker\epsilon\colon\ \bar{L}^i_{j\,(1)}(a) = 0 \text{ for all } i, j \text{ and for all } (1)\}$. Thus $R_M \subseteq \{a \in \ker\epsilon\colon\ \bar{L}^i_j(ab) = 0 \text{ for all } i,j = 1,2 \text{ and for all } b \in \mathcal{A}\}$. Conversely, if $a \in \ker\epsilon$ and $\bar{L}^i_j(ab) = 0$ for all $b \in \mathcal{A}$, then $\sum \bar{L}^i_{j\,(1)}(a)\bar{L}^i_{j\,(2)} = 0$ for all i, j, so we find that $\bar{L}^i_{j\,(1)}(a) = 0$ for all i, j, (1). Thus

$$R_M = \{a \in \ker\epsilon\colon\ \bar{L}^i_j(ab) = 0 \text{ for all } i,j = 1,2 \text{ and for all } b \in \mathcal{A}\}.$$

On the other hand, by definition we have $\beta(a)^i_j = \bar{L}^i_j(a)$ for $a \in \ker\epsilon$. Therefore $R_M = \{a \in \ker\epsilon\colon\ \beta(ab) = 0 \text{ for all } b \in \mathcal{A}\}$. It is well-known that the set $\{a \in \ker\epsilon\colon\ \beta(a) = 0\}$ is the fundamental ideal associated to the 4D-calculus over $\mathcal{A}$ (see [6]), thus it is a right ideal of $\ker\epsilon$. So we find that $\{a \in \ker\epsilon\colon\ \beta(ab) = 0 \text{ for all } b \in \mathcal{A}\} = \{a \in \ker\epsilon\colon\ \beta(a) = 0\}$ (since for $b \in \mathcal{A}$ we have $\beta(ab) = \beta(a\bar{b}) + \epsilon(b)\beta(a)$). Hence,

$$R_M = \{a \in \ker\epsilon\colon\ \beta(a) = 0\} = \{a \in \ker\epsilon\colon\ \bar{L}^i_j(a) = 0 \text{ for all } i,j = 1,2\}.$$

Thus the proof of (4.10) is now complete and since the fundamental ideal of Γ_M is equal with the fundamental ideal of the 4D-calculus, we conclude that these two l.c.FODC's coincide. Next, let

$$T' := \operatorname{span}\Big\{X^{i,j}_b := \sum_{k,l}\langle S(L^{-i}_k)L^{+l}_j, b\rangle \bar{L}^k_l - \langle \bar{L}^i_j, b\rangle 1\colon\ i,j = 1,2,\ b \in \mathcal{A}\Big\}.$$

By using (4.7), if we set $R' := \{a \in \ker\epsilon\colon\ X(a) = 0 \text{ for all } X \in T'\}$ then we have $R' = \{a \in \ker\epsilon\colon\ \bar{L}^i_j(ab) = 0 \text{ for all } i,j = 1,2 \text{ and for all } b \in \mathcal{A}\}$. Thus R' is a right ideal of $\ker\epsilon$ and therefore there exists a unique l.c.FODC over $\mathcal{A}$ such that its fundamental ideal is R' and its tangent space is T' (see [7] or the proof of Proposition 5 of Chapter 14 in [4]). But since $R' = R_M$, we find that indeed this latter FODC is Γ_M, which is in turn the 4D-calculus, hence $T_M = T'$. On the other hand, it is obvious that $T' \subseteq \operatorname{span}\{\bar{L}^i_j - \langle \bar{L}^i_j, 1\rangle 1\colon i,j = 1,2\}$. But since T_M is four-dimensional we conclude that T' is also four-dimensional and we find that $T' = \operatorname{span}\{\bar{L}^i_j - \langle \bar{L}^i_j, 1\rangle 1\colon i,j = 1,2\}$. So, we recovered the 4D-calculus over $\mathcal{A} = \mathrm{SL}_q(2)$ via our method of covariantization. □

5. Example: The l.c.FODC associated with the Dirac-Kulish-Bibikov operator of $\mathrm{SU}_q(2)$

Let $A = \mathrm{SU}_q(2)$ and $U = U_q(su_2)$. Here, we use the notation of [1]. Hence, we denote the generators of U by k, e, f, k^{-1}. There is a standard nondegenerate dual pairing $\langle , \rangle\colon U \otimes A \to \mathbb{C}$ between U and A which enables us to regard U as a subalgebra of A°. Thus we regard each $u \in U$ as a linear functional over A and write $u(a)$ instead of $\langle u, a\rangle$. Let $\pi_1\colon U \to L(\mathbb{C}^2)$ be the spin $\frac{1}{2}$-representation. That is

$$\pi_1(k) = \begin{bmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{bmatrix}, \quad \pi_1(e) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \pi_1(f) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{5.1}$$

Also we have another representation (4.8) of U induced from the dual pairing

$$\pi_2 : U \to L(A), \quad \pi_2(u)(a) = a_{(1)} u(a_{(2)}). \tag{5.2}$$

Thus, we obtain a representation $\pi\colon U \to L(\mathbb{C}^2 \otimes A)$, $\pi(u) = \pi_1(u_{(1)}) \otimes \pi_2(u_{(2)})$. We set $K := \pi(k)$, $K^{-1} := \pi(k^{-1})$, $E := \pi(e)$, $F := \pi(f)$. Now let $C \in U$ denote the Casimir element. The Dirac operator is defined by

$$D_{KB} = \lambda^{-2}(\pi(C) - \mu \operatorname{id}_{\mathbb{C}^2} \otimes \pi_2(C)) \in L(\mathbb{C}^2 \otimes A), \tag{5.3}$$

where $\lambda = q - q^{-1}$ and $\mu = (q^2 - q^{-2})/(q - q^{-1})$. Next, we represent A on the vector space $\mathbb{C}^2 \otimes A$ by the left regular representation in the second component, i.e.

$$\theta\colon A \to L(\mathbb{C}^2 \otimes A), \quad \theta(a)(x \otimes y) := x \otimes ay, \quad x \in \mathbb{C}^2,\ y \in A. \tag{5.4}$$

Theorem 5.1. *For the Dirac operator $D = D_{KB}$, the associated fundamental form is*

$$\omega_{KB}(a) = \lambda^{-2}(C_{(2)}(a) - \epsilon_U(C_{(2)})\epsilon_A(a))(\pi_1(C_{(1)}) - \mu\epsilon_U(C_{(1)}) \operatorname{id}_{\mathbb{C}^2}) \otimes \pi_2(C_{(3)}), \tag{5.5}$$

the associated fundamental ideal is

$$R_{KB} = \{a \in \ker \epsilon_A \colon\ C(bac) = 0 \text{ for all } b, c \in A\}, \tag{5.6}$$

the associated tangent space is

$$T_{KB} = \operatorname{span}\{X_{b,c} := C_{(1)}(b)C_{(3)}(c)C_{(2)} - C(bc)\epsilon_A \colon\ b, c \in A\}, \tag{5.7}$$

and the resulted l.c.FODC is 8*-dimensional.*

Proof. We have

$$\begin{aligned}\omega_{KB}(a) &= \theta(S(a_{(1)}))[D, \theta(a_{(2)})] = \theta(S(a_{(1)}))D\theta(a_{(2)}) - \theta(S(a_{(1)}))\theta(a_{(2)})D \\ &= \theta(S(a_{(1)}))D\theta(a_{(2)}) - \theta(S(a_{(1)})a_{(2)})D = \theta(S(a_{(1)}))D\theta(a_{(2)}) - \epsilon_A(a)D.\end{aligned}$$

Thus for $a \in \ker \epsilon_A$,

$$\begin{aligned}\lambda^2\omega_{KB}(a)(x \otimes y) &= \lambda^2\theta(S(a_{(1)}))D(x \otimes a_{(2)}y) \\ &= \theta(S(a_{(1)}))(\pi_1(C_{(1)})(x) \otimes a_{(2)}y_{(1)}C_{(2)}(a_{(3)}y_{(2)}) \\ &\quad - \mu x \otimes a_{(2)}y_{(1)}C(a_{(3)}y_{(2)})) \\ &= \pi_1(C_{(1)})(x) \otimes y_{(1)}C_{(2)}(ay_{(2)}) - \mu x \otimes y_{(1)}C(ay_{(2)}) \\ &= \pi_1(C_{(1)})(x) \otimes y_{(1)}C_{(2)}(a)C_{(3)}(y_{(2)}) - \mu x \otimes y_{(1)}C_{(1)}(a)C_{(2)}(y_{(2)}) \\ &= C_{(2)}(a)\pi_1(C_{(1)})(x) \otimes y_{(1)}C_{(3)}(y_{(2)}) - \mu C_{(1)}(a)x \otimes y_{(1)}C_{(2)}(y_{(2)}) \\ &= (C_{(2)}(a)\pi_1(C_{(1)}) \otimes \pi_2(C_{(3)}) - \mu C_{(1)}(a)\,\mathrm{id} \otimes \pi_2(C_{(2)}))(x \otimes y) \\ &= C_{(2)}(a)((\pi_1(C_{(1)}) - \mu\epsilon_U(C_{(1)})) \otimes \pi_2(C_{(3)}))(x \otimes y).\end{aligned}$$

Now, since for $a \in A$ we have $\overline{a} \in \ker \epsilon_A$, $\omega_{KB}(a) = \omega_{KB}(\overline{a})$ and for $u \in U$ we have $u(\overline{a}) = u(a - \epsilon_A(a)1) = u(a) - \epsilon_A(a)\epsilon_U(u)$, we get $\omega_{KB}(a) = \lambda^{-2}(C_{(2)}(a) - \epsilon_U(C_{(2)})\epsilon_A(a))(\pi_1(C_{(1)}) - \mu\epsilon_U(C_{(1)})\,\mathrm{id}) \otimes \pi_2(C_{(3)})$. Thus the proof of (5.5) is complete.

Now we prove (5.6). Let $C_{(1)(1)'} \otimes C_{(1)(2)'} \otimes C_{(2)} \in U^{\otimes 3}$ be a presentation of $\Delta_U^2(C) = \Delta_U(\Delta_U \otimes \mathrm{id}_U)(C)$ such that the set $\{C_{(2)} \colon \text{ for all } (2)\}$ is linearly independent and for each fixed index (1), the set $\{\pi_1(C_{(1)(1)'}) - \mu\epsilon_U(C_{(1)(1)'})\,\mathrm{id}_{\mathbb{C}^2} \colon \text{ for all } (1)'\}$ is also linearly independent (we call such presentation an *extraordinary* presentation and the existence of such presentation will be shown below). Note that this assumption implies that for each fixed index (1), the set $\{C_{(1)(1)'} \colon \text{ for all } (1)'\}$ is linearly independent: for in general the image of a set of linearly dependent vectors under any linear operator is also linearly dependent. Now since the representation π_2 is faithful (see previous section), we conclude that the set $\{\pi_2(C_{(2)}) \colon \text{ for all } (2)\}$ is also linearly independent. Now let $a \in R_{KB}$, i.e. $a \in \ker \epsilon_A$ and $\omega_{KB}(a) = 0$, and let

$P_1, P_2\colon\ \mathbb{C}^2 \to \mathbb{C}$ be the canonical projections. So by combining the operators $P_i \otimes \mathrm{id}_A$ with the operator $\omega_{KB}(a)$, we get

$$C_{(1)\,(2)'}(a) P_i(\pi_1(C_{(1)\,(1)'}) - \mu\epsilon_U(C_{(1)\,(1)'})\,\mathrm{id})\pi_2(C_{(2)}) = 0, \quad i = 1, 2.$$

Thus for each fixed index (1) we have $C_{(1)\,(2)'}(a)(\pi_1(C_{(1)\,(1)'}) - \mu\epsilon_U(C_{(1)\,(1)'})\,\mathrm{id}) = 0$, and by our assumption we find that $C_{(1)\,(2)'}(a) = 0$ for each (1) and $(2)'$. The converse is obviously true, i.e., if $a \in \ker \epsilon_A$ and $C_{(1)\,(2)'}(a) = 0$ for each (1) and $(2)'$, then $\omega_{KB}(a) = 0$. Thus the fundamental ideal is

$$\begin{aligned} R_{KB} &= \{a \in \ker \epsilon_A\colon\ \omega_{KB}(a) = 0\} = \{a \in \ker \epsilon_A\colon\ C_{(1)\,(2)'}(a) = 0 \text{ for all } (1), (2)'\} \\ &\subseteq \{a \in \ker \epsilon_A\colon\ C(bac) = 0 \text{ for all } b, c \in A\}. \end{aligned}$$

Conversely, let $a \in \ker \epsilon_A$ such that $C(bac) = 0$ for all $b, c \in A$. Then

$$C_{(1)\,(1)'}(b) C_{(1)\,(2)'}(a) C_{(2)} = 0 \quad \forall\, b \in A,$$

but since $\{C_{(2)}\colon$ for all $(2)\}$ is linearly independent, we find that for each fixed index (1) and for all $b \in A$ we have $C_{(1)\,(1)'}(b) C_{(1)\,(2)'}(a) = 0$. Thus for each fixed index (1), $C_{(1)\,(2)'}(a) C_{(1)\,(1)'} = 0$. But since $\{C_{(1)\,(1)'}\colon$ for all $(1)'\}$ is linearly independent, we find that $C_{(1)\,(2)'}(a) = 0$. Hence,

$$\{a \in \ker \epsilon_A\colon\ C(bac) = 0 \text{ for all } b, c \in A\} \subseteq \{a \in \ker \epsilon_A\colon\ C_{(1)\,(2)'}(a) = 0\} = R_{KB}.$$

Thus the proof of (5.6) is complete and we have also shown that under an extraordinary presentation of $\Delta_U^2(C)$ we have

$$R_{KB} = \{a \in \ker \epsilon_A\colon\ C_{(1)\,(2)'}(a) = 0 \text{ for all } (1), (2)'\}. \tag{5.8}$$

Now we prove (5.7). It is known that if T is a finite-dimensional vector space of linear functionals on a Hopf algebra A such that $X(1) = 0$ for all $X \in T$ and the set $R = \{a \in \ker \epsilon_A\colon\ X(a) = 0 \text{ for all } X \in T\}$ is a right ideal of $\ker \epsilon_A$, then there exists a *unique* l.c.FODC Γ over A such that its fundamental ideal is R and its tangent space is T (see [4], Chapter 14, the proof of Proposition 5 for the existence and Proposition 1, part (ii) for the uniqueness). Now let $\sum C_{(1)} \otimes C_{(2)} \otimes C_{(3)}$ be an ordinary presentation of $\Delta_U^2(C) \in U^{\otimes 3}$ and let $T = \mathrm{span}\{X_{b,c} := C_{(1)}(b) C_{(3)}(c) C_{(2)} - \epsilon_U(C_{(2)})\epsilon_A\colon\ b, c \in A\}$. We have $T \subset \mathrm{span}\{C_{(2)} - \epsilon_U(C_{(2)})\epsilon_A\colon$ for all $(2)\}$ and thus T is finite-dimensional and

$$\begin{aligned} R &:= \{a \in \ker \epsilon_A\colon\ X(a) = 0 \text{ for all } X \in T\} \\ &= \{a \in \ker \epsilon_A\colon\ C(bac) = 0 \text{ for all } b, c \in A\}. \end{aligned}$$

Thus R is a right ideal of $\ker \epsilon_A$ and since $R = R_{KB}$, we conclude that the l.c.FODC obtained from T is Γ_{KB}, so $T_{KB} = T$. To find the dimension of Γ_{KB} we find a basis for T_{KB}. Above we showed that $T \subset T' := \operatorname{span}\{C_{(2)} - \epsilon_U(C_{(2)})\epsilon_A\colon \text{ for all } (2)\}$ and in the previous paragraph we also showed that for an extraordinary presentation of $\Delta_U^2(C)$, the right ideal $R' := \{a \in \ker \epsilon_A\colon\ X(a) = 0 \text{ for all } a \in T'\}$ is equal with R_{KB}. Thus by the uniqueness, we conclude that the l.c.FODC obtained from T' is isomorphic with Γ_{KB} and thus $T_{KB} = T'$. Therefore the dimension of Γ_{KB} is the dimension of

$$T' := \operatorname{span}\{C_{(1)(2)'} - \epsilon_U(C_{(1)(2)'})\epsilon_A\colon \text{ for all } (1), (2)'\} \tag{5.9}$$

under an extraordinary presentation of $C_{(1)(1)'} \otimes C_{(1)(2)'} \otimes C_{(2)} \in U^{\otimes 3}$. To complete the proof and to find the dimension of this calculus, we find an extraordinary presentation of $\Delta_U^2(C)$ for $q \neq -1, 0, 1$. The Casimir element is given by $C = q^{-1}k^2 + qk^{-2} + \lambda^2 fe$. We have

$$\begin{aligned}
\Delta_U^2(C) &= (\Delta \otimes \mathrm{id})\Delta(C) = C_{(1)(1)'} \otimes C_{(1)(2)'} \otimes C_{(2)} \\
&= ((q^{-1}k^2 + \lambda^2 fe) \otimes k^2 + k^{-2} \otimes \lambda^2 fe + fk^{-1} \otimes \lambda^2 ke + k^{-1}e \otimes \lambda^2 fk) \otimes k^2 \\
&\quad + k^{-2} \otimes qk^{-2} \otimes k^{-2} + (k^{-2} \otimes \lambda^2 fk^{-1} + fk^{-1} \otimes \lambda^2 \cdot 1) \otimes ke \\
&\quad + (k^{-2} \otimes \lambda^2 k^{-1}e + k^{-1}e \otimes \lambda^2 1) \otimes fk + k^{-2} \otimes \lambda^2 k^{-2} \otimes fe.
\end{aligned}$$

Thus the set of all $C_{(2)}$'s is $\{k^2, k^{-2}, ke, fk, fe\}$, which is linearly independent because it is a subset of the standard basis of U, and we have four sets of the elements $C_{(1)(1)}$'s,

$$\begin{gathered}
S_1 = \{q^{-1}k^2 + \lambda^2 fe, k^{-2}, fk^{-1}, k^{-1}e\}, \quad S_2 = \{k^{-2}\}, \\
S_3 = \{k^{-2}, fk^{-1}\}, \quad S_4 = \{k^{-2}, k^{-1}e\}.
\end{gathered}$$

Let $\tau := \pi_1 - \mu\epsilon_U \operatorname{id}_{\mathbb{C}^2}$. We should show that each of the sets $\tau(S_i)$, $i = 1, \ldots, 4$, is linearly independent. A simple calculation shows that $\tau(S_1)$ is

$$\left\{ \begin{bmatrix} q^{-2} - q^{-1}\mu & 0 \\ 0 & 1 + \lambda^2 - q^{-1}\mu \end{bmatrix}, \begin{bmatrix} q - \mu & 0 \\ 0 & q^{-1} - \mu \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & q^{-1/2} \\ 0 & 0 \end{bmatrix} \right\}.$$

Since $(q^{-2} - q^{-1}\mu)(q - \mu)^{-1} \neq (1 + \lambda^2 - q^{-1}\mu)(q^{-1} - \mu)^{-1}$ for $q \neq \pm 1, 0$, this set is linearly independent. Similarly the other sets $\tau(S_i)$, $i = 2, 3, 4$, are linearly independent. Hence the proof now is complete and the dimension of the associated l.c.FODC is the dimension of the vector space $\operatorname{span}\{C_{(1)(2)'} - \epsilon_U(C_{(1)(2)'})\epsilon_A\colon$ for all $(1), (2)'\} = \operatorname{span}\{k^2 - \mu\epsilon_A, fe, ke, fk, k^{-2} - \mu\epsilon_A, fk^{-1}, (1 - \mu)\epsilon_A, k^{-1}e\} = \operatorname{span}\{k^2, fe, ke, fk, k^{-2}, fk^{-1}, 1, k^{-1}e\}$, which is 8-dimensional. □

R e m a r k 5.1. Comparing Majid's Dirac operator with Kulish-Bibikov's Dirac operator, we observe that the former gives better l.c.FODC than the latter and the natural question arises that given a quantum group, which Dirac operator gives the most suitable covariant FODC on this quantum group?

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