146 (2021) MATHEMATICA BOHEMICA No. 3, 263–278

# EXISTENCE OF NONOSCILLATORY SOLUTIONS TO THIRD ORDER NEUTRAL TYPE DIFFERENCE EQUATIONS WITH DELAY AND ADVANCED ARGUMENTS

Srinivasan Selvarangam, Sethurajan A. Rupadevi,
Ethiraju Thandapani, Chennai,
Sandra Pinelas, Amadora

Received July 10, 2019. Published online September 30, 2020.
Communicated by Leonid Berezansky

*Abstract.* In this paper, we present several sufficient conditions for the existence of nonoscillatory solutions to the following third order neutral type difference equation

$$\Delta^3(x_n + a_n x_{n-l} + b_n x_{n+m}) + p_n x_{n-k} - q_n x_{n+r} = 0, \quad n \geqslant n_0$$

via Banach contraction principle. Examples are provided to illustrate the main results. The results obtained in this paper extend and complement some of the existing results.

*Keywords*: third order; nonoscillation; delay and advanced arguments; neutral difference equation

*MSC 2020*: 39A10

## 1. Introduction

This paper deals with the existence of nonoscillatory solutions of third order neutral type difference equations of the form

$$\Delta^3(x_n + a_n x_{n-l} + b_n x_{n+m}) + p_n x_{n-k} - q_n x_{n+r} = 0, \quad n \geqslant n_0 \tag{1.1}$$

where $n_0$, a nonnegative integer, is subject to the following conditions:

This work was partially supported by FCT and CEMAT by the projects UIDB/04621/2020 and UIDP/04621/2020.

DOI: 10.21136/MB.2020.0099-19

(H$_1$) $\{a_n\}$ and $\{b_n\}$ are real sequences;
(H$_2$) $\{p_n\}$ and $\{q_n\}$ are non-negative real sequences;
(H$_3$) $l$ and $m$ are positive integers and $k$ and $r$ are non-negative integers.

Let $\theta = \max\{l, k\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geqslant n_0 - \theta$, and satisfying the equation (1.1) for all $n \geqslant n_0$. A nontrivial solution of equation (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and oscillatory otherwise.

Recently, many researchers have been interested in investigating the existence of nonoscillatory solutions of neutral type difference equations; see for example [4]–[7], [9], [10], [13], [15], [16], and the references cited therein.

In [10], the authors discussed the existence of nonoscillatory solutions for the equation

$$\Delta^3(x_n + p_n x_{n-k}) + q_n f(x_{n-l}) = h_n.$$

They established sufficient conditions for the existence of nonoscillatory solutions depending on the different ranges of $\{p_n\}$.

In [7], the authors discussed the existence of nonoscillatory solutions of the third order difference equation

$$\Delta(a_n \Delta(b_n \Delta(x_n + p x_{n-m}))) + p_n f(x_{n-k}) - q_n f(x_{n-l}) = 0,$$

when $p \geqslant 0$ or $p \leqslant 0$.

In [16], the existence of nonoscillatory solutions of a higher order nonlinear neutral difference equation

$$\Delta^m(x(n) + p(n)x(\tau(n))) + f_1(n, x(\sigma_1(n))) - f_2(n, x(\sigma_2(n))) = 0$$

was studied.

On the other hand, there has been great interest in studying the oscillatory behavior of third and higher order neutral type difference equations with delay and advanced terms; see for example [1], [2], [3], [8], [11], [12], [14], and the references cited therein. To the best of our knowledge, only a few results are available for third order nonlinear difference equations with delay and advanced terms. This is due mainly to the technical difficulties arising in their analysis.

In view of the above observation, in this paper we obtain some new sufficient conditions for the existence of nonoscillatory solutions for the equation (1.1) using the Banach fixed point theorem. Examples are provided to illustrate the main results.

## 2. Existence of nonoscillatory solutions

In this section, we present some sufficient conditions for the existence of nonoscillatory solutions of equation (1.1) using the Banach contraction principle. For convenance we use the following notation:

$$(n)^{(m)} = n(n-1)\dots(n-m+1) \quad \text{for any positive integer } m.$$

We begin with the following theorem.

**Theorem 2.1** (Banach's contraction mapping principle)**.** *A contraction mapping on a complete metric space has a unique fixed point.*

**Theorem 2.2.** *Assume that* $0 \leqslant a_n \leqslant a < 1$, *and* $0 \leqslant b_n \leqslant b \leqslant 1 - a$ *for all* $n \geqslant n_0$. *If*

$$\sum_{n=n_0}^{\infty} (n+2)^2 p_n < \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} (n+2)^2 q_n < \infty, \tag{2.1}$$

*then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. From condition (2.1), one can choose an integer $N > n_0$ so that

$$N \geqslant n_0 + \theta \tag{2.2}$$

sufficiently large such that

$$\sum_{s=n}^{\infty}(s+2)^2 p_s \leqslant \frac{M_2 - \alpha}{M_2}, \quad n \geqslant N, \tag{2.3}$$

and

$$\sum_{s=n}^{\infty}(s+2)^2 q_s \leqslant \frac{\alpha - (a+b)M_2 - M_1}{M_2}, \quad n \geqslant N, \tag{2.4}$$

where $M_1$ and $M_2$ are positive constants such that

$$(a+b)M_2 + M_1 < M_2, \quad \text{and} \quad \alpha \in ((a+b)M_2 + M_1, M_2).$$

Let $B$ be the set of all bounded real sequences $\{x_n\}$ defined for all $n \geqslant n_0$ with supremum norm $\|x\| = \sup\limits_{n \geqslant n_0} |x|$. Clearly $B$ is a Banach space. Set

$$S = \{x \in B\colon\ M_1 \leqslant x_n \leqslant M_2,\ n \geqslant n_0\}.$$

It is clear that $S$ is a bounded, closed and convex subset of $B$. Define an operator $T\colon S \to B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} \\ \qquad\qquad + \displaystyle\sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2}(p_s x_{s-k} - q_s x_{s+r}), & n \geqslant N, \\ (Tx)_N, & n_0 \leqslant n \leqslant N. \end{cases}$$

Clearly $Tx$ is continuous. For $n \geqslant N$ and $x \in S$, we have

$$(Tx)_n \leqslant \alpha + \sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2} p_s x_{s-k}.$$

Since $x_{n-k} \in B$, we have $x_{n-k} \leqslant M_2$ and $s - n \leqslant s$ for all $s \geqslant n$, and $(s+2)^{(2)} \leqslant (s+2)^2$ and using these inequalities, we obtain

$$(Tx)_n \leqslant \alpha + M_2 \sum_{s=n}^{\infty} (s+2)^2 p_s \leqslant \alpha + M_2 - \alpha \leqslant M_2,$$

and

$$\begin{aligned} (Tx)_n &\geqslant \alpha - a_n x_{n-l} - b_n x_{n+m} - \sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2} q_s x_{s+r} \\ &\geqslant \alpha - aM_2 - bM_2 - M_2 \sum_{s=n}^{\infty} (s+2)^2 q_s \geqslant M_1, \end{aligned}$$

where we have used (2.3) and (2.4). Thus $TS \subset S$.

Next, we show that $T$ is a contraction mapping on $S$. Let $x, y \in S$, and $n \geqslant N$. Then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leqslant a_n |x_{n-l} - y_{n-l}| + b_n |x_{n+m} - y_{n+m}| \\ &\qquad + \sum_{s=n}^{\infty} \frac{(s+2)^2}{2} (p_s |x_{s-k} - y_{s-k}| + q_s |x_{s+r} - y_{s+r}|) \\ &\leqslant \|x - y\| \Big( a + b + \sum_{s=n}^{\infty} (s+2)^2 (p_s + q_s) \Big) \\ &\leqslant \|x - y\| \Big( a + b + \frac{M_2 - \alpha}{M_2} + \frac{\alpha - (a+b)M_2 - M_1}{M_2} \Big) \\ &\leqslant \|x - y\| \Big( \frac{M_2 - M_1}{M_2} \Big) \leqslant \lambda_1 \|x - y\|, \end{aligned}$$

where $\lambda_1 = (1 - M_1/M_2)$. This implies that $\|Tx - Ty\| \leqslant \lambda_1\|x - y\|$. Since $\lambda_1 = (1 - M_1/M_2) < 1$, $T$ is a contraction mapping on $S$. By Theorem 2.1, $T$ has a unique fixed point, that is $T(x_n) = x_n$. Now for $n \geqslant n_0$,

$$x_n = \alpha - a_n x_{n-l} - b_n x_{n+m} + \sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2}(p_s x_{s-k} - q_s x_{s+r}),$$

then

$$\Delta(x_n + a_n x_{n-l} + b_n x_{n+m}) = -\sum_{s=n}^{\infty}(s-n+1)(p_s x_{s-k} - q_s x_{s+r}),$$

and

$$\Delta^2(x_n + a_n x_{n-l} + b_n x_{n+m}) = \sum_{s=n}^{\infty}(p_s x_{s-k} - q_s x_{s+r}),$$

and hence

$$\Delta^3(x_n + a_n x_{n-l} + b_n x_{n+m}) + p_n x_{n-k} - q_n x_{n+r} = 0.$$

Thus $\{x_n\}$ is a positive and bounded solution of equation (1.1). The proof is now completed. □

**Theorem 2.3.** *Assume that* $0 \leqslant a_n \leqslant a < 1$, $a - 1 < b \leqslant b_n \leqslant 0$ *for all* $n \geqslant n_0$. *If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), we can choose an integer $N > n_0$ sufficiently large satisfying (2.2) such that

$$\sum_{s=n}^{\infty}(s+2)^2 p_s \leqslant \frac{(1+b)M_4 - \alpha}{M_4}, \quad n \geqslant N, \tag{2.5}$$

and

$$\sum_{s=n}^{\infty}(s+2)^2 q_s \leqslant \frac{\alpha - aM_4 - M_3}{M_4}, \quad n \geqslant N, \tag{2.6}$$

where $M_3$ and $M_4$ are positive constants such that

$$M_3 + aM_4 < (1+b)M_4, \quad \text{and} \quad \alpha \in (M_3 + aM_4, (1+b)M_4).$$

Let $B$ be the Banach space as defined in Theorem 2.2. Set

$$S = \{x \in B\colon\ M_3 \leqslant x_n \leqslant M_4,\ n \geqslant n_0\}.$$

It is clear that $S$ is a bounded, closed and convex subset of $B$. Define an operator $T\colon S\to B$ as follows:

$$(Tx)_n=\begin{cases}\alpha-a_nx_{n-l}-b_nx_{n+m}\\ \qquad\qquad\qquad+\displaystyle\sum_{s=n}^{\infty}\frac{(s-n+2)^{(2)}}{2}(p_sx_{s-k}-q_sx_{s+r}), & n\geqslant N,\\ (Tx)_N, & n_0\leqslant n\leqslant N.\end{cases}$$

Clearly $Tx$ is continuous. For $n\geqslant N$ and $x\in S$, we have from (2.5) and (2.6)

$$(Tx)_n\leqslant \alpha-bM_4+M_4\sum_{s=n}^{\infty}(s+2)^2p_s\leqslant M_4,$$

and

$$(Tx)_n\geqslant \alpha-aM_4-M_4\sum_{s=n}^{\infty}(s+2)^2q_s\geqslant M_3.$$

This proves that $TS\subset S$.

Next we show that $T$ is a contraction mapping. Let $x,y\in S$ and $n\geqslant N$. Then

$$|(Tx)_n-(Ty)_n|\leqslant \|x-y\|\left(a-b+\sum_{s=n}^{\infty}(s+2)^2(p_s+q_s)\right)\leqslant \lambda_2\|x-y\|,$$

where $\lambda_2=(1-M_3/M_4)$. This implies that $\|Tx-Ty\|\leqslant\lambda_2\|x-y\|$. Since $\lambda_2<1$, $T$ is a contraction mapping on $S$. Hence by Theorem 2.1, $T$ has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. □

**Theorem 2.4.** *Assume that* $1<a\leqslant a_n\leqslant d<\infty$, *and* $0\leqslant b_n\leqslant b<a-1$ *for all* $n\geqslant n_0$. *If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

P r o o f. In view of condition (2.1), one can choose an integer $N>n_0$ so that

$$N+l\geqslant n_0+k \tag{2.7}$$

sufficiently large such that

$$\sum_{s=n}^{\infty}(s+2)^2p_s\leqslant\frac{aM_6-\alpha}{M_6},\quad n\geqslant N, \tag{2.8}$$

and

$$\sum_{s=n}^{\infty}(s+2)^2 q_s \leqslant \frac{\alpha - dM_5 - (1+b)M_6}{M_6}, \quad n \geqslant N, \tag{2.9}$$

where $M_5$ and $M_6$ are positive constants such that

$$dM_5 + (1+b)M_6 < aM_6, \quad \text{and} \quad \alpha \in (dM_5 + (1+b)M_6, aM_6).$$

Let $B$ be the Banach space as defined in Theorem 2.2. Set

$$S = \{x \in B\colon\ M_5 \leqslant x_n \leqslant M_6,\ n \geqslant n_0\}.$$

Obviously $S$ is a bounded, closed and convex subset of $B$. Define a mapping $T\colon S \to B$ as follows:

$$(Tx)_n = \begin{cases} \dfrac{1}{a_{n+l}}\Big\{\alpha - x_{n+l} - b_n x_{n+l+m} \\ \qquad + \displaystyle\sum_{s=n+l}^{\infty} \frac{(s+2-n-l)^{(2)}}{2}(p_s x_{s-k} - q_s x_{s+r})\Big\}, & n \geqslant N, \\ (Tx)_N, & n_0 \leqslant n \leqslant N. \end{cases}$$

Clearly, $Tx$ is continuous. For $n \geqslant N$ and $x \in S$, we have from (2.8), and (2.9), respectively, that

$$(Tx)_n \leqslant \frac{1}{a_{n+l}}\left(\alpha + M_6 \sum_{s=n}^{\infty}(s+2)^2 p_s\right) \leqslant M_6,$$

and

$$(Tx)_n \geqslant \frac{1}{a_{n+l}}\left(\alpha - M_6 - bM_6 - M_6 \sum_{s=n}^{\infty}(s+2)^2 q_s\right) \geqslant M_5.$$

Thus $TS \subset S$. Next we show that $T$ is a contraction mapping on $S$. If $x, y \in S$ and $n \geqslant N$, then

$$|(Tx)_n - (Ty)_n| \leqslant \frac{1}{a}\|x-y\|\left(1 + b + \sum_{s=n}^{\infty}(s+2)^2(p_s + q_s)\right) \leqslant \lambda_3\|x-y\|,$$

where $\lambda_3 = (1 - dM_5/M_6)$. This implies that $\|Tx - Ty\| \leqslant \lambda_3\|x-y\|$. Since $\lambda_3 < 1$, $T$ is a contraction mapping on $S$. Therefore by Theorem 2.1, $T$ has a unique fixed point which is a positive and bounded solution of equation (1.1). The proof is now completed. □

**Theorem 2.5.** *Assume that* $1 < a \leqslant a_n \leqslant d < \infty$, $1 - a < b \leqslant b_n \leqslant 0$ *for all* $n \geqslant n_0$. *If condition* (2.1) *holds then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.7) such that

$$\sum_{s=n}^{\infty}(s+2)^2 p_s \leqslant \frac{(a+b)M_8 - \alpha}{M_8}, \quad n \geqslant N, \tag{2.10}$$

and

$$\sum_{s=n}^{\infty}(s+2)^2 q_s \leqslant \frac{\alpha - dM_7 - M_8}{M_8}, \quad n \geqslant N, \tag{2.11}$$

where $M_7$ and $M_8$ are positive constants such that

$$dM_7 + M_8 < (a+b)M_8, \quad \text{and} \quad \alpha \in (dM_7 + M_8, (a+b)M_8).$$

Let $B$ be the Banach space as defined in Theorem 2.2. Set

$$S = \{x \in B\colon\ M_7 \leqslant x_n \leqslant M_8,\ n \geqslant n_0\}.$$

Clearly $S$ is a bounded, closed and convex subset of $B$. Define a mapping $T\colon\ S \to B$ as follows:

$$(Tx)_n = \begin{cases} \dfrac{1}{a_{n+l}}\Big\{\alpha - x_{n+l} - b_{n+l}x_{n+l+m} \\ \qquad + \displaystyle\sum_{s=n+l}^{\infty}\frac{(s-n-l+2)^{(2)}}{2}(p_s x_{s-k} - q_s x_{s+r})\Big\}, & n \geqslant N, \\ (Tx)_N, & n_0 \leqslant n \leqslant N. \end{cases}$$

It is clearly that $Tx$ is continuous. For $n \geqslant N$ and $x \in S$, we have from (2.10) and (2.11), respectively, that

$$(Tx)_n \leqslant \frac{1}{a}\left(\alpha - bM_8 + M_8\sum_{s=n}^{\infty}(s+2)^2 p_s\right) \leqslant M_8,$$

and

$$(Tx)_n \geqslant \frac{1}{d}\left(\alpha - M_8 - M_8\sum_{s=n}^{\infty}(s+2)^2 q_s\right) \geqslant M_7.$$

This implies that $TS \subset S$. Further, if $x, y \in S$ and $n \geqslant N$, then

$$|(Tx)_n - (Ty)_n| \leqslant \frac{1}{a}\|x-y\|\left(1 - b + \sum_{s=n}^{\infty}(s+2)^2(p_s+q_s)\right) = \lambda_4\|x-y\|,$$

where $\lambda_4 = (1 - dM_7/M_8)$. This implies that $\|Tx - Ty\| \leqslant \lambda_4\|x-y\|$. Since $\lambda_4 < 1$, $T$ is a contraction mapping on $S$. By Theorem 2.1, $T$ has a unique fixed point, that is $T(x_n) = x_n$. Now for $n \geqslant n_0$,

$$x_n = \frac{1}{a_{n+l}}\left\{\alpha - x_{n+l} - b_{n+l}x_{n+l+m} + \sum_{s=n+l}^{\infty}\frac{(s-n-l+2)^{(2)}}{2}(p_s x_{s-k} - q_s x_{s+r})\right\},$$

or

$$a_{n+l}x_n = \alpha - x_{n+l} - b_{n+l}x_{n+l+m} + \sum_{s=n+l}^{\infty}\frac{(s-n-l+2)^{(2)}}{2}(p_s x_{s-k} - q_s x_{s+r}),$$

and by replacing $n$ by $n-l$, we have

$$x_n + a_n x_{n-l} + b_n x_{n+m} = \alpha + \sum_{s=n}^{\infty}\frac{(s-n+2)^{(2)}}{2}(p_s x_{s-k} - q_s x_{s+r}),$$

then arguing as in the proof of Theorem 2.2, we obtain

$$\Delta^3(x_n + a_n x_{n-l} + b_n x_{n+m}) + p_n x_{n-k} - q_n x_{n+r} = 0.$$

Thus, $\{x_n\}$ is a positive and bounded solution of equation (1.1). The proof is now completed. □

**Theorem 2.6.** *Assume that* $-1 < a \leqslant a_n \leqslant 0$, $0 \leqslant b_n \leqslant b \leqslant 1 + a$ *for all* $n \geqslant n_0$. *If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

P r o o f. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.2) such that

$$\sum_{s=n}^{\infty}(s+2)^2 p_s \leqslant \frac{(1+a)M_{10} - \alpha}{M_{10}}, \quad n \geqslant N, \tag{2.12}$$

and

$$\sum_{s=n}^{\infty}(s+2)^2 q_s \leqslant \frac{\alpha - bM_{10} - M_9}{M_{10}}, \quad n \geqslant N, \tag{2.13}$$

where $M_9$ and $M_{10}$ are positive constants such that

$$M_9 + bM_{10} < (1+a)M_{10}, \quad \text{and} \quad \alpha \in (M_9 + bM_{10}, (1+a)M_{10}).$$

Let $B$ be the Banach space as defined in Theorem 2.1. Set

$$S = \{x \in B\colon\ M_9 \leqslant x_n \leqslant M_{10},\ n \geqslant n_0\}.$$

Clearly $S$ is a bounded, closed and convex subset of $B$. Define a mapping $T\colon\ S \to B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} \\ \qquad\qquad + \displaystyle\sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2} \left(p_s x_{s-k} - q_s x_{s+r}\right), & n \geqslant N, \\ (Tx)_N, & n_0 \leqslant n \leqslant N. \end{cases}$$

Obviously, $Tx$ is continuous. For $n \geqslant N$ and $x \in S$, from (2.12) and (2.13) it follows that

$$(Tx)_n \leqslant \alpha - aM_{10} + M_{10} \sum_{s=n}^{\infty} (s+2)^2 p_s \leqslant M_{10},$$

and

$$(Tx)_n \geqslant \alpha - bM_{10} - M_{10} \sum_{s=n}^{\infty} (s+2)^2 q_s \geqslant M_9.$$

Thus, $TS \subset S$. Next we show that $T$ is a contraction mapping on $S$. If $x, y \in S$, and $n \geqslant N$, then

$$|(Tx)_n - (Ty)_n| \leqslant \|x-y\| \left(-a + b + \sum_{s=n}^{\infty} (s+2)^2 (p_s + q_s)\right) = \lambda_5 \|x-y\|,$$

where $\lambda_5 = (1 - M_9/M_{10})$. This implies that $\|Tx - Ty\| \leqslant \lambda_5 \|x-y\|$. Since $\lambda_5 < 1$, $T$ is a contraction mapping on $S$. By Theorem 2.1, $T$ has a unique fixed point which is a positive and bounded solution of equation (1.1). The proof is now completed. □

**Theorem 2.7.** *Assume that* $-1 < a \leqslant a_n \leqslant 0$, $-1 - a < b \leqslant b_n \leqslant 0$ *for all* $n \geqslant n_0$. *If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.2) such that

$$\sum_{s=n}^{\infty} (s+2)^2 p_s \leqslant \frac{(1+a+b)M_{12} - \alpha}{M_{12}}, \quad n \geqslant N, \tag{2.14}$$

and

$$\sum_{s=n}^{\infty}(s+2)^2 q_s \leqslant \frac{\alpha - M_{11}}{M_{12}}, \quad n \geqslant N, \tag{2.15}$$

where $M_{11}$ and $M_{12}$ are positive constants such that

$$M_{11} < (1+a+b)M_{12}, \quad \text{and} \quad \alpha \in (M_{11}, (1+a+b)M_{12}).$$

Let $B$ be the Banach space as defined in Theorem 2.1. Set

$$S = \{x \in B\colon\ M_{11} \leqslant x_n \leqslant M_{12},\ n \geqslant n_0\}.$$

It is clear that $S$ is a bounded, closed and convex subset of $B$. Define an operator $T\colon\ S \to B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} \\ \qquad\qquad + \displaystyle\sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2}(p_s x_{s-k} - q_s x_{s+r}), & n \geqslant N, \\ (Tx)_N, & n_0 \leqslant n \leqslant N. \end{cases}$$

Clearly $Tx$ is continuous. For $n \geqslant N$ and $x \in S$, from (2.14) and (2.15), it follows that

$$(Tx)_n \leqslant \alpha - aM_{12} - bM_{12} + M_{12}\sum_{s=n}^{\infty}(s+2)^2 p_s \leqslant M_{12},$$

and

$$(Tx)_n \geqslant \alpha - M_{12}\sum_{s=n}^{\infty}(s+2)^2 q_s \geqslant M_{11}.$$

This implies that $TS \subset S$. If $x, y \in S$ and $n \geqslant N$, then we have

$$|(Tx)_n - (Ty)_n| \leqslant \|x-y\|\Big(-a-b+\sum_{s=n}^{\infty}(s+2)^2(p_s+q_s)\Big) = \lambda_6\|x-y\|,$$

where $\lambda_6 = (1 - M_{11}/M_{12})$. This implies that $\|Tx - Ty\| \leqslant \lambda_6\|x-y\|$. Since $\lambda_6 < 1$, $T$ is a contraction mapping on $S$. By Theorem 2.1, $T$ has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. □

**Theorem 2.8.** *Assume that* $-\infty < d \leqslant a_n \leqslant a < -1$, $0 \leqslant b_n \leqslant b < -a-1$ *for all* $n \geqslant n_0$. *If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

P r o o f. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.7) such that

$$\sum_{s=n}^{\infty}(s+2)^2 p_s \leqslant \frac{dM_{13}+\alpha}{M_{14}}, \quad n \geqslant N, \tag{2.16}$$

and

$$\sum_{s=n}^{\infty}(s+2)^2 q_s \leqslant \frac{(-a-1-b)M_{14}-\alpha}{M_{14}}, \quad n \geqslant N, \tag{2.17}$$

where $M_{13}$ and $M_{14}$ are positive constants such that

$$-dM_{13} < (-a-1-b)M_{14}, \quad \text{and} \quad \alpha \in (-dM_{13}, (-a-1-b)M_{14}).$$

Let $B$ be the Banach space as defined in Theorem 2.2. Set

$$S = \{x \in B\colon\ M_{13} \leqslant x_n \leqslant M_{14},\ n \geqslant n_0\}.$$

Clearly $S$ is a bounded, closed and convex subset of $B$. Define a mapping $T\colon\ S \to B$ as follows:

$$(Tx)_n = \begin{cases} -\dfrac{1}{a_{n+l}}\Big\{\alpha + x_{n+l} + b_{n+l}x_{n+l+m} \\ \qquad\qquad -\displaystyle\sum_{s=n+l}^{\infty}\frac{(s-n-l+2)^{(2)}}{2}(p_s x_{n-k} - q_s x_{s+r})\Big\}, & n \geqslant N, \\ (Tx)_N, & n_0 \leqslant n \leqslant N. \end{cases}$$

Clearly $Tx$ is continuous. For $n \geqslant N$ and $x \in S$, from (2.16) and (2.17) we see that

$$(Tx)_n \leqslant -\frac{1}{a}\Big(\alpha + M_{14} + bM_{14} - M_{14}\sum_{s=n}^{\infty}(s+2)^2 p_s\Big) \leqslant M_{14},$$

and

$$(Tx)_n \geqslant -\frac{1}{d}\Big(\alpha + M_{14}\sum_{s=n}^{\infty}(s+2)^2 q_s\Big) \geqslant M_{13}.$$

Thus $TS \subset S$. If $x, y \in S$ and $n \geqslant N$, then we have

$$|(Tx)_n - (Ty)_n| \leqslant -\frac{1}{a}\|x-y\|\Big(1 + b + \sum_{s=n}^{\infty}(s+2)^2(p_s+q_s)\Big) = \lambda_7\|x-y\|,$$

where $\lambda_7 = (1 - M_{13}/M_{14})$. This implies that $\|Tx - Ty\| \leqslant \lambda_7\|x-y\|$. Since $\lambda_7 < 1$, $T$ is a contraction mapping on $S$. By Theorem 2.1, $T$ has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. □

**Theorem 2.9.** *Assume that* $-\infty < d \leqslant a_n \leqslant a < -1$, $a+1 < b \leqslant b_n \leqslant 0$ *for all* $n \geqslant n_0$. *If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

P r o o f. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.7) such that

$$\sum_{s=n}^{\infty}(s+2)^2 p_s \leqslant \frac{dM_{15} + bM_{16} + \alpha}{M_{16}}, \quad n \geqslant N, \tag{2.18}$$

and

$$\sum_{s=n}^{\infty}(s+2)^2 q_s \leqslant \frac{(-a-1)M_{16} - \alpha}{M_{16}}, \quad n \geqslant N, \tag{2.19}$$

where $M_{15}$ and $M_{16}$ are positive constants such that

$$-dM_{15} - bM_{16} < (-a-1)M_{16}, \quad \text{and} \quad \alpha \in (-dM_{15} - bM_{16}, (-a-1)M_{16}).$$

Let $B$ be the Banach space as defined in Theorem 2.2. Set

$$S = \{x \in B\colon\ M_{15} \leqslant x_n \leqslant M_{16},\ n \geqslant n_0\}.$$

It is clear that $S$ is a bounded, closed and convex subset of $B$. Define a mapping $T\colon S \to B$ as follows:

$$(Tx)_n = \begin{cases} -\dfrac{1}{a_{n+l}}\Big\{\alpha + x_{n+l} + b_{n+l}x_{n+l+m} \\ \qquad\qquad - \displaystyle\sum_{s=n+l}^{\infty}\frac{(s-n-l+2)^{(2)}}{2}(p_s x_{n-k} - q_s x_{s+r})\Big\}, & n \geqslant N, \\ (Tx)_N, & n_0 \leqslant n \leqslant N. \end{cases}$$

Clearly $Tx$ is continuous. For $n \geqslant N$ and $x \in S$, we have from (2.18) and (2.19) that

$$(Tx)_n \leqslant -\frac{1}{a}\Big(\alpha + M_{16} - M_{16}\sum_{s=n}^{\infty}(s+2)^2 p_s\Big) \leqslant M_{16},$$

and

$$(Tx)_n \geqslant -\frac{1}{d}\Big(\alpha + bM_{16} + M_{16}\sum_{s=n}^{\infty}(s+2)^2 q_s\Big) \geqslant M_{15}.$$

This implies that $TS \subset S$. If $x, y \in S$ and $n \geqslant N$, then

$$|(Tx)_n - (Ty)_n| \leqslant -\frac{1}{a}\|x-y\|\Big(1 - b + \sum_{s=n}^{\infty}(s+2)^2(p_s+q_s)\Big) = \lambda_8\|x-y\|,$$

where $\lambda_8 = (1 - dM_{15}/M_{16})$. This implies that $\|Tx - Ty\| \leqslant \lambda_8 \|x - y\|$. Since $\lambda_8 < 1$, $T$ is a contraction mapping on $S$. By Theorem 2.1, $T$ has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. $\square$

## 3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the neutral difference equation of the form

$$\begin{aligned}(3.1)\quad &\Delta^3\Big(x_n + \frac{1}{2}x_{n-1} + \frac{1}{3}x_{n+2}\Big) + \Big(\frac{9n+3}{n^2(n+1)(n+2)(n+3)} \\ &+ \frac{(3n^2+21n+38)(n-1)}{n(n+2)(n+3)^2(n+4)(n+5)}\Big)x_{n-1} - \frac{1}{(n+3)^4}x_{n+2} = 0, \quad n \geqslant 1.\end{aligned}$$

Here $a_n = \frac{1}{2}$, $b_n = \frac{1}{3}$,

$$p_n = \frac{9n+3}{n^2(n+1)(n+2)(n+3)} + \frac{(3n^2+21n+38)(n-1)}{n(n+2)(n+3)^2(n+4)(n+5)}, \quad q_n = \frac{1}{(n+3)^4}.$$

One can easily verify that all conditions of Theorem 2.2 are satisfied, and hence equation (3.1) has a bounded nonoscillatory solution. In fact, $\{x_n\} = \{(n+1)/n\}$ is one such solution of equation (3.1).

Example 3.2. Consider a neutral difference equation of the form

$$(3.2)\quad \Delta^3\Big(x_n + \frac{1}{4}x_{n-3} - \Big(\frac{3}{4} - \frac{1}{3^n}\Big)x_{n+2}\Big) + \frac{36}{27}\frac{1}{3^n}x_{n-2} - \frac{28}{27}\frac{1}{3^n}x_{n+1} = 0, \quad n \geqslant 1.$$

Here $a_n = \frac{1}{4}$, $b_n = -(\frac{3}{4} - 3^{-n})$, $p_n = \frac{36}{27}3^{-n}$, $q_n = \frac{28}{27}3^{-n}$. A straight-forward verification shows that all conditions of Theorem 2.3 are satisfied, and hence equation (3.2) has a bounded nonoscillatory solution. In fact $\{x_n\} = \{2 + (-1)^n\}$ is one such solution of equation (3.2).

Example 3.3. Consider a neutral difference equation of the form

$$(3.3)\quad \Delta^3\Big(x_n - \frac{1}{2}\Big(\frac{3}{4} - \frac{1}{2^n}\Big)x_{n-2} - \frac{1}{4}x_{n+2}\Big) + \frac{217}{384}\frac{1}{2^n}x_{n-1} - \frac{55}{96}\frac{1}{2^n}x_{n+1} = 0, \quad n \geqslant 1.$$

Here $a_n = -\frac{1}{2}(\frac{3}{4} - 2^{-n})$, $b_n = -\frac{1}{4}$, $p_n = \frac{217}{384}2^{-n}$, $q_n = \frac{55}{96}2^{-n}$. It is easy to verify that all conditions of Theorem 2.7 are satisfied. In fact $\{x_n\} = \{1 + 2^{-n}\}$ is a bounded nonoscillatory solution of equation (3.3).

Example 3.4. Consider a neutral difference equation of the form

$$(3.4)\qquad \Delta^3(x_n - 4x_{n-1} - 2x_{n+1}) + \frac{1}{2^{n+2}(2+2^n)}x_{n-1} - \frac{1}{2^n}x_{n+2} = 0, \quad n \geqslant 1.$$

Here $a_n = -4$, $b_n = -2$, $p_n = 1/(2^{n+2}(2+2^n))$, and $q_n = 2^{-n}$. One can easily verify that all conditions of Theorem 2.9 are valid. Hence equation (3.4) has a bounded nonoscillatory solution. In fact $\{x_n\} = \{1 + 2^{-n}\}$ is one such solution of equation (3.4).

*References*


[1] *R. P. Agarwal*: Difference Equations and Inequalities: Theory, Methods and Applications. Pure and Applied Mathematics, Marcel Dekker 228. Marcel Dekker, New York, 2000. zbl MR doi

[2] *R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan*: Discrete Oscillation Theory. Hindawi Publishing, New York, 2005. zbl MR doi

[3] *R. P. Agarwal, S. R. Grace, E. Akin-Bohner*: On the oscillation of higher order neutral difference equations of mixed type. Dyn. Syst. Appl. *11* (2002), 459–469. zbl MR

[4] *M. P. Chen, B. G. Zhang*: The existence of the bounded positive solutions of delay difference equations. Panam. Math. J. *3* (1993), 79–94. zbl MR

[5] *B. S. Lalli, B. G. Zhang*: On existence of positive solutions and bounded oscillations for neutral difference equations. J. Math. Anal. Appl. *166* (1992), 272–287. zbl MR doi

[6] *B. S. Lalli, B. G. Zhang, J. Z. Li*: On the oscillation of solutions and existence of positive solutions of neutral difference equations. J. Math. Anal. Appl. *158* (1991), 213–233. zbl MR doi

[7] *Q. Li, H. Liang, W. Dong, Z. Zhang*: Existence of nonoscillatory solutions of higher-order difference equations with positive and negative coefficients. Bull. Korean Math. Soc. *45* (2008), 23–31. zbl MR doi

[8] *S. Selvarangam, S. Geetha, E. Thandapani*: Oscillation and asymptotic behavior of third order nonlinear neutral difference equations with mixed type. J. Nonlinear Funct. Anal. *2017* (2017), Article ID 2, 17 pages. doi

[9] *E. Thandapani, J. R. Graef, P. W. Spikes*: On existence of positive solutions and oscillations of neutral difference equations of odd order. J. Differnce Equ. Appl. *2* (1996), 175–183. zbl MR doi

[10] *E. Thandapani, R. Karunakaran, I. M. Arockiasamy*: Existence results for nonoscillatory solutions of third order nonlinear neutral difference equations. Sarajevo J. Math. *5(17)* (2009), 73–87. zbl MR

[11] *E. Thandapani, N. Kavitha*: Oscillatory behavior of solutions of certain third order mixed neutral difference equations. Acta Math. Sci., Ser. B, Engl. Ed. *33* (2013), 218–226. zbl MR doi

[12] *E. Thandapani, S. Selvarangam, D. Seghar*: Oscillatory behavior of third order nonlinear difference equation with mixed neutral terms. Electron. J. Qual. Theory Differ. Equ. *2014* (2014), Article ID 53, 11 pages. zbl MR doi

[13] *K. S. Vidhyaa, C. Dharuman, J. R. Graef, E. Thandapani*: Existence of nonoscillatory solutions to third order nonlinear neutral difference equations. Filomat *32* (2018), 4981–4991. MR doi

[14] *B. Zhang*: Oscillatory behavior of solutions of general third order mixed neutral difference equations. Acta Math. Appl. Sin., Engl. Ser. *31* (2015), 467–474. zbl MR doi

[15] *Y. Zhou*: Existence of nonoscillatory solutions of higher-order neutral difference equations with general coefficients. Appl. Math. Lett. *15* (2002), 785–791. zbl MR doi

[16] *Y. Zhou, Y. Q. Huang*: Existence for nonoscillatory solutions of higher-order nonlinear neutral difference equations. J. Math. Anal. Appl. *280* (2003), 63–76. zbl MR doi

*Authors' addresses*: *Srinivasan Selvarangam*, *Sethurajan Ammaiappan Rupadevi*, Department of Mathematics, Presidency College (Autonomous), 100 Kamarajar Salai, Chepauk, Chennai 600005, India, e-mail: selvarangam.9962@gmail.com, rupalect@gmail.com; *Ethiraju Thandapani*, Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Wallajah Road, Navalar Nagar, Chepauk, Triplicane, Chennai 600005, India, e-mail: ethandapani@yahoo.co.in; *Sandra Pinelas*, Departamento de Ciências Exatas e Engenharia, Academia Militar, Av. Conde Castro Guimarães, 2720-113 Amadora, Portugal, e-mail: sandra.pinelas@gmail.com.