THE NON-UNIQUENESS OF THE LIMIT SOLUTIONS OF THE SCALAR CHERN-SIMONS EQUATIONS WITH SIGNED MEASURES

Adilson Eduardo Presoto, São Carlos

Received December 14, 2018. Published online September 25, 2020.
Communicated by Michela Eleuteri

Abstract. We investigate the effect of admitting signed measures as a datum at the scalar Chern-Simons equation

$$-\Delta u + \mathrm{e}^u(\mathrm{e}^u - 1) = \mu \quad \text{in } \Omega$$

with the Dirichlet boundary condition. Approximating μ by a sequence $(\mu_n)_{n\in\mathbb{N}}$ of L^1 functions or finite signed measures such that this equation has a solution u_n for each $n \in \mathbb{N}$, we are interested in establishing the convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ to a function $u^{\#}$ and describing the form of the measure which appears on the right-hand side of the scalar Chern-Simons equation solved by $u^{\#}$.

Keywords: elliptic equation; exponential nonlinearity; scalar Chern-Simons equation; signed measure

MSC 2020: 35R06, 35J25, 35J61

1. Introduction

The main purpose of this paper is to understand the phenomenon caused by admitting signed measures on the convergence and stability of solutions of the scalar Chern-Simons problem

$$\text{(CS)} \qquad \begin{cases} -\Delta u + \mathrm{e}^u(\mathrm{e}^u - 1) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and μ is a finite signed Borel measure (equivalently, a Radon measure) on Ω. By the solution of (CS), we mean a function $u \in W_0^{1,1}(\Omega)$ such that $\mathrm{e}^u(\mathrm{e}^u - 1) \in L^1(\Omega)$ and u satisfies the equation in the sense

DOI: 10.21136/MB.2020.0165-18

of distributions,

$$-\int_\Omega u\Delta\varphi + \int_\Omega \mathrm{e}^u(\mathrm{e}^u - 1)\varphi = \int_\Omega \varphi\,\mathrm{d}\mu \quad \forall\,\varphi\in\mathcal{C}_c^\infty(\Omega).$$

In two dimensions the energy functional associated to the equation (CS), namely

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega \Big(\frac{\mathrm{e}^{2u}}{2} - \mathrm{e}^u\Big) - \int_\Omega u\mu,$$

achieves its minimun on $W_0^{1,2}(\Omega)$ for every $\mu\in L^p(\Omega)$, $1\leqslant p<\infty$. Thus the scalar Chern-Simons equation always has a solution with the datum $\mu\in L^p(\Omega)$ for any $1<p\leqslant\infty$ (see [9], Chapter 3). The existence in the case of the datum $\mu\in L^1(\Omega)$ can be obtained by an approximation using the L^∞ data (see [5], Corollary 12; [9], Chapter 4). An important result in the proofs below is the characterization by Vázquez (see [12]) of measures for which (CS) has a solution: μ is a good measure for (CS) (that is the Dirichlet problem (CS) has a solution) if and only if for every $x\in\Omega$, one has

$$\mu(\{x\})\leqslant 2\pi. \tag{1.1}$$

For measures satisfying the above inequality, the procedure is similar to the $L^1(\Omega)$ case as we can see in [1], Theorem 1, and [9], Chapter 14. On the other hand, μ charges a common mass a with the density larger than 2π. The Poisson problem

$$\begin{cases} -\Delta v = 2\pi\delta_a & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega \end{cases}$$

has a solution $v\in L^1(\Omega)$, whose mean on circumferences in a certain neigbourhood of a behaves like the fundamental solution

$$v(x)\sim\log\frac{1}{|x-a|}.$$

By the comparasion principle (see [3], Corollary B.2, and [9], Chapter 14), we have $u\geqslant v$. Then by using the Jensen inequality (see [2], Problem 4.9), we obtain $\mathrm{e}^u\notin L^1(\Omega)$, i.e. u is not a solution of (CS) in the above sense. The detailed proofs are presented in [1], Section 5 and [12], Section 5.

We now rewrite μ in an appropriate way, which allows us to easily identify it as a good measure or not. We note that the total mass of μ is finite, consequently, the set of massful points is countable. Thus, we write

$$\mu = \overline{\mu} + \sum_{i=1}^\infty \alpha_i\delta_{a_i},$$

where $\overline{\mu}$ is the non-atomic part $\overline{\mu}$, i.e. $\overline{\mu}(\{x\}) = 0$ for all $x \in \Omega$, the points a_i are distinct and δ_{a_i} is the Dirac measure at a_i. Hence the largest measure $\mu^\star \leqslant \mu$ for which (CS) has solution is described by

$$\mu^\star = \overline{\mu} + \sum_{i=1}^{\infty} \min\{\alpha_i, 2\pi\}\delta_{a_i}.$$

The set of points for which (1.1) fails is clearly finite, then the measure μ is cut off exactly on the finite set

$$A = \{a \in \Omega\colon\ \mu(\{a\}) > 2\pi\} \subset \{a_1, a_2, \ldots\},$$

i.e. the measure $\mu - \mu^\star$ is supported on A, and

$$\mu^\star(\{a\}) = \min\{\mu(\{a\}), 2\pi\}.$$

In virtue of Vázquez's result mentioned before, the problem

$$\text{(CS}^\star\text{)} \qquad \begin{cases} -\Delta u^\star + \mathrm{e}^{u^\star}(\mathrm{e}^{u^\star} - 1) = \mu^\star & \text{in } \Omega, \\ u^\star = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u^\star \in L^1(\Omega)$ which, by the maximum principle, is the largest subsolution of (CS).

An interesting question arises when we force the problem to have a solution by an approximate scheme and we wonder what happens with the convergence of the sequence of solutions.

Let $\mathcal{M}(\Omega)$ be the vector space of (finite) measures in Ω equipped with the norm

$$\|\mu\|_{\mathcal{M}(\Omega)} = |\mu|(\Omega) = \int_\Omega \mathrm{d}|\mu|.$$

We recall that the sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(\Omega)$ converges to μ in the weak-$*$ sense in $\mathcal{M}(\Omega)$, if for every continuous function $\zeta\colon \overline{\Omega} \to \mathbb{R}$ such that $\zeta = 0$ on $\partial\Omega$,

$$\lim_{n\to\infty} \int_\Omega \zeta\,\mathrm{d}\mu_n = \int_\Omega \zeta\,\mathrm{d}\mu.$$

We denote this convergence by $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$.

Let (μ_n) be a sequence of measures such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$, and let u_n be the solution of

$$\text{(CS}_n\text{)} \qquad \begin{cases} -\Delta u_n + \mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1) = \mu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Passing to a subsequence (u_{n_k}), we will show that the latter converges in $L^1(\Omega)$ to a limit $u^\# \in L^1(\Omega)$, this limit solves

$$(\mathrm{CS}^\#) \qquad \begin{cases} -\Delta u^\# + \mathrm{e}^{u^\#}(\mathrm{e}^{u^\#} - 1) = \mu^\# & \text{in } \Omega, \\ u^\# = 0 & \text{on } \partial\Omega \end{cases}$$

for a measure $\mu^\# \leqslant \mu$, which is called a reduced limit or a reduced measure of (μ_n). This definition was originally introduced in [8]. In general, this measure is not unique, as we will see in an example in Section 4. This measure has the property

$$\mathrm{e}^{u_{n_k}}(\mathrm{e}^{u_{n_k}} - 1) \stackrel{*}{\rightharpoonup} \mathrm{e}^{u^\#}(\mathrm{e}^{u^\#} - 1) + \tau \quad \text{in } \mathcal{M}(\Omega)$$

for a nonnegative measure τ with support on A. Hence, we will establish a close relationship between the measures μ^* and $\mu^\#$ by the formula $\mu^\# = \mu^* - \tau$. By naming the points of A as $r_1, r_2, \ldots, r_m$, we obtain

$$\mu^\# = \mu^* - \sum_{i=1}^{m} c_i \delta_{r_i}$$

for positive constants c_i's.

In [10], we have focused on approximating the datum μ by nonnegative measures. We proved that in this situation one always has

$$\mu^\# = \mu^*. \tag{1.2}$$

As a consequence, the reduced measure for the scalar Chern-Simons problem depends only on the measure μ (which might not exist from the beginning) and the sequence of approximated solutions converges to the largest subsolution of (CS). Thus we have the surprising fact that the limit of a sequence of solutions is independent of how the datum is approximated.

Here we carry out the study of the problem (CS) for signed-measures. The main novelty in this case is that the equality (1.2), in general, does not hold anymore. In fact, we show that any measure obtained from μ^* by a subtraction of a linear combination of Dirac measures with positive coeficients can be produced as a reduced limit. Hence we characterize all the reduced limits for the Chern-Simons equation.

At the end of the paper, we give an example of sequences of measures (μ_n) and (ν_n) such that $\mu_n, \nu_n \stackrel{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$ and the respective reduced limits are different. The key point is that by handling the convergence speed of ν_n^+ and ν_n^- to μ^+ and μ^-, respectively, different resulting Dirac measures are produced.

2. Preliminary results

We start by stating an order relation between the measures $\mu^{\#}$ and $\mu^{\star}$.

Proposition 2.1. *Let* (μ_n) *be a sequence of good measures such that* $\mu_n \overset{*}{\rightharpoonup} \mu$ *in* $\mathcal{M}(\Omega)$*. Then*

$$\mu^{\#} \leqslant \mu^{\star} \leqslant \mu$$

for all reduced limits $\mu^{\#}$ *of* (μ_n)*.*

Proof. Let $\mu^{\#}$ be a reduced limit of $(\mu_n)_{n\in\mathbb{N}}$, that is, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of the sequence of solutions of (CS_n) converging to the solution of $(\mathrm{CS}^{\#})$. We start with proving that

$$\mu^{\#} \leqslant \mu.$$

Recall that $u_{n_k} \in W_0^{1,1}(\Omega)$ and that for every $\varphi \in C_c^{\infty}(\Omega)$,

$$-\int_\Omega u_{n_k}\Delta\varphi + \int_\Omega \mathrm{e}^{u_{n_k}}(\mathrm{e}^{u_{n_k}} - 1)\varphi = \int_\Omega \varphi \,\mathrm{d}\mu_n. \tag{2.1}$$

Notice that the nonlinear term in the equation verified by u_{n_k} is bounded from below, for every $t \in \mathbb{R}$,

$$\mathrm{e}^t(\mathrm{e}^t - 1) \geqslant -1.$$

If the test function satisfies $\varphi \geqslant 0$, then by Fatou's lemma (see [2], Lemma 4.1),

$$\int_\Omega \mathrm{e}^v(\mathrm{e}^u - 1)\varphi \leqslant \liminf_{k\to\infty} \int_\Omega \mathrm{e}^{u_{n_k}}(\mathrm{e}^{u_{n_k}} - 1)\varphi.$$

As we let k tend to infinity in (2.1), we get

$$\int_\Omega \varphi \,\mathrm{d}\mu^{\#} = -\int_\Omega u\Delta\varphi + \int_\Omega \mathrm{e}^v(\mathrm{e}^u - 1)\varphi \leqslant \int_\Omega \varphi \,\mathrm{d}\mu.$$

Since this property holds for every $\varphi \in C_c^{\infty}(\Omega)$ such that $\varphi \geqslant 0$, we deduce that $\mu^{\#} \leqslant \mu$. Finally, since $\mu^{\star}$ is the largest good measure less than or equal to μ (see [8], Theorem 1, and [9], Proposition 17.9) we achieve

$$\mu^{\#} \leqslant \mu^{\star},$$

finishing the proof. □

In what follows, we give a lemma based on the Brezis-Merle inequality (see [4], Theorem 1, [9], Proposition 11.7), which plays an important role in the proofs of Theorems 3.1 and 3.5 below.

Lemma 2.2. *Let (μ_n) be a sequence of good measures in $\mathcal{M}(\Omega)$, and let u_n be the solution of* (CS_n). *Suppose that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$ and also that the sequence of solutions $(u_n)_{n\in\mathbb{N}}$ converges to u in $L^1(\Omega)$. Then there exists a measure $\tau \in \mathcal{M}(\Omega)$ such that u solves* $(\mathrm{CS}^{\#})$ *with*

$$\mu^{\#} = \mu - \tau.$$

Moreover, τ is supported on the set $A = \{x \in \Omega\colon \mu(\{x\}) \geqslant 2\pi\}$, thus there exist finitely many points $r_1, r_2, \ldots, r_m \in \Omega$ and $c_1, c_2, \ldots, c_m \in \mathbb{R}$ satisfying

$$\tau = \sum_{i=1}^{m} c_i \delta_{r_i}.$$

P r o o f. By a standard property of elliptic equations with absorption term (see [9], Lemma 14.2), for every $n \in \mathbb{N}$,

$$\|\mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1)\|_{L^1(\Omega)} \leqslant \|\mu_n\|_{\mathcal{M}(\Omega)}, \tag{2.2}$$

whence $(\mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1))_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$. Passing to a further subsequence if necessary, we may assume that there exists a finite measure $\tau \in \mathcal{M}(\Omega)$ such that

$$\mathrm{e}^{u_{n_k}}(\mathrm{e}^{u_{n_k}} - 1) \overset{*}{\rightharpoonup} \mathrm{e}^{u}(\mathrm{e}^{u} - 1) + \tau \quad \text{in } \mathcal{M}(\Omega) \tag{2.3}$$

and u_{n_k} converges to u a.e. in Ω. Thus, u satisfies the scalar Chern-Simons problem

$$\begin{cases} -\Delta u + \mathrm{e}^{u}(\mathrm{e}^{u} - 1) = \mu - \tau & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $A = \{r_1, \ldots, r_m\}$ be as above, where each $r_i \in \Omega$ satisfies $\mu(\{r_i\}) \geqslant 2\pi$. Since μ is a finite measure, the set A is finite. To finish the proof, it remains to show that τ is supported on A. In what follows, we use arguments similar to those contained in [10], Theorem 1.1.

Let $N(\mu_n^+)$ be the Newtonian potential generated by μ_n^+,

$$N(\mu_n^+)(x) = \frac{1}{2\pi} \int_{\Omega} \log \frac{d}{|x-y|} \, \mathrm{d}\mu_n^+(y),$$

where $d \geqslant \operatorname{diam}\Omega$. Given $b \in \Omega$ and $r > 0$, we write the Newtonian potential of μ_n as

$$N(\mu_n^+) = N(\mu_n^+\lfloor_{B_r(b)}) + N(\mu_n^+\lfloor_{\Omega\setminus B_r(b)}).$$

Assume for the moment that there exist $\varepsilon > 0$ and $m \in \mathbb{N}$ such that for every $n \geqslant m$,

$$\mu_n^+(B_r(b)) \leqslant 2\pi - \varepsilon. \tag{2.4}$$

By the Brezis-Merle inequality (see [4], Theorem 1, and [9], Proposition 11.7), there exist $p > 1$ and $C_1 > 0$ such that for every $n \geqslant m$,

$$\|\mathrm{e}^{2N(\mu_n^+\lfloor_{B_r(b)})}\|_{L^p(\Omega)} \leqslant C_1.$$

Since the functions $N(\mu_n^+\lfloor_{\Omega\setminus B_r(b)})$ are harmonic in $B_r(b)$ and have a uniformly bounded L^1 norm in $B_r(b)$, consequently, the sequence $(N(\mu_n^+\lfloor_{\Omega\setminus B_r(b)}))$ is uniformly bounded in $B_{r/2}(b)$. We conclude that there exists $C_2 > 0$ such that for every $n \geqslant m$,

$$\|\mathrm{e}^{2N(\mu_n^+)}\|_{L^p(B_{r/2}(b))} \leqslant C_2. \tag{2.5}$$

Note that if $b \in \Omega \setminus A$, i.e. $\mu(\{b\}) < 2\pi$, then $\mu^+(\{b\}) = \max\{\mu(\emptyset), \mu(\{b\})\} < 2\pi$. Thus, there exist $\varepsilon > 0$ and $r > 0$ satisfying (2.4). Indeed, let $\overline{\varepsilon} > 0$ and $R > 0$ such that

$$\mu^+(B_R(b)) \leqslant 2\pi - \overline{\varepsilon}.$$

Then, by weak convergence of the sequence (μ_n), given $0 < r < R$ and $0 < \varepsilon < \overline{\varepsilon}$ the property (2.4) is ensured for n large enough (see [6], Section 1.9).

Let U_n be the solution of the linear Dirichlet problem

$$\begin{cases} -\Delta U_n = \mu_n^+ & \text{in } \Omega, \\ U_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

By the comparison estimate (see [3], Corollary B.2, and [9], Chapter 14), for every $n \in \mathbb{N}$, we have

$$u_n \leqslant U_n \quad \text{in } \Omega.$$

By the weak maximum principle (see [9], Proposition 6.1), $U_n \leqslant N(\mu_n^+)$ in Ω. Hence,

$$u_n \leqslant N(\mu_n^+) \quad \text{in } \Omega.$$

It follows from (2.5) that the sequence $(\mathrm{e}^{u_n}(\mathrm{e}^{u_n} - 1))_{n\in\mathbb{N}}$ is uniformly bounded in $L^p(B_{r/2}(b))$. Since $u_{n_k} \to u$ a.e. in $B_{r/2}(b)$, by Egoroff's theorem (see [6], Theorem 3) we obtain

$$\mathrm{e}^{u_{n_k}}(\mathrm{e}^{u_{n_k}} - 1) \to \mathrm{e}^{u}(\mathrm{e}^{u} - 1) \quad \text{in } L^1(B_{r/2}(b)).$$

We deduce that $\tau = 0$ in $B_{r/2}(b)$. Since $b \in \Omega \setminus A$ is arbitrary, we conclude that τ is supported on A. $\square$

We also need a result obtained as a particular case of [7], Lemma 8.1.

Lemma 2.3. *Let μ be a Radon measure and $f \in L^1(\Omega)$. Then*

$$\begin{cases} -\Delta u + \mathrm{e}^u(\mathrm{e}^u - 1) = \mu & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

has solution if and only if (2.7) *has solution with* (μ^+, f^+) *and* (μ^-, f^-) *as data.*

3. Reduced limit

Combining Proposition 2.1 and Lemma 2.2 we get the first of the two characterization results for reduced limits.

Theorem 3.1. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of good measures in $\mathcal{M}(\Omega)$ and let u_n be the solution of* (CS_n). *If the sequence $(\mu_n)_{n\in\mathbb{N}}$ is such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$ and the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $L^1(\Omega)$ then there exists a Radon measure $\tau \geqslant 0$ such that u solves* $(\mathrm{CS}^{\#})$ *with*

$$\mu^{\#} = \mu^{\star} - \tau. \tag{3.1}$$

Moreover, τ is supported on the set $A = \{x \in \Omega;\ \mu(\{x\}) \geqslant 2\pi\}$, so that there exist finitely many points $r_1, r_2, \ldots, r_m \in \Omega$ and $c_1, c_2, \ldots, c_m \geqslant 0$ satisfying

$$\tau = \sum_{i=1}^{m} c_i \delta_{r_i}.$$

P r o o f. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of solutions of (CS_n). Using Lemma 2.2, we obtain that there exist $c_1, c_2, \ldots, c_m \in \mathbb{R}$ and $r_1, r_2, \ldots, r_m \in \Omega$ such that

$$\begin{cases} -\Delta u + \mathrm{e}^u(\mathrm{e}^u - 1) = \mu - \tau & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $\tau = \sum\limits_{i=1}^{m} c_i \delta_{r_i}$. Since $\mu - \tau$ is a reduced limit of μ, then $\mu - \tau \leqslant \mu$ by Proposition 2.1. Therefore, $\tau \geqslant 0$ and this concludes the proof. $\square$

The next two technical lemmas will allow us to compute reduced limits in particular situations. They also play role in important steps of the proof of the second characterization theorem.

Lemma 3.2. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(\Omega)$ such that $\mu_n = \tau - \nu_n$ with $\nu_n \geqslant 0$ for each $n \in \mathbb{N}$ and $\nu_n \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}(\Omega)$. If $\tau(\{x\}) \leqslant 2\pi$ for all $x \in \Omega$ then $(\mu_n)_{n\in\mathbb{N}}$ has a unique reduced limit given by*

$$(\tau - \nu)^{\#} = \tau - \nu.$$

P r o o f. Let $(\mu_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(\mu_n)_{n\in\mathbb{N}}$. Since $(\tau - \nu_{n_k})^+ \leqslant \tau^+$, by passing to a subsequence if necessary, it follows from the Banach-Alaoglu-Bourbaki theorem (see [2], Theorem 3.16) that there exists $\kappa \in \mathcal{M}(\Omega)$ such that $(\tau-\nu_{n_k})^+ \overset{*}{\rightharpoonup} \kappa$ in $\mathcal{M}(\Omega)$. Since $\kappa \leqslant \tau^+$ and τ^+ is a good measure, by [10], Theorem 1, $(\tau - \nu_{n_k})^+$ has κ for its unique reduced limit. On the other hand, due to the boundedness from above of the exponential function, the reduced limit of $-(\tau - \nu_{n_k})^-$ is unique and equal to its weak limit $-(\tau-\nu)^-$. Therefore, the conclusion follows from [8], Proposition 7.3, which ensures that (μ_n) has a reduced limit $\mu^{\#}$ if and only if (μ_n^+) and $(-\mu_n^-)$ have reduced limits $\mu_1^{\#}$ and $\mu_2^{\#}$, respectively, and, moreover, $\mu^{\#} = \mu_1^{\#} + \mu_2^{\#}$. □

Lemma 3.3. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of good measures with $\mu_n = \nu_n - \tau$, $\tau \geqslant 0$ and $\tau(\{x\}) = 0$ for all $x \in \Omega$. If $\nu_n \overset{*}{\rightharpoonup} \nu$ and $\nu_n^+ \overset{*}{\rightharpoonup} \nu^+$ in $\mathcal{M}(\Omega)$ then $(\mu_n)_{n\in\mathbb{N}}$ has a unique reduced limit given by*

$$(\nu - \tau)^{\#} = \nu^{\star} - \tau.$$

P r o o f. Let $\mu^{\#}$ be a reduced limit of (μ_n), i.e. there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of (u_n) converging in $L^1(\Omega)$ to a function u which solves (CS$^{\#}$). By Lemma 2.2, $\mu - \mu^{\#}$ is concentrated on the set $A = \{x \in \Omega\colon\ \mu^+(\{x\}) \geqslant 2\pi\}$.

We first prove that $(\nu^{\star} - \tau) \leqslant \mu^{\#}$. If $p \in \Omega$ satisfies $\nu^+(\{p\}) < 2\pi$ then

$$\mu^{\#}(\{p\}) = \mu(\{p\}) \geqslant (\nu^{\star} - \tau)(\{p\}).$$

On the other hand, if $\nu^+(\{p\}) \geqslant 2\pi$, we take $\alpha < 2\pi/\nu^+(\{p\})$ and consider the solutions of

$$\begin{cases} -\Delta v_n + \mathrm{e}^{v_n}(\mathrm{e}^{v_n} - 1) = \alpha\nu_n - \tau & \text{in } \Omega,\\ v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By the comparison principle (see [3], Corollary B.2, and [9], Chapter 14), $v_{n_k} \leqslant u_{n_k}$ for all $k \in \mathbb{N}$. Since $\alpha\nu^+(\{p\}) < 2\pi$, using a comparison result for reduced limits (see [8], Theorem 7.1), we obtain

$$(\alpha\nu - \tau)(\{p\}) = (\alpha\nu - \tau)^{\#}(\{p\}) \leqslant \mu^{\#}(\{p\}).$$

Since $\alpha < 2\pi/\nu^+(\{p\})$ is arbitrary, $2\pi \leqslant \mu^\#(\{p\})$, whence

$$(\nu^\star - \tau)(\{p\}) = 2\pi \leqslant \mu^\#(\{p\}).$$

But $\mu^\#$ differs from $\mu = \nu - \tau$ only on the set A and we obtain the desired inequality.

For the reverse inequality, we need the following property for mutually singular measures μ_1 and μ_2 (see [8], Theorem 8)

$$(\mu_1 + \mu_2)^\star = \mu_1^\star + \mu_2^\star.$$

From $\tau(\{x\}) = 0$ for all $x \in \Omega$, it follows

$$(\nu - \tau)^\star = \left(\overline{\nu} + \sum_{i=1}^{\infty} b_i \delta_{q_i} - \tau\right)^\star = (\overline{\nu} + \tau) + \sum_{i=1}^{\infty} \min\{b_i, 2\pi\}\delta_{q_i} = \nu^\star - \tau,$$

where we decomposed, as before, ν into nonatomic and atomic parts, $\nu = \overline{\nu} + \sum\limits_{i=1}^{\infty} b_i \delta_{q_i}$. Applying Lemma 2.1, we have

$$\mu^\# \leqslant (\nu^\star - \tau).$$

Thus we conclude that the reduced limit has necessarily the form $\nu^\star - \tau$. □

We now show that the unique form that can be assumed by reduced limits is that one expressed in (3.1). The proof is based on the Cantor diagonal argument. The result was previously announced in [10] without proof.

Theorem 3.4. *Let $\mu \in \mathcal{M}(\Omega)$, $c_1, \ldots, c_m \geqslant 0$ and $r_1, \ldots, r_m \in \Omega$. Then there exists a sequence $(\mu_n) \subset \mathcal{C}_c^\infty(\Omega)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$. If u_n is the solution of* (CS$_n$)*, then (u_n) converges to the solution of* (CS$^\#$) *where*

$$\mu^\# = \mu^\star - \sum_{i=1}^{m} c_i \delta_{r_i}.$$

Proof. We rewrite the expressions for $\sum\limits_{i=1}^{m} c_i \delta_{r_i}$ and the atomic part of μ in order that the sums share the same sequence of Dirac measures

$$\sum_{i=1}^{\infty} a_i \delta_{p_i} = \sum_{i=1}^{\infty} a_i' \delta_{q_i} \quad \text{and} \quad \sum_{i=1}^{m} c_i \delta_{r_i} = \sum_{i=1}^{\infty} c_i' \delta_{q_i}.$$

The first k points q_i are $r_i \in \{p_i\colon\ i \in \mathbb{N}\}$, the next $m-k$ points q_i are the r_i which do not belong to $\{p_i\colon\ i \in \mathbb{N}\}$, and the last q_i are the p_i which do not appear in $\{r_1, \dots, r_m\}$. We now define

$$\mu_{n_\varepsilon} = \varrho_\varepsilon * \overline{\mu} + \sum_{i=1}^{\infty} b_i \varrho_\varepsilon(x - q_i) - \sum_{i=1}^{\infty} d_i \varrho_{1/2n}(x - q_i - (1/n)e_1),$$

where $e_1 = (1,0)$ and for $i = 1, \dots, m$,

$$b_i = \begin{cases} a_i' + c_i' & \text{if } a_i' \geqslant 2\pi, \\ 2\pi + c_i' & \text{if } a_i' < 2\pi, \end{cases} \qquad d_i = \begin{cases} c_i' & \text{if } a_i' \geqslant 2\pi, \\ 2\pi - a_i' + c_i' & \text{if } a_i' < 2\pi, \end{cases}$$

and $b_i = a_i'$ if $i > m$. Since $\mu_{n,\varepsilon} \in \mathcal{C}_0^\infty(\Omega)$, the Dirichlet problem

$$\begin{cases} -\Delta u_{n,\varepsilon} + \mathrm{e}^{u_{n,\varepsilon}}(\mathrm{e}^{u_{n,\varepsilon}} - 1) = \mu_{n,\varepsilon} & \text{in } \Omega, \\ u_{n,\varepsilon} = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution $u_{n,\varepsilon}$. Let (ε_k) be a sequence converging to zero. Due to Lemma 3.3, $(u_{n,\varepsilon_k})_{k\in\mathbb{N}}$ converges to the solution u_n of the scalar Chern-Simons equation with the datum

$$\overline{\mu} + \sum_{i=1}^{\infty} \min\{b_i, 2\pi\}\delta_{q_i} - \sum_{i=1}^{\infty} d_i \varrho_{1/2n}\Big(x - q_i - \frac{1}{n}e_1\Big).$$

For each $n \in \mathbb{N}$, we take k_n satisfying

$$\|u_{n,k_n} - u_n\|_{L^1(\Omega)} \leqslant \frac{1}{n}.$$

Applying Lemma 3.2, we deduce that $(u_n)_{n\in\mathbb{N}}$ converges to the solution of the scalar Chern-Simons problem with the datum

$$\overline{\mu} + \sum_{i=1}^{\infty} \min\{b_i, 2\pi\}\delta_{q_i} - \sum_{i=1}^{\infty} d_i \delta_{q_i}.$$

According to the choices of b_i and d_i, we have $\min\{b_i, 2\pi\} = 2\pi$ and $2\pi - d_i = \min\{2\pi, a_i'\} - c_i'$. Then

$$\overline{\mu} + \sum_{i=1}^{\infty} \min\{b_i, 2\pi\}\delta_{q_i} - \sum_{i=1}^{\infty} d_i\delta_{q_i} = \overline{\mu} + \sum_{i=1}^{\infty} \min\{2\pi, a_i'\}\delta_{q_i} - \sum_{i=1}^{\infty} c_i'\delta_{q_i} = \mu^\star - \sum_{i=1}^{m} c_i \delta r_i.$$

Therefore, the conclusion follows from taking $\mu_n = \mu_{n,k_n}$. □

If we consider the set consisting of all reduced measures for (CS), we then easily see that it is not itself a vector space, but it is closer under addition.

As a final result we obtain the independence of reduced limit with respect to the approximating sequence in signed-measure framework whenever the convergence of positive and negative parts is also taken as hypotheses.

Theorem 3.5. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of good measures in $\mathcal{M}(\Omega)$ and let u_n be the solution of* (CS_n). *If the sequences $(\mu_n^+)_{n\in\mathbb{N}}$ and $(\mu_n^-)_{n\in\mathbb{N}}$ are such that*

$$\mu_n^+ \overset{*}{\rightharpoonup} \mu^+ \quad \textit{and} \quad \mu_n^- \overset{*}{\rightharpoonup} \mu^-,$$

then $(u_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$ to the solution of $(\mathrm{CS}^\star)$.

Proof. By estimate (2.2) and the triangle inequality,

$$\|\Delta u_n\|_{\mathcal{M}(\Omega)} \leqslant 2\|\mu_n\|_{\mathcal{M}(\Omega)}.$$

Since the sequence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}(\Omega)$, the sequence $(\Delta u_n)_{n\in\mathbb{N}}$ is also bounded in $\mathcal{M}(\Omega)$. From Stampacchia's linear regularity theory (see [11], Theorem 9.1, and [9], Proposition 5.8), the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,q}(\Omega)$ for every $1 \leqslant q < 2$. By the Rellich-Kondrachov compactness theorem (see [2], Theorem 9.16), there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging to some function u in $L^1(\Omega)$ and a.e. in Ω. By Lemma 2.2, u solves

$$\begin{cases} -\Delta u + \mathrm{e}^u(\mathrm{e}^u - 1) = \mu - \tau & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tau = \sum\limits_{i=1}^{m} c_i \delta_{r_i}$, $c_1, c_2, \dots c_m \in \mathbb{R}$, $r_1, r_2, \dots, r_m \in A$, and

$$A = \{x \in \Omega\colon\ \mu(\{x\}) \geqslant 2\pi\}.$$

If the set A is empty, the conclusion of the theorem follows with $\mu^\star = \mu$. We may assume that A is nonempty, so that

$$A = \{x_1, \dots, x_l\},$$

where the points $x_i \in \Omega$ are distinct.

In view of Lemma 2.3, we can consider the solutions of

$$\begin{cases} -\Delta v_n + \mathrm{e}^{v_n}(\mathrm{e}^{v_n} - 1) = \mu_n^+ & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Using the known result for nonnegative measures (see [10], Theorem 1.1), (v_n) converges to some function $v \in L^1(\Omega)$ satisfying

$$\begin{cases} -\Delta v + \mathrm{e}^v(\mathrm{e}^v - 1) = (\mu^+)^\star & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Given $i \in \{1, \dots, l\}$, let $r > 0$ be such that $B_r(x_i) \cap A = \{x_i\}$. By the comparison estimate between the solutions v_n of (3.2) and u_n of (CS_n) (see [9], Chapter 14), for every $n \in \mathbb{N}$, we have

$$u_n \leqslant v_n \quad \text{in } \Omega.$$

From

$$\begin{cases} \mathrm{e}^{v_n}(\mathrm{e}^{v_n} - 1) \overset{*}{\rightharpoonup} \mathrm{e}^v(\mathrm{e}^v - 1) + \mu^+ - (\mu^+)^\star, \\ \mathrm{e}^{u_{n_k}}(\mathrm{e}^{u_{n_k}} - 1) \overset{*}{\rightharpoonup} \mathrm{e}^u(\mathrm{e}^u - 1) + \tau \end{cases} \quad \text{in } \mathcal{M}(\Omega),$$

it follows that

$$\mathrm{e}^u(\mathrm{e}^u - 1) + \tau \leqslant \mathrm{e}^v(\mathrm{e}^v - 1) + \mu^+ - (\mu^+)^\star.$$

Evaluating both sides at the set $\{x_i\}$, we get

$$\tau(\{x_i\}) \leqslant (\mu^+ - (\mu^+)^\star)(\{x_i\}).$$

Therefore,

$$\mu(\{x_i\}) - \tau(\{x_i\}) \geqslant \mu^+(\{x_i\}) - (\mu^+ - (\mu^+)^\star)(\{x_i\}) = (\mu^+)^\star(\{x_i\}) = 2\pi = \mu^\star(\{x_i\}).$$

On the other hand, by Vázquez's nonexistence result (see [12], Section 5, and [1], Section 5), we also have $(\mu - \tau)(\{x_i\}) \leqslant 2\pi$. We conclude that

$$(\mu - \tau)(\{x_i\}) = 2\pi = \mu^\star(\{x_i\})$$

for every $i \in \{1, \dots, l\}$. Besides, $\mu = \mu^\star$ in $\Omega \backslash \{x_1, \dots, x_l\}$. Hence u is the solution $u^\star$ of ($\mathrm{CS}^\star$). Since the measure $\mu^\star$ does not depend on the taken subsequence of (u_n) and the solution of ($\mathrm{CS}^\star$) is unique, we deduce that the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$ to the solution $u^\star$ of ($\mathrm{CS}^\star$). □

The particular case $\mu_n = \varrho_n * \mu$, where ϱ_n is a mollifier sequence such that $\operatorname{supp} \varrho_n \subset B_{1/n}$, stated in [3], Theorem 11, is then extended for the larger class of sequences of measures fulfilling the conditions given in the above theorem.

4. A non-uniqueness example

To illustrate the existence of multiple reduced limits for the Chern-Simons equation, we will construct two sequences of measures converging in the weak-$*$ sense to zero with different reduced limits.

Theorem 3.5 implies that $\mu_n = (-1/n)\delta_0$ has zero as reduced limit. We now consider the solution $u_{n,k}$ of the scalar Chern-Simons equation with the datum

$$\mu_{n,k} = 4\pi\varrho_{\varepsilon_k}(x) - 4\pi\varrho_{1/n}\Bigl(x - \Bigl(\frac{1}{n},0\Bigr)\Bigr),$$

where (ε_k) is a sequence of positive numbers converging to zero. By Lemma 3.3, $(u_{n,k})$ converges to the solution u_n of (CS_n) with

$$\mu_n = 2\pi\delta_0 - 4\pi\varrho_{1/n}\Bigl(x - \Bigl(\frac{1}{n},0\Bigr)\Bigr)$$

as k goes to infinity. Applying Lemma 3.2, we have that the solution u_n of (CS_n) converges to the solution of the scalar Chern-Simons equation with the datum

$$\mu = -2\pi\delta_0.$$

Thus, using Cantor's diagonal argument, we take $\nu_n = \mu_{n,k_n}$, where k_n is chosen in order to have

$$\|u_{n,k_n} - u_n\|_{L^1(B_1(0))} \leqslant \frac{1}{k}.$$

Therefore, for $\mu_n = -(1/n)\delta_0$ and $\nu_n = 4\pi\varrho_{1/k_n}(x) - 4\pi\varrho_{1/n}(x - (1/n,0))$ (for an appropriate subsequence (k_n) of integer numbers), the corresponding solutions u_n and v_n converge to the solution of the scalar Chern-Simons equation with 0 and $-2\pi\delta_0$ as data, respectively.

5. Chern-Simons system

The authors have also approached the approximation scheme in [10] for the Chern-Simons system

$$\begin{cases} -\Delta u + \mathrm{e}^v(\mathrm{e}^u - 1) = \mu & \text{in } \Omega,\\ -\Delta v + \mathrm{e}^u(\mathrm{e}^v - 1) = \nu & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

with nonnegative measures. An additional hypothesis $\mu(\{x\}), \nu(\{x\}) \leqslant 4\pi$ for all $x \in \Omega$ is necessary to ensure the stability of the solutions, i.e. if $\mu_n \overset{*}{\rightharpoonup} \mu$ and $\nu_n \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}(\Omega)$ then the pair of the solutions (u_n, v_n) converges in $L^1(\Omega) \times L^1(\Omega)$ to a solution of the Chern-Simons system with prescribed data. In a future work, we will precisely elaborate the results for the system with signed measures as we intend to investigate the general case when the measures μ and ν are not restricted on unitary sets by the value of 4π.

Acknowledgments. The author thanks the CAPES and FAPESP for the financial support, the Université Catholique de Louvain and Augusto Ponce for their hospitality during the development of part of this work, and the referee for his/her valuable suggestions.

References

[1] *D. Bartolucci, F. Leoni, L. Orsina, A. C. Ponce*: Semilinear equations with exponential nonlinearity and measure data. Ann. Inst. Henri Poincaré, Anal. Non Linéaire *22* (2005), 799–815. zbl MR doi

[2] *H. Brezis*: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, Springer, New York, 2011. zbl MR doi

[3] *H. Brezis, M. Marcus, A. C. Ponce*: Nonlinear elliptic equations with measures revisited. Mathematical Aspects of Nonlinear Dispersive Equations (J. Bourgain et al., eds.). Lectures of the CMI/IAS workshop on Mathematical aspects of nonlinear PDEs, Princeton, 2004. Ann. Math. Stud. 163. Princeton Univ. Press, Princeton, pp. 55–109. zbl MR doi

[4] *H. Brezis, F. Merle*: Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. Commun. Partial Differ. Equations *16* (1991), 1223–1253. zbl MR doi

[5] *H. Brezis, W. A. Strauss*: Semi-linear second-order elliptic equations in L^1. J. Math. Soc. Japan *25* (1973), 565–590. zbl MR doi

[6] *L. C. Evans, R. F. Gariepy*: Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992. zbl MR doi

[7] *C.-S. Lin, A. C. Ponce, Y. Yang*: A system of elliptic equations arising in Chern-Simons field theory. J. Funct. Anal. *247* (2007), 289–350. zbl MR doi

[8] *M. Marcus, A. C. Ponce*: Reduced limits for nonlinear equations with measures. J. Funct. Anal. *258* (2010), 2316–2372. zbl MR doi

[9] *A. C. Ponce*: Elliptic PDEs, Measures and Capacities. From the Poisson Equation to Nonlinear Thomas-Fermi Problems. EMS Tracts in Mathematics 23. EMS, Zürich, 2016. zbl MR doi

[10] *A. C. Ponce, A. E. Presoto*: Limit solutions of the Chern-Simons equation. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *84* (2013), 91–102. zbl MR doi

[11] *G. Stampacchia*: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Ann. Inst. Fourier *15* (1965), 189–257. (In French.) zbl MR doi

[12] *J. L. Vázquez*: On a semilinear equation in $\mathbb{R}^2$ involving bounded measures. Proc. R. Soc. Edinb., Sect. A *95* (1983), 181–202. zbl MR doi

Author's address: *Adilson Eduardo Presoto*, Federal University of São Carlos, Rodovia Washington Luís, Km 235, 13565-905 São Carlos-SP, Brazil, e-mail: `presoto@dm.ufscar.br`.