OSCILLATION OF DEVIATING DIFFERENTIAL EQUATIONS

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Abstract. Consider the first-order linear delay (advanced) differential equation

$$x'(t)+p(t)x(\tau(t))=0 \quad (x'(t)-q(t)x(\sigma(t))=0), \quad t\geqslant t_0,$$

where p (q) is a continuous function of nonnegative real numbers and the argument $\tau(t)$ ($\sigma(t)$) is not necessarily monotone. Based on an iterative technique, a new oscillation criterion is established when the well-known conditions

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)\,\mathrm{d}s>1 \quad \Bigl(\limsup_{t\to\infty}\int_{t}^{\sigma(t)} q(s)\,\mathrm{d}s>1\Bigr)$$

and

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)\,\mathrm{d}s>\frac{1}{\mathrm{e}} \quad \Bigl(\liminf_{t\to\infty}\int_{t}^{\sigma(t)} q(s)\,\mathrm{d}s>\frac{1}{\mathrm{e}}\Bigr)$$

are not satisfied. An example, numerically solved in MATLAB, is also given to illustrate the applicability and strength of the obtained condition over known ones.

Keywords: differential equation; non-monotone argument; oscillatory solution; nonoscillatory solution; Grönwall inequality

MSC 2020: 34K06, 34K11

1. Introduction

Consider the first-order delay differential equation (DDE)

$$x'(t)+p(t)x(\tau(t))=0, \quad t\geqslant t_0, \tag{E}$$

and the (dual) advanced differential equation (ADE)

$$x'(t)-q(t)x(\sigma(t))=0, \quad t\geqslant t_0, \tag{E$'$}$$

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where $p(t) \geqslant 0$, $q(t) \geqslant 0$, and τ, σ are Lebesgue measurable functions satisfying

$$\tau(t) < t, \quad t \geqslant t_0 \quad \text{and} \quad \lim_{t\to\infty} \tau(t) = \infty \tag{1.1}$$

and

$$\sigma(t) > t, \quad t \geqslant t_0, \tag{1.1$'$}$$

respectively.

Definition 1. A solution of (E) is a function absolutely continuous on $[t_0, \infty)$ and satisfying (E) for all $t \geqslant t_0$. By a solution of (E$'$) we mean a function absolutely continuous on $[t_0, \infty)$ and satisfying (E$'$) for all $t \geqslant t_0$.

Definition 2. A solution of (E) or (E$'$) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory.* An equation is oscillatory if all its solutions oscillate.

The problem of establishing sufficient conditions for the oscillation of all solutions of equations (E) and (E$'$) has been the subject of many investigations. The reader is referred to [1]–[8], [10], [11], [13]–[17], [19]–[23] and the references cited therein. Most of these papers concern a special case where the arguments are nondecreasing. Some of these papers study the general case where the arguments are not necessarily monotone. See, for example, [2]–[8], [14], and the references cited therein. For the general theory of differential equations, the reader is referred to the monographs [9], [12], [18].

1.1. DDEs. In 1972, Ladas et al. (see [16]), and in 1982, Koplatadze and Chanturija (see [13]) proved that if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s > 1 \tag{1.2}$$

or

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{1.3}$$

respectively, then all solutions of (E) are oscillatory.

Assume that the argument $\tau(t)$ is not necessarily monotone. Set

$$h(t) := \sup_{s\leqslant t} \tau(s), \quad t \geqslant t_0. \tag{1.4}$$

Clearly, the function $h(t)$ is nondecreasing and $\tau(t) \leqslant h(t) < t$ for all $t \geqslant t_0$.

In 2011, Braverman and Karpuz (see [1]) proved that if

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\Bigl(\int_{\tau(s)}^{h(t)} p(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}s > 1, \tag{1.5}$$

then all solutions of (E) are oscillatory.

Several improvements were made to the above condition, see [2]–[6] to arrive at the recent form (see [5])

$$\liminf_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\Bigl(\int_{\tau(s)}^{h(s)} p(u)\exp\Bigl(\int_{\tau(u)}^{u} d_l(\xi)\,\mathrm{d}\xi\Bigr)\,\mathrm{d}u\Bigr)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{1.6}$$

where

$$d_l(t) = p(t)\Bigl(1+\int_{\tau(t)}^{t} p(s)\exp\Bigl(\int_{\tau(s)}^{t} p(u)\exp\Bigl(\int_{\tau(u)}^{u} d_{l-1}(\xi)\,\mathrm{d}\xi\Bigr)\,\mathrm{d}u\Bigr)\,\mathrm{d}s\Bigr)$$

with

$$d_0(t) = p(t)\Bigl(1+\int_{\tau(t)}^{t} p(s)\exp\Bigl(\int_{\tau(s)}^{t} p(\omega)\exp\Bigl(\lambda_0\int_{\tau(\omega)}^{\omega} p(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}\omega\Bigr)\,\mathrm{d}s\Bigr)$$

and λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\alpha\lambda}$.

1.2. ADEs. By Theorem 2.4.3 of [18], if

$$\limsup_{t\to\infty}\int_{t}^{\sigma(t)} q(s)\,\mathrm{d}s > 1, \tag{1.7}$$

then all solutions of (E′) are oscillatory.

In 1983, Fukagai and Kusano (see [11]) proved that if

$$\beta := \liminf_{t\to\infty}\int_{t}^{\sigma(t)} q(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{1.8}$$

then all solutions of (E′) are oscillatory.

Assume that the argument $\sigma(t)$ is not necessarily monotone. Set

$$\varrho(t) = \inf_{s\geqslant t}\sigma(s), \quad t\geqslant t_0. \tag{1.9}$$

Clearly, the function $\varrho(t)$ is nondecreasing and $\sigma(t)\geqslant\varrho(t)>t$ for all $t\geqslant t_0$.

In 2015, Chatzarakis and Ocalan (see [7]) proved that if

$$\limsup_{t\to\infty}\int_t^{\varrho(t)} q(s)\exp\Big(\int_{\varrho(t)}^{\sigma(s)} q(u)\,\mathrm{d}u\Big)\,\mathrm{d}s > 1, \tag{1.10}$$

then all solutions of (E′) oscillate.

Several improvements were made to the above condition, see [2]–[7] to arrive at the recent form (see [5])

$$\liminf_{t\to\infty}\int_t^{\varrho(t)} q(s)\exp\Big(\int_{\varrho(s)}^{\sigma(s)} q(u)\exp\Big(\int_u^{\sigma(u)} \varphi_l(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}u\Big)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{1.11}$$

where

$$\varphi_l(t) = q(t)\Big(1+\int_t^{\sigma(t)} q(s)\exp\Big(\int_t^{\sigma(s)} q(u)\exp\Big(\int_u^{\sigma(u)} \varphi_{l-1}(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}u\Big)\,\mathrm{d}s\Big)$$

with

$$\varphi_0(t) = q(t)\Big(1+\int_t^{\sigma(t)} q(s)\exp\Big(\int_t^{\sigma(s)} q(\omega)\exp\Big(\lambda_0\int_\omega^{\sigma(\omega)} q(u)\,\mathrm{d}u\Big)\,\mathrm{d}\omega\Big)\,\mathrm{d}s\Big).$$

The motivation for considering equations in the form of (E) or (E′), with non-monotone arguments, is justified not only by its pure mathematical interestingness, but also because such equations describe in a more realistic way a wide class of natural phenomena of natural disturbances (e.g. noise in communication systems) affecting parameters of the equation cause non-monotone deviations in the argument of the solutions. Thus, an interesting question arising is that of obtaining oscillation criteria in the case where the argument $\tau(t)$ or $\sigma(t)$ is not necessarily monotone. In the present paper, we achieve this goal by establishing criteria which, up to our knowledge, essentially improve all other known results in the literature.

2. Main results

2.1. DDEs. The proof of our main result is essentially based on the following lemmas.

Lemma 1 ([9], Lemma 2.1.1)**.** *Assume that* $h(t)$ *is defined by* (1.4)*. Then*

$$\alpha := \liminf_{t\to\infty}\int_{\tau(t)}^t p(s)\,\mathrm{d}s = \liminf_{t\to\infty}\int_{h(t)}^t p(s)\,\mathrm{d}s.$$

Lemma 2 ([23], [9] (Lemma 2.1.2)). *Assume that $h(t)$ is defined by* (1.4), $\alpha \in (0, 1/\mathrm{e}]$ *and $x(t)$ is an eventually positive solution of* (E). *Then*

$$\liminf_{t\to\infty} \frac{x(h(t))}{x(t)} \geqslant \lambda_0, \tag{2.1}$$

where λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\alpha\lambda}$.

Theorem 1. *Assume that $h(t)$ is defined by* (1.4) *and for some $l \in \mathbb{N}$*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\biggl(\int_{\tau(s)}^{h(s)} p(u) \exp\biggl(\int_{\tau(u)}^{u} b_l(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.2}$$

where

$$b_l(t) = p(t)\biggl(1 + \int_{\tau(t)}^{t} p(s) \exp\biggl(\int_{\tau(s)}^{t} p(u) \exp\biggl(\int_{\tau(u)}^{u} b_{l-1}(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s\biggr)$$

with

$$b_0(t) = p(t)\biggl(1 + \int_{\tau(t)}^{t} p(s) \exp\biggl(\int_{\tau(s)}^{t} p(y) \exp\biggl(\int_{\tau(y)}^{y} p(\omega) \times \exp\biggl(\lambda_0 \int_{\tau(\omega)}^{\omega} p(u)\,\mathrm{d}u\biggr)\,\mathrm{d}\omega\biggr)\,\mathrm{d}y\biggr)\,\mathrm{d}s\biggr)$$

and λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\alpha\lambda}$. Then all solutions of (E) *are oscillatory.*

P r o o f. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (E). Since $-x(t)$ is also a solution of (E), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists $t_1 > t_0$ such that $x(t)$ and $x(\tau(t)) > 0$ for all $t \geqslant t_1$. Thus, from (E) we have $x'(t) = -p(t)x(\tau(t)) \leqslant 0$ for all $t \geqslant t_1$, which means that $x(t)$ is an eventually nonincreasing function of positive numbers.

Now we divide (E) by $x(t) > 0$ and integrate on $[\tau(t), t]$, so

$$\int_{\tau(t)}^{t} \frac{x'(u)}{x(u)}\,\mathrm{d}u = -\int_{\tau(t)}^{t} p(u)\frac{x(\tau(u))}{x(u)}\,\mathrm{d}u$$

or

$$x(\tau(t)) = x(t) \exp\biggl(\int_{\tau(t)}^{t} p(u)\frac{x(\tau(u))}{x(u)}\,\mathrm{d}u\biggr). \tag{2.3}$$

Combining (E) and (2.3), we have

$$x'(t) + p(t)x(t) \exp\left(\int_{\tau(t)}^{t} p(u)\frac{x(\tau(u))}{x(u)}\,\mathrm{d}u\right) = 0. \tag{2.4}$$

We divide (2.4) by $x(t) > 0$ and integrate on $[\tau(t), t]$, so

$$\int_{\tau(t)}^{t} \frac{x'(\omega)}{x(\omega)}\,\mathrm{d}\omega = -\int_{\tau(t)}^{t} p(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} p(u)\frac{x(\tau(u))}{x(u)}\,\mathrm{d}u\right)\mathrm{d}\omega$$

or

$$x(\tau(t)) = x(t) \exp\left(\int_{\tau(t)}^{t} p(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} p(u)\frac{x(\tau(u))}{x(u)}\,\mathrm{d}u\right)\mathrm{d}\omega\right). \tag{2.5}$$

Combining (E) and (2.5), we have

$$x'(t) + p(t)x(t) \exp\left(\int_{\tau(t)}^{t} p(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} p(u)\frac{x(\tau(u))}{x(u)}\,\mathrm{d}u\right)\mathrm{d}\omega\right) = 0. \tag{2.6}$$

We divide (2.6) by $x(t) > 0$ and integrate on $[\tau(s), t]$, so

$$\int_{\tau(s)}^{t} \frac{x'(y)}{x(y)}\,\mathrm{d}y = -\int_{\tau(s)}^{t} p(y) \exp\left(\int_{\tau(y)}^{y} p(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} p(u)\frac{x(\tau(u))}{x(u)}\,\mathrm{d}u\right)\mathrm{d}\omega\right)\mathrm{d}y$$

or

(2.7)

$$x(\tau(s)) = x(t) \exp\left(\int_{\tau(s)}^{t} p(y) \exp\left(\int_{\tau(y)}^{y} p(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} p(u)\frac{x(\tau(u))}{x(u)}\,\mathrm{d}u\right)\mathrm{d}\omega\right)\mathrm{d}y\right).$$

Integrating (E) from $\tau(t)$ to t, using (2.7), multiplying by $p(t)$ and taking into account the fact that $x'(t) = -p(t)x(\tau(t))$, we obtain

$$\begin{aligned} 0 = x'(t) + p(t)x(t) + p(t)x(t) \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(y) \exp\left(\int_{\tau(y)}^{y} p(\omega)\right.\right.& \\ \left.\left.\times \exp\left(\int_{\tau(\omega)}^{\omega} p(u)\frac{x(\tau(u))}{x(u)}\,\mathrm{d}u\right)\mathrm{d}\omega\right)\mathrm{d}y\right)\mathrm{d}s.& \end{aligned}$$

Since $\tau(u) \leqslant h(u)$, clearly

$$\begin{aligned} 0 \geqslant x'(t) + p(t)x(t) + p(t)x(t) \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(y) \exp\left(\int_{\tau(y)}^{y} p(\omega)\right.\right.& \\ \left.\left.\times \exp\left(\int_{\tau(\omega)}^{\omega} p(u)\frac{x(h(u))}{x(u)}\,\mathrm{d}u\right)\mathrm{d}\omega\right)\mathrm{d}y\right)\mathrm{d}s.& \end{aligned}$$

Taking into account the fact that Lemmas 1 and 2 are satisfied, the last inequality becomes

$$0 \geqslant x'(t) + p(t)\biggl(1 + \int_{\tau(t)}^{t} p(s)\exp\biggl(\int_{\tau(s)}^{t} p(y)\exp\biggl(\int_{\tau(y)}^{y} p(\omega) \times \exp\biggl((\lambda_0 - \varepsilon)\int_{\tau(\omega)}^{\omega} p(u)\,\mathrm{d}u\biggr)\,\mathrm{d}\omega\biggr)\,\mathrm{d}y\biggr)\,\mathrm{d}s\biggr)x(t)$$

or

$$x'(t) + b_0(t,\varepsilon)x(t) \leqslant 0 \tag{2.8}$$

with

$$b_0(t,\varepsilon) = p(t)\biggl(1 + \int_{\tau(t)}^{t} p(s)\exp\biggl(\int_{\tau(s)}^{t} p(y)\exp\biggl(\int_{\tau(y)}^{y} p(\omega) \times \exp\biggl((\lambda_0 - \varepsilon)\int_{\tau(\omega)}^{\omega} p(u)\,\mathrm{d}u\biggr)\,\mathrm{d}\omega\biggr)\,\mathrm{d}y\biggr)\,\mathrm{d}s\biggr).$$

Applying the Grönwall inequality in (2.8), we obtain

$$x(\tau(u)) \geqslant x(u)\exp\biggl(\int_{\tau(u)}^{u} b_0(\xi,\varepsilon)\,\mathrm{d}\xi\biggr). \tag{2.9}$$

Now we divide (E) by $x(t) > 0$, integrate on $[\tau(s), t]$ and use (2.9), so

$$-\int_{\tau(s)}^{t} \frac{x'(u)}{x(u)}\,\mathrm{d}u = \int_{\tau(s)}^{t} p(u)\frac{x(\tau(u))}{x(u)}\,\mathrm{d}u \geqslant \int_{s}^{t} p(u)\exp\biggl(\int_{\tau(u)}^{u} b_0(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u$$

or

$$x(\tau(s)) \geqslant x(t)\exp\biggl(\int_{\tau(s)}^{t} p(u)\exp\biggl(\int_{\tau(u)}^{u} b_0(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr). \tag{2.10}$$

Integrating (E) from $\tau(t)$ to t and using (2.10), we obtain

$$x(t) - x(\tau(t)) + x(t)\int_{\tau(t)}^{t} p(s)\exp\biggl(\int_{\tau(s)}^{t} p(u)\exp\biggl(\int_{\tau(u)}^{u} b_0(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s \leqslant 0.$$

Multiplying the last inequality by $p(t)$, we have

$$p(t)x(t) - p(t)x(\tau(t)) + p(t)x(t)\int_{\tau(t)}^{t} p(s)\exp\biggl(\int_{\tau(s)}^{t} p(u) \times \exp\biggl(\int_{\tau(u)}^{u} b_0(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s \leqslant 0,$$

which, in view of (E), becomes

$$x'(t)+p(t)x(t)+p(t)x(t)\int_{\tau(t)}^{t}p(s)\exp\biggl(\int_{\tau(s)}^{t}p(u)\exp\biggl(\int_{\tau(u)}^{u}b_0(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s\leqslant 0.$$

Hence, for sufficiently large t

$$x'(t)+p(t)\biggl(1+\int_{\tau(t)}^{t}p(s)\exp\biggl(\int_{\tau(s)}^{t}p(u)\exp\biggl(\int_{\tau(u)}^{u}b_0(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s\biggr)x(t)\leqslant 0$$

or

$$x'(t)+b_1(t,\varepsilon)x(t)\leqslant 0,$$

where

$$b_1(t,\varepsilon)=p(t)\biggl(1+\int_{\tau(t)}^{t}p(s)\exp\biggl(\int_{\tau(s)}^{t}p(u)\exp\biggl(\int_{\tau(u)}^{u}b_0(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s\biggr).$$

Following the above procedure, we can inductively construct the inequalities

$$x'(t)+b_l(t,\varepsilon)x(t)\leqslant 0,\quad l\in\mathbb{N},$$

where

$$b_l(t,\varepsilon)=p(t)\biggl(1+\int_{\tau(t)}^{t}p(s)\exp\biggl(\int_{\tau(s)}^{t}p(u)\exp\biggl(\int_{\tau(u)}^{u}b_{l-1}(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s\biggr)$$

and

$$x(\tau(s))\geqslant x(h(t))\exp\biggl(\int_{\tau(s)}^{h(t)}p(u)\exp\biggl(\int_{\tau(u)}^{u}b_l(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr). \tag{2.11}$$

Now, dividing (E) by $x(t)$ and integrating from $h(t)$ to t we have

$$\ln\frac{x(h(t))}{x(t)}=\int_{h(t)}^{t}p(s)\frac{x(\tau(s))}{x(s)}\,\mathrm{d}s,$$

from which, in view of $\tau(s)\leqslant h(s)$ and by (2.11), we obtain

$$\ln\frac{x(h(t))}{x(t)}\geqslant\int_{h(t)}^{t}p(s)\frac{x(h(s))}{x(s)}\exp\biggl(\int_{\tau(s)}^{h(s)}p(u)\exp\biggl(\int_{\tau(u)}^{u}b_l(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s.$$

Taking into account that x is nonincreasing and $h(s)<s$, the last inequality leads to

$$\ln\frac{x(h(t))}{x(t)}\geqslant\int_{h(t)}^{t}p(s)\exp\biggl(\int_{\tau(s)}^{h(s)}p(u)\exp\biggl(\int_{\tau(u)}^{u}b_l(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s. \tag{2.12}$$

From (2.2), it follows that there exists a constant $c > 0$ such that for a sufficiently large t it holds that

$$\int_{h(t)}^{t} p(s)\exp\biggl(\int_{\tau(s)}^{h(s)} p(u)\exp\biggl(\int_{\tau(u)}^{u} b_l(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s \geqslant c > \frac{1}{\mathrm{e}}.$$

Choose c' such that $c > c' > 1/\mathrm{e}$. For every $\varepsilon > 0$ such that $c - \varepsilon > c'$ we have

$$\int_{h(t)}^{t} p(s)\exp\biggl(\int_{\tau(s)}^{h(s)} p(u)\exp\biggl(\int_{\tau(u)}^{u} b_l(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s \geqslant c-\varepsilon > c' > \frac{1}{\mathrm{e}}. \tag{2.13}$$

Combining inequalities (2.12) and (2.13), we obtain

$$\ln\frac{x(h(t))}{x(t)} \geqslant c'$$

or

$$\frac{x(h(t))}{x(t)} \geqslant \mathrm{e}^{c'} \geqslant \mathrm{e}c' > 1,$$

which implies

$$x(h(t)) \geqslant (\mathrm{e}c')x(t).$$

Repeating the above procedure, it follows by induction that for any positive integer k,

$$\frac{x(h(t))}{x(t)} \geqslant (\mathrm{e}c')^k \quad \text{for sufficiently large } t.$$

Since $\mathrm{e}c' > 1$, there is $k \in \mathbb{N}$ satisfying $k > 2(\ln 2 - \ln c')/(1+\ln c')$ such that for t sufficiently large

$$\frac{x(h(t))}{x(t)} \geqslant (\mathrm{e}c')^k > \Bigl(\frac{2}{c'}\Bigr)^2. \tag{2.14}$$

Taking the integral on $[h(t),t]$, which is not less than c', we split the interval into two parts where integrals are not less than $\frac{1}{2}c'$. Let $t_m \in (h(t),t)$ be the splitting point:

$$\begin{aligned}\int_{h(t)}^{t_m} p(s)\exp\biggl(\int_{\tau(s)}^{h(s)} p(u)\exp\biggl(\int_{\tau(u)}^{u} b_l(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s &\geqslant \frac{c'}{2},\\ \int_{t_m}^{t} p(s)\exp\biggl(\int_{\tau(s)}^{h(s)} p(u)\exp\biggl(\int_{\tau(u)}^{u} b_l(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s &\geqslant \frac{c'}{2}.\end{aligned} \tag{2.15}$$

Integrating (E) from $h(t)$ to t_m, using (2.11) and the fact that $x(t_m) > 0$, we obtain

$$x(h(t)) > x(h(t_m))\int_{h(t)}^{t_m} p(s)\exp\biggl(\int_{\tau(s)}^{h(s)} p(u)\exp\biggl(\int_{\tau(u)}^{u} b_l(\xi,\varepsilon)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s,$$

which, in view of the first inequality in (2.15), implies

$$x(h(t)) > \frac{c'}{2} x(h(t_m)). \tag{2.16}$$

Similarly, integrating (E) from t_m to t, using (2.11) and the fact that $x(t) > 0$, we have

$$x(t_m) > x(h(t)) \int_{t_m}^{t} p(s) \exp\biggl(\int_{\tau(s)}^{h(s)} p(u) \exp\biggl(\int_{\tau(u)}^{u} b_l(\xi, \varepsilon)\, \mathrm{d}\xi\biggr)\, \mathrm{d}u\biggr)\, \mathrm{d}s,$$

which, in view of the second inequality in (2.15), implies

$$x(t_m) > \frac{c'}{2} x(h(t)). \tag{2.17}$$

Combining the inequalities (2.16) and (2.17), we obtain

$$x(h(t_m)) < \frac{2}{c'} x(h(t)) < \Bigl(\frac{2}{c'}\Bigr)^2 x(t_m),$$

which contradicts (2.14).

The proof of the theorem is complete. □

2.2. ADEs. The corresponding theorem is stated below while its proof is omitted, as it is quite similar to this for Theorem 1.

Theorem 2. *Assume that $\varrho(t)$ is defined by* (1.9) *and for some $l \in \mathbb{N}$*

$$\liminf_{t\to\infty} \int_{t}^{\varrho(t)} q(s) \exp\biggl(\int_{\varrho(s)}^{\sigma(s)} q(u) \exp\biggl(\int_{u}^{\sigma(u)} g_l(\xi)\, \mathrm{d}\xi\biggr)\, \mathrm{d}u\biggr)\, \mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.18}$$

where

$$g_l(t) = q(t)\biggl(1 + \int_{t}^{\sigma(t)} q(s) \exp\biggl(\int_{t}^{\sigma(s)} q(u) \exp\biggl(\int_{u}^{\sigma(u)} g_{l-1}(\xi)\, \mathrm{d}\xi\biggr)\, \mathrm{d}u\biggr)\, \mathrm{d}s\biggr)$$

with

$$g_0(t) = q(t)\biggl(1 + \int_{t}^{\sigma(t)} q(s) \exp\biggl(\int_{t}^{\sigma(s)} q(y) \exp\biggl(\int_{y}^{\sigma(y)} q(\omega) \times \exp\biggl(\lambda_0 \int_{\omega}^{\sigma(\omega)} q(u)\, \mathrm{d}u\biggr)\, \mathrm{d}\omega\biggr)\, \mathrm{d}y\biggr)\, \mathrm{d}s\biggr)$$

and λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\beta\lambda}$. Then all solutions of (E$'$) *are oscillatory.*

E x a m p l e 1. Consider the advanced differential equation

$$x'(t) - \frac{373}{1250}x(\sigma(t)) = 0, \quad t \geqslant 0, \tag{2.19}$$

with (see Figure 1 (a))

$$\sigma(t) = \begin{cases} t+1 & \text{if } t \in [3.5k, 3.5k+1], \\ 3t-7k-1 & \text{if } t \in [3.5k+1, 3.5k+1.5], \\ -t+7k+5 & \text{if } t \in [3.5k+1.5, 3.5k+2], \\ t+1 & \text{if } t \in [3.5k+2, 3.5k+2.5], \\ 3t-7k-4 & \text{if } t \in [3.5k+2.5, 3.5k+3], \\ -t+7k+8 & \text{if } t \in [3.5k+3, 3.5k+3.5], \end{cases}$$

where $k \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

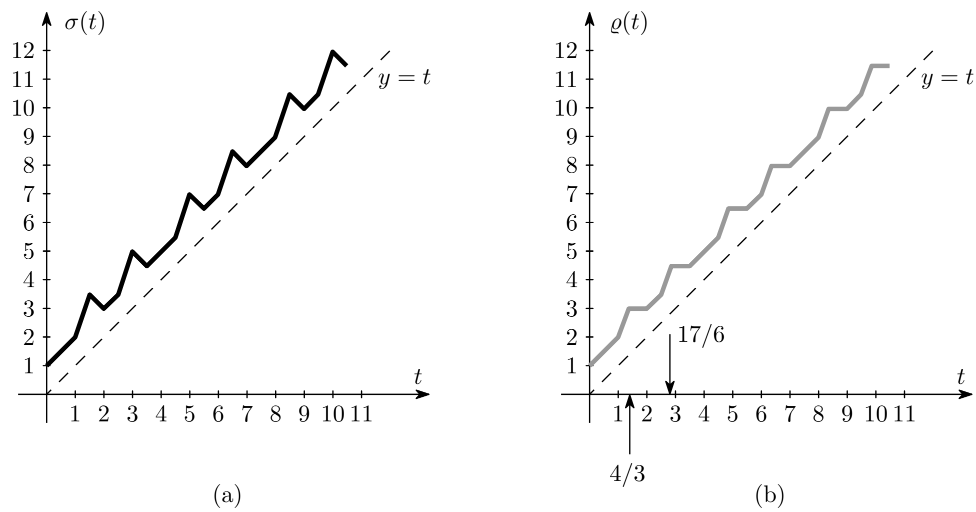

Figure 1. The graphs of $\sigma(t)$ and $\varrho(t)$.

By (1.9), we see (Figure 1 (b)) that

$$\varrho(t) = \begin{cases} t+1 & \text{if } t \in [3.5k, 3.5k+1], \\ 3t-7k-1 & \text{if } t \in [3.5k+1, 3.5k+4/3], \\ 3.5k+3 & \text{if } t \in [3.5k+4/3, 3.5k+2], \\ t+1 & \text{if } t \in [3.5k+2, 3.5k+2.5], \\ 3t-7k-4 & \text{if } t \in [3.5k+2.5, 3.5k+17/6], \\ 3.5k+4.5 & \text{if } t \in [3.5k+17/6, 3.5k+3.5]. \end{cases}$$

It is easy to see that

$$\beta = \liminf_{t\to\infty} \int_t^{\varrho(t)} q(s)\,\mathrm{d}s = \liminf_{k\to\infty} \int_{3.5k+1}^{3.5k+2} \frac{373}{1250}\,\mathrm{d}s = 0.2984$$

and therefore the smaller root of $\mathrm{e}^{0.2984\lambda} = \lambda$ is $\lambda_0 = 1.62308$.

Observe that the function $F_l\colon \mathbb{R}_0 \to \mathbb{R}_+$ defined as

$$F_l(t) = \int_t^{\varrho(t)} q(s) \exp\biggl(\int_{\varrho(s)}^{\sigma(s)} q(u) \exp\biggl(\int_u^{\sigma(u)} g_l(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s$$

attains its minimum at $t = 3.5k+1$, $k \in \mathbb{N}_0$, for every $l \in \mathbb{N}$. Specifically, by using an algorithm on MATLAB software, we obtain

$$F_1(t = 3.5k+1) = \int_{3.5k+1}^{3.5k+2} q(s) \exp\biggl(\int_{\varrho(s)}^{\sigma(s)} q(u) \exp\biggl(\int_u^{\sigma(u)} g_1(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s \simeq 0.3685$$

and therefore condition (2.18) of Theorem 2 is satisfied for $l = 1$. Thus, all solutions of (2.19) are oscillatory.

Observe, however, that

$$\limsup_{t\to\infty} \int_t^{\varrho(t)} p(s)\,\mathrm{d}s = \limsup_{k\to\infty} \int_{3.5k+4/3}^{3.5k+3} \frac{373}{1250}\,\mathrm{d}s \simeq 0.4973 < 1,$$

$$\beta = 0.2984 < \frac{1}{\mathrm{e}},$$

$$\begin{aligned}
\limsup_{t\to\infty} \int_t^{\varrho(t)} & q(s) \exp\biggl(\int_{\varrho(t)}^{\sigma(s)} q(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s \\
&= \limsup_{k\to\infty} \int_{3.5k+4/3}^{3.5k+3} q(s) \exp\biggl(\int_{3.5k+3}^{\sigma(s)} q(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s \\
&= \limsup_{k\to\infty} \biggl(\int_{3.5k+4/3}^{3.5k+1.5} q(s) \exp\biggl(\int_{3.5k+3}^{3s-7k-1} q(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s \\
&\quad + \int_{3.5k+1.5}^{3.5k+2} q(s) \exp\biggl(\int_{3.5k+3}^{-s+7k+5} q(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s \\
&\quad + \int_{3.5k+2}^{3.5k+2.5} q(s) \exp\biggl(\int_{3.5k+3}^{s+1} q(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s \\
&\quad + \int_{3.5k+2.5}^{3.5k+3} q(s) \exp\biggl(\int_{3.5k+3}^{3s-7k-4} q(u)\,\mathrm{d}u\biggr)\,\mathrm{d}s\biggr) \simeq 0.5939 < 1
\end{aligned}$$

and

$$\liminf_{t\to\infty} \int_t^{\varrho(t)} q(s) \exp\biggl(\int_{\varrho(s)}^{\sigma(s)} q(u) \exp\biggl(\int_u^{\sigma(u)} \varphi_1(\xi) d\xi\biggr)\,\mathrm{d}u\biggr)\,\mathrm{d}s \simeq 0.3603 < \frac{1}{\mathrm{e}}.$$

That is, none of the conditions (1.7), (1.8), (1.10) and (1.11) (for $l = 1$) is satisfied.

Comments. It is worth noting that the improvement of condition (2.18) to the corresponding condition (1.8) is significant, approximately 23.5%, if we compare the values on the left-side of these conditions. Also, observe that condition (1.11) does not lead to oscillation for the first iteration. On the contrary, condition (2.18) is satisfied from the first iteration. This means that our condition is better and much faster than (1.11).

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