ON AN ENTIRE FUNCTION REPRESENTED BY MULTIPLE DIRICHLET SERIES

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Abstract. Consider the space L of entire functions represented by multiple Dirichlet series that becomes a non uniformly convex Banach space which is also proved to be dense, countable and separable. Continuing further, for the given space L the characterization of bounded linear transformations in terms of matrix and characterization of linear functional has been obtained.

Keywords: Dirichlet series; Banach algebra

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1. Introduction

A series of the form

$$f(s)=\sum_{n=1}^{\infty}a_n\mathrm{e}^{\lambda_n s},\quad s=\sigma+\mathrm{i}t,\ \sigma,t\in\mathbb{R},$$

where a_n's belong to $\mathbb{C}$ and λ_n is a strictly increasing sequence of positive numbers in $\mathbb{R}$ which satisfies

$$0<\lambda_1<\lambda_2<\lambda_3<\ldots<\lambda_n\ldots,\quad \lambda_n\to\infty \text{ as } n\to\infty,$$

is called a Dirichlet series. Originally the above series in the form $\sum\limits_{n=1}^{\infty}a_nn^{-s}$ was first instigated by Dirichlet for his studies in the number theory. Dirichlet and Dedekind considered only the real values of the variable s and obtained many results. The initial results were obtained by Cahen (see [6]) who involved the complex values of s and determined the nature of the region of convergence of the series (1.1). Further

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Littlewood in [9] succeeded in proving that the Dirichlet series could be used in the study of entire functions. Moving a step ahead in the given field it was established in [19], that the Dirichlet series could also be used in the study of meromorphic functions.

Let

$$(1.1)\quad f(s_1, s_2, \ldots, s_n) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \ldots \sum_{m_n=1}^{\infty} a_{m_1,m_2,\ldots,m_n} \mathrm{e}^{(\lambda_{1_{m_1}} s_1 + \lambda_{2_{m_2}} s_2 + \ldots + \lambda_{n_{m_n}} s_n)}$$

be a multiple Dirichlet series, where $s_j = \sigma_j + \mathrm{i}t_j$, $j \in \{1, 2, \ldots, n\}$, and $a_{m_1,m_2,\ldots,m_n} \in \mathbb{C}$. Also

$$0 < \lambda_{p_1} < \lambda_{p_2} < \ldots < \lambda_{p_k} \to \infty \quad \text{as } k \to \infty \text{ for } p = 1, 2, \ldots, n.$$

To simplify the form of an n-tuple Dirichlet series we use the following notations:

$$\begin{aligned} s &= (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n, \\ m &= (m_1, m_2, \ldots, m_n) \in \mathbb{C}^n, \\ \lambda_{n_{m_n}} &= (\lambda_{1_{m_1}}, \lambda_{2_{m_2}}, \ldots, \lambda_{n_{m_n}}) \in \mathbb{R}^n, \\ \lambda_{n_{m_n}} s &= \lambda_{1_{m_1}} s_1 + \lambda_{2_{m_2}} s_2 + \ldots + \lambda_{n_{m_n}} s_n, \\ |\lambda_{n_{m_n}}| &= \lambda_{1_{m_1}} + \lambda_{2_{m_2}} + \ldots + \lambda_{n_{m_n}}, \\ |m| &= m_1 + m_2 + \ldots + m_n. \end{aligned}$$

Thus, the series (1.1) can be written as

$$(1.2)\qquad f(s) = \sum_{m=1}^{\infty} a_m \mathrm{e}^{\lambda_{n_{m_n}} s}.$$

Janusauskas in [10] showed that if there exists a tuple $p > \bar{0} = (0, 0, \ldots, 0)$ such that

$$(1.3)\qquad \limsup_{|m| \to \infty} \frac{\sum_{k=1}^{\infty} \log m_k}{p \lambda_{n_{m_n}}} = 0,$$

then the domain of absolute convergence of (1.2) coincides with its domain of convergence. Sarkar in [27] proved that the necessary and sufficient condition for the series (1.2), where $a_m \in \mathbb{C}$ satisfies (1.3), to be entire is that

$$(1.4)\qquad \lim_{|m| \to \infty} \frac{\log |a_m|}{|\lambda_{n_{m_n}}|} = -\infty.$$

Meili and Zongsheng in [20] investigated the convergence and growth of an n-tuple Dirichlet series and thus established the equivalence relation between the order and its coefficients. Vaish in [31] proved a necessary and sufficient condition so that the Goldberg order of a multiple Dirichlet series defining an entire function remained unaltered under the rearrangements of coefficients of the series.

Let $u(s) = \sum\limits_{m=1}^{\infty} \alpha_m e^{\lambda_{n_{m_n}} s}$ be a fixed Dirichlet series having none of α_m's equal to zero and exponents satisfying (1.3). Let L be the class of all functions f having the same sequence $\{\lambda_{n_{m_n}}\}$ of exponents as that of u and $|a_m/\alpha_m| \to 0$ as $|m| \to \infty$. Moreover, if u represents an entire function then L includes entire functions only.

The norm in L is defined as

$$\|f\| = \sup_{|m|} \left| \frac{a_m}{\alpha_m} \right|. \tag{1.5}$$

Since ages many researchers have worked in the field of Dirichlet series in one variable which can be seen in [14] and [15]. Various results have been proved for different classes of entire Dirichlet series and a few of them may be found in [1]–[5], [7], [8], [11], [14]–[16], [20]–[31].

Further in [16] Kumar and Manocha studied results for a Dirichlet series having complex frequencies. Very recently Akanksha and Srivastava in [1] studied the vector-valued Dirichlet series in a half-plane and thus proved certain fruitful results. In [13] Kumar and Lakshika worked on Dirichlet series in two variables thus expanding the field further. Kong and Gan in [12] and Kong in [11] studied facts on order and type of Dirichlet series.

In [5] emphasis was laid on the bornological properties of the space of entire functions of several complex variables which further widens the field of the Dirichlet series.

In the present paper multiple Dirichlet series (1.1) are considered, their form is reduced to (1.2) and certain aspects of bounded linear transformations in the form of a matrix and characterization of a linear functional are obtained. This further expands the field taking it to different heights. The theory of Dirichlet series was further expanded when eloquent developments were made by Tanaka (see [28]–[30]), Azpeitia (see [2]–[4]), Rahman (see [21]–[26]) and Dagene (see [7]).

2. Main results

In this section main results are proved. For the definitions of terms used refer to [17] and [18].

Theorem 1. *L is a non uniformly convex Banach space which is also separable.*

P r o o f. In order to prove this theorem we need to show that L is complete under the norm defined in (1.5). Let $\{f_p\}$ be any Cauchy sequence in L where

$$f_p(s) = \sum_{m=1}^{\infty} a_m^{(p)} \mathrm{e}^{\lambda_{n_{m_n}} s}.$$

Then for a given $\varepsilon > 0$ we have

$$\|f_p - f_q\| < \varepsilon \quad \forall\, |p|, |q| \geqslant |M|,$$

that is,

$$\sup_{|m|} \left| \frac{a_m^{(p)} - a_m^{(q)}}{\alpha_m} \right| < \varepsilon \quad \forall\, |p| \geqslant |M|.$$

This shows that $\{a_m^{(p)}\}$ forms a Cauchy sequence in $\mathbb{C}$ for all values of $|m| \geqslant 1$. Hence

$$\lim_{|p| \to \infty} a_m^{(p)} = a_m \quad \forall\, |m| \geqslant 1.$$

Letting $|q| \to \infty$ above we get

$$\sup_{|m|} \left| \frac{a_m^{(p)} - a_m}{\alpha_m} \right| < \varepsilon \quad \forall\, |p| \geqslant |M|.$$

Thus $f_p \to f$ as $|p| \to \infty$. Also

$$\left| \frac{a_m}{\alpha_m} \right| \leqslant \left| \frac{a_m^{(p)} - a_m}{\alpha_m} \right| + \left| \frac{a_m^{(p)}}{\alpha_m} \right| \to 0.$$

Thus L is complete, therefore a Banach space. Further consider f and g defined as

$$f(s) = \sum_{m=1}^{\infty} \alpha_m \mathrm{e}^{\lambda_{n_{m_n}} s} \quad \text{and} \quad g(s) = \sum_{m=1}^{\infty} \dot{\alpha}_m \mathrm{e}^{\lambda_{\dot{n}_{\dot{m}_{\dot{n}}}} s} + \alpha_m \mathrm{e}^{\lambda_{n_{m_n}} s},$$

where $|m|, |n|, |\dot{m}|, |\dot{n}|$ are fixed positive integers.

Clearly $f, g \in L$ and $\|f\| = \|g\| = 1$, $\|f - g\| = 1 > \varepsilon$, but $\|f + g\| = 2 \nleq 2 - 2\delta$ for any positive $\delta(\varepsilon)$ which shows that L is not uniformly convex.

Further, L is proved to be separable. For this consider the set consisting of the functions f represented as $f(s) = \sum\limits_{m=1}^{k} c_m \mathrm{e}^{\lambda_{n_{m_n}} s}$, where $|k|$ is a positive integer, and define $c_m = r_m + q_m$ such that r_m, q_m are rational numbers.

Clearly the given set is countable and dense in L. We know that $f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{n_{m_n}} s} \in L$ if and only if $|a_m/\alpha_m| \to 0$, $|m| \to \infty$. This implies $|a_m/\alpha_m| \leqslant \frac{1}{2}\varepsilon$ for $|m| \geqslant |\widetilde{M}|$.

Define $h(s) \in L$ as $h(s) = \sum_{m=1}^{\infty} c_m e^{\lambda_{n_{m_n}} s}$ such that $b_m = 0$ for $|m| \geqslant |\widetilde{M}|$ and

$$\left|\frac{a_m - c_m}{\alpha_m}\right| \leqslant \frac{1}{2}\varepsilon \quad \text{for } |m| = 1, 2, 3, \ldots, |\widetilde{M}|.$$

Then

$$\|f - h\| \leqslant \sup_{|m| \leqslant |\widetilde{M}|} \left|\frac{a_m - c_m}{\alpha_m}\right| + \sup_{|m| \geqslant |\widetilde{M}|} \left|\frac{a_m}{\alpha_m}\right| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus L is separable which proves the theorem. □

Theorem 2. *Every bounded linear functional φ defined for $f \in L$ is of the form*

$$\varphi(f) = \sum_{m=1}^{\infty} a_m p_m \quad \textit{where} \quad \sum_{m=1}^{\infty} |\alpha_m p_m| < \infty$$

and $\{p_m\}$ is a sequence of real numbers.

To prove the theorem we need the following lemma.

Lemma 1. *$f_{\widehat{m}} \to f$ where $f_{\widehat{m}}(s) = \sum_{m=1}^{\widehat{m}} a_m e^{\lambda_{n_{\widehat{m}_n}} s}$ and $f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{n_{m_n}} s}$ if and only if $|a_m/\alpha_m| \to 0$, $|m| \to \infty$, i.e. $f \in L$.*

Proof. If $f \in L$, where $f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{n_{m_n}} s}$, we have $|a_m/\alpha_m| \to 0$ as $|m| \to \infty$. Now $\|f - f_{\widehat{m}}\| = \sup_{|m| > |\widehat{m}|} |a_m/\alpha_m| \to 0$ as $|m| \to \infty$. Conversely, if $f \notin L$ then

$$\|f_{pq}\| = \max_{|p| \leqslant |m| \leqslant |q|} \left|\frac{a_m}{\alpha_m}\right|, \quad \text{where } f_{pq}(s) = \sum_{m=p}^{q} a_m e^{\lambda_{n_{m_n}} s},$$

so that $\{f_{pq}\}$ is not even a Cauchy sequence. □

Proof of Theorem 2. Let φ be defined on L as $\varphi(f) = \sum_{m=1}^{\infty} a_m p_m$. It is

$$\sum_{m=1}^{\infty} |a_m p_m| \leqslant \sup_{|m|} \left|\frac{a_m}{\alpha_m}\right| \sum_{m=1}^{\infty} |\alpha_m p_m| = \|f\| \sum_{m=1}^{\infty} |\alpha_m p_m| < \infty.$$

Hence φ is a well defined functional on L. Clearly

$$|\varphi(f)| \leqslant \sum_{m=1}^{\infty} |a_m p_m| \leqslant \|f\| \sum_{m=1}^{\infty} |\alpha_m p_m|.$$

This implies $\|\varphi\| \leqslant \sum\limits_{m=1}^{\infty} |\alpha_m p_m|$.

Therefore φ is a bounded linear functional on L which also belongs to L^*, the dual space of L.

Conversely, if $\varphi \in L^*$ and is defined as $\varphi(\delta_m) = p_m$, where $\delta_m(s) = \mathrm{e}^{\lambda_{n_{m_n}} s}$ for each $|m|$. Then for any $f \in L^*$

$$f(s) = \sum_{m=1}^{\infty} a_m \mathrm{e}^{\lambda_{n_{m_n}} s} = \sum_{m=1}^{\infty} a_m \delta_m(s),$$

$$\varphi(f) = \varphi\Big(\lim_{|m^-|\to\infty} f_{m^-}\Big) = \varphi\Big(\lim_{|t|\to\infty} \sum_{m=1}^{|t|} a_m \delta_m\Big) = \lim_{|t|\to\infty} \sum_{m=1}^{|t|} a_m \varphi(\delta_m) = \sum_{m=1}^{\infty} a_m p_m.$$

Further we prove that $\sum\limits_{m=1}^{\infty} |\alpha_m p_m| \leqslant \|\varphi\|$ so that $\sum\limits_{m=1}^{\infty} |\alpha_m p_m| < \infty$. Let $|d| \geqslant 1$, then define

$$a_m = \begin{cases} |\alpha_m| \operatorname{sgn}(p_m) & \text{for } 1 \leqslant |m| \leqslant |d|, \\ 0 & \text{for } |m| > |d|. \end{cases}$$

If $f(s) = \sum\limits_{m=1}^{\infty} a_m \mathrm{e}^{\lambda_{n_{m_n}} s}$ then $f \in L$ and $\|f\| = 1$, and hence

$$|\varphi(f)| = \left| \sum_{m=1}^{|d|} |\alpha_m| \operatorname{sgn}(p_m) \varphi(\delta_m) \right| = \sum_{m=1}^{|d|} |\alpha_m p_m|,$$

whereas $|\varphi(f)| \leqslant \|\varphi\| \cdot \|f\| = \|\varphi\|$ so that $\sum\limits_{m=1}^{|d|} |\alpha_m p_m| \leqslant \|\varphi\|$ and $\sum\limits_{m=1}^{\infty} |\alpha_m p_m| = \sup\limits_{|d|} \sum\limits_{m=1}^{|d|} |\alpha_m p_m| \leqslant \sup\limits_{|d|} \|\varphi\| \leqslant \|\varphi\|$. Thus from the above stated we conclude the theorem. □

Theorem 3. *Let $(\zeta_{m,k})_{|m|,|k|\in\mathbb{N}}$ be an infinite matrix of complex entries and B be a transformation on L defined as $B(f)(s) = \sum\limits_{m=1}^{\infty} B_m(f)\mathrm{e}^{\lambda_{n_{m_n}} s}$ where $B_m(f) = \sum\limits_{k=1}^{\infty} \zeta_{m_k} a_k$ where $f(s) = \sum\limits_{k=1}^{\infty} a_k \mathrm{e}^{\lambda_{n_{k_n}} s}$. Let*

(i) *$\zeta_{m_k}/\alpha_m \to 0$ as $|m| \to \infty$ where $|k|$ are fixed.*

(ii) *$O = \sup\limits_{|m|} \sum\limits_{k=1}^{\infty} |\alpha_k \zeta_{m_k}/\alpha_m| < \infty$.*

Then B is a bounded linear operator on L such that $\|B\| = O$.

P r o o f. Let $f \in L$ then $B(f) \in L$ provided that $|B_m/\alpha_m| \to 0$ as $|m| \to \infty$. Thus

$$\Big|\frac{B_m}{\alpha_m}\Big| = \sum_{k=1}^{|d|}\Big|\frac{\zeta_{m_k} a_k}{\alpha_m}\Big| + \sum_{k=|d|+1}^{\infty}\Big|\frac{\zeta_{m_k} a_k}{\alpha_m}\Big| < \|f\| \sum_{k=1}^{|d|}\Big|\frac{\zeta_{m_k}\alpha_k}{\alpha_m}\Big| + \Big(\max_{|k|>|d|+1}\Big|\frac{a_k}{\alpha_k}\Big|\Big)O.$$

If we assume $|d|$ large enough so that $\max\limits_{|k|>|d|+1} |a_k/\alpha_k| < \varepsilon$ for $\varepsilon > 0$, choose $|d|$ so large that $\sum\limits_{k=1}^{|d|} |\zeta_{m_k}/\alpha_m \alpha_k| < \varepsilon$.

Thus from (i) and using the above statements one can make $|B_m/\alpha_m|$ small and conclude that $|B_m/\alpha_m| \to 0$ as $|m| \to \infty$. Clearly B is linear. Also

$$\|B(f)\| \leqslant \|f\| \sup_{|m|} \sum_{k=1}^{\infty}\Big|\frac{\zeta_{m_k}}{\alpha_m}\alpha_k\Big| = O\|f\|, \tag{3.1}$$

proving that O is bounded and

$$\|B\| \leqslant O. \tag{3.2}$$

Since $\varepsilon > 0$ is given then there exists a positive integer $|m'|$ such that

$$\sum_{k=1}^{\infty}\Big|\frac{\zeta_{m'_k}}{\alpha_{m'}}\alpha_k\Big| > G - \frac{1}{2}\varepsilon.$$

But $\sum\limits_{k=1}^{\infty} |\alpha_k \zeta_{m'_k}/\alpha_{m'}|$ is finite so there exists a positive integer $|j|$ such that

$$\sum_{|k|>|p|}\Big|\frac{\zeta_{m'_k}}{\alpha_{m'}}\alpha_k\Big| < \frac{1}{2}\varepsilon.$$

Define

$$a_k = \begin{cases} \alpha_k \operatorname{sgn}(\zeta_{m'_k}) & \text{for } 1 \leqslant |k| \leqslant |j|, \\ 0 & \text{for } |k| > |j|. \end{cases}$$

Then $f(s) = \sum\limits_{k=1}^{\infty} a_k e^{\lambda_{n_{k_n}} s} \in L$ and $\|f\| = 1$.

Also

$$\frac{\|B(f)\|}{\|f\|} = \|B(f)\| = \sup_{|m|} \Bigl|\frac{B_m(f)}{\alpha_m}\Bigr| \geqslant \Bigl|\frac{B_{m'}(f)}{\alpha_{m'}}\Bigr| > O - \varepsilon$$

but $\|B\| = \sup \|B(f)\|/\|f\|$ provided $f \neq \gamma$ where in $\gamma(s) = \sum\limits_{m=1}^{\infty} a_m e^{\lambda_{n_{m_n}} s}$ it is such that $a_m = 0$ for each $|m|$ and therefore

$$\|B\| \geqslant O. \tag{3.3}$$

Therefore from (3.2) and (3.3) we have $\|B\| = O$ which concludes the theorem. □

Theorem 4. *Let B be a bounded linear transformation on L, then it determines a matrix (ζ_{m_k}), $|m| = 1, 2, \ldots$ such that $B_m(f) = \sum\limits_{k=1}^{\infty} \zeta_{m_k} a_k$ and the conditions* (i) *and* (ii) *of Theorem* 3 *hold where $(B(f)(s)) = \sum\limits_{m=1}^{\infty} B_m(f) e^{\lambda_{n_{m_n}} s}$, $f \in L$ is given as $f(s) = \sum\limits_{k=1}^{\infty} a_k e^{\lambda_{n_{k_n}} s}$.*

Proof. Consider the set $\{\delta_k, \delta_k(s) = e^{\lambda_{n_{k_n}} s}, |k| = 1, 2, \ldots\}$ and let B be defined on it as $B(\delta_k(s)) = \sum\limits_{m=1}^{\infty} \zeta_{m_k} e^{\lambda_{n_{m_n}} s}$. Since B is linear and bounded it follows

$$B(f)(s) = \sum_{k=1}^{\infty} a_k B(\delta_k(s)) = \sum_{m=1}^{\infty} \Bigl(\sum_{k=1}^{\infty} \zeta_{m_k} a_k\Bigr) e^{\lambda_{n_{m_n}} s} = \sum_{m=1}^{\infty} B_m(f) e^{\lambda_{n_{m_n}} s}$$

where

$$B_m(f) = \sum_{k=1}^{\infty} \zeta_{m_k} a_k.$$

Also

$$\frac{B_m(f)}{\alpha_m} = \sum_{k=1}^{\infty} \frac{\zeta_{m_k}}{\alpha_m} a_k.$$

Since $B(\delta_k(s)) \in L$ for each $|k|$ this implies $\zeta_{m_k}/\alpha_m \to 0$ as $|m| \to \infty$.

Next we prove that $\|B\| = O$. Since

$$\Bigl|\frac{B_m}{\alpha_m}\Bigr| \leqslant \|B(f)\| \leqslant \|B\|\|f\|,$$

$\{B_m(f)/\alpha_m\}$ clearly is a sequence of bounded linear functionals on L and it also follows that $\lim_{|m|\to\infty} B_m(f)/\alpha_m = 0$ since

$$\left\|\frac{B_m}{\alpha_m}\right\| = \sum_{k=1}^{\infty}\left|\frac{\zeta_{m_k}}{\alpha_m}\alpha_k\right| < \infty.$$

It is also shown that $O = \sup_{|m|}\Big(\sum_{k=1}^{\infty}|\alpha_k\zeta_{m_k}/\alpha_m|\Big) < \infty$. Thus from the above proof it follows that $\|B\| = O$.

Hence the theorem has been proved. □

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