OSCILLATORY AND NON-OSCILLATORY CRITERIA FOR LINEAR FOUR-DIMENSIONAL HAMILTONIAN SYSTEMS

Gevorg A. Grigorian, Yerevan

Received October 21, 2019. Published online October 1, 2020.
Communicated by Josef Diblík

Abstract. The Riccati equation method is used to study the oscillatory and non-oscillatory behavior of solutions of linear four-dimensional Hamiltonian systems. One oscillatory and three non-oscillatory criteria are proved. Examples of the obtained results are compared with some well known ones.

Keywords: Riccati equation; oscillation; non-oscillation; conjoined (prepared, preferred) solution; Liouville's formula

MSC 2020: 34C10

1. Introduction

Let $A(t) \equiv (a_{jk}(t))_{j,k=1}^2$, $B(t) \equiv (b_{jk}(t))_{j,k=1}^2$, $C(t) \equiv (c_{jk}(t))_{j,k=1}^2$, $t \geqslant t_0$, be complex-valued continuous matrix functions on $[t_0, \infty)$ and let $B(t)$ and $C(t)$ be Hermitian, i.e. $B(t) = B^*(t)$, $C(t) = C^*(t)$, $t \geqslant t_0$. Consider the four-dimensional Hamiltonian system

$$\begin{cases} \varphi' = A(t)\varphi + B(t)\psi, \\ \psi' = C(t)\varphi - A^*(t)\psi, \quad t \geqslant t_0. \end{cases} \tag{1.1}$$

Here $\varphi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2)$ are unknown continuously differentiable vector functions on $[t_0, \infty)$. Along with the system (1.1) consider the linear system of matrix equations

$$\begin{cases} \Phi' = A(t)\Phi + B(t)\Psi, \\ \Psi' = C(t)\Phi - A^*(t)\Psi, \quad t \geqslant t_0, \end{cases} \tag{1.2}$$

where $\Phi(t)$ and $\Psi(t)$ are unknown continuously differentiable matrix functions of dimension 2×2 on $[t_0, \infty)$.

DOI: 10.21136/MB.2020.0149-19

Definition 1.1. A solution $(\Phi(t), \Psi(t))$ of the system (1.2) is called *conjoined* (or *prepared, preferred*) if $\Phi^*(t)\Psi(t) = \Psi^*(t)\Phi(t)$, $t \geqslant t_0$.

Definition 1.2. A solution $(\Phi(t), \Psi(t))$ of the system (1.1) is called *oscillatory* if $\det \Phi(t)$ has arbitrarily large zeros.

Definition 1.3. The system (1.1) is called *oscillatory* if all conjoined solutions of the system (1.2) are oscillatory, otherwise it is called *non-oscillatory*.

The study of the oscillatory and non-oscillatory behavior of Hamiltonian systems (in particular of the system (1.1)) is an important problem of qualitative theory of differential equations and many works are devoted to it (see, e.g., [1], [4], [11]–[14], [16], [18]–[20] and works cited therein). For any Hermitian matrix H, we denote by $H \geqslant 0$, $H > 0$, its nonnegative (positive) definiteness. In the works [1], [4], [12]–[14], [16], [18]–[20], the oscillatory behavior of general Hamiltonian systems is studied under the condition that the coefficient corresponding to $B(t)$ is assumed to be positive definite. In this paper we study the oscillatory and non-oscillatory behavior of the system (1.1) in the case where the assumption $B(t) > 0$, $t \geqslant t_0$, may be violated.

2. Auxiliary propositions

Let $f(t)$, $g(t)$, $h(t)$, $h_1(t)$ be real-valued continuous functions on $[t_0, \infty)$. Consider the Riccati equations

$$y' + f(t)y^2 + g(t)y + h(t) = 0, \quad t \geqslant t_0, \tag{2.1}$$

$$y' + f(t)y^2 + g(t)y + h_1(t) = 0, \quad t \geqslant t_0. \tag{2.2}$$

Theorem 2.1. *Let equation* (2.2) *have a real-valued solution* $y_1(t)$ *on* $[t_1, t_2)$ $(t_0 \leqslant t_1 < t_2 \leqslant \infty)$, *and let* $f(t) \geqslant 0$, $h(t) \leqslant h_1(t)$, $t \in [t_1, t_2)$. *Then for each* $y_{(0)} \geqslant y_1(t_0)$ *equation* (2.1) *has a solution* $y_0(t)$ *on* $[t_1, t_2)$ *with* $y_0(t_0) = y_{(0)}$, *and* $y_0(t) \geqslant y_1(t)$, $t \in [t_1, t_2)$.

Proof. A proof for a more general theorem is presented in [6] (see also [7]). □

Denote

$$I_{g,h}(\xi; t) \equiv \int_\xi^t \exp\left(-\int_\tau^t g(s)\,\mathrm{d}s\right) h(\tau)\,\mathrm{d}\tau, \quad t \geqslant \xi \geqslant t_0.$$

Let $t_0 < \tau_0 \leqslant \infty$ and let $t_0 < t_1 < \ldots$ be a finite or infinite sequence such that $t_k \in [t_0, \tau_0]$, $k = 1, 2, \ldots$ We assume that if $\{t_k\}$ is finite then the maximum of t_k is equal to τ_0 and if $\{t_k\}$ is infinite then $\lim\limits_{k\to\infty} t_k = \tau_0$.

Theorem 2.2. *Let $f(t) \geqslant 0$, $t \in [t_0, \tau_0)$, and*

$$\int_{t_k}^{t} \exp\Bigl(\int_{t_k}^{\tau} (g(s) - I_{g,h}(t_k; s))\,\mathrm{d}s\Bigr) h(\tau)\,\mathrm{d}\tau \leqslant 0, \quad t \in [t_k, t_{k+1}),\ k = 0, 1, \ldots$$

Then for every $y_{(0)} \geqslant 0$ equation (2.1) *has a solution $y_0(t)$ on $[t_0, \tau_0)$ satisfying the initial condition $y_0(t_0) = y_{(0)}$ and $y_0(t) \geqslant 0$, $t \in [t_0, \tau_0)$.*

P r o o f. See the proof in [7]. □

Consider the matrix Riccati equation

$$Z' + ZB(t)Z + A^*(t)Z + ZA(t) - C(t) = 0, \quad t \geqslant t_0. \tag{2.3}$$

The solutions $Z(t)$ of this equation existing on an interval $[t_1, t_2)$ ($t_0 \leqslant t_1 < t_2 \leqslant \infty$) are connected with solutions $(\varphi(t), \Psi(t))$ of the system (1.2) by the following relations (see [11]):

$$\Phi'(t) = (A(t) + B(t)Z(t))\Phi(t), \quad \Phi(t_1) \neq 0,\ \Psi(t) = Z(t)\Phi(t),\ t \in [t_1, t_2). \tag{2.4}$$

Let $Z_0(t)$ be a solution to equation (2.3) on $[t_1, t_2)$.

Definition 2.1. We say that $[t_1, t_2)$ is the *maximum existence interval for* $Z_0(t)$ if $Z_0(t)$ cannot be continued to the right of t_2 as a solution of equation (2.3).

Lemma 2.1. *Let $Z_0(t)$ be a solution of equation* (2.3) *on $[t_1, t_2)$ and let $t_2 < \infty$. Then $[t_1, t_2)$ cannot be the maximum existence interval for $Z_0(t)$ provided the function $G(t) \equiv \int_{t_1}^{t} \operatorname{tr}(B(\tau)Z_0(\tau))\,\mathrm{d}\tau$, $t \in [t_1, t_2)$, is bounded from below on $[t_1, t_2)$.*

P r o o f. The proof is similar to that of Lemma 2.1 in [11]. □

Assume $B(t) = \operatorname{diag}\{b_1(t), b_2(t)\}$, $t \geqslant t_0$. Then it is not difficult to verify that for Hermitian unknowns $Z = \begin{pmatrix} z_{11} & z_{12} \\ \overline{z}_{12} & z_{22} \end{pmatrix}$, equation (2.3) is equivalent to the following nonlinear system:

$$\begin{cases} z'_{11} + b_1(t)z_{11}^2 + 2\operatorname{Re} a_{11}(t)z_{11} + b_2(t)|z_{12}|^2 \\ \qquad\qquad + a_{21}(t)z_{12} + \bar{a}_{21}(t)\overline{z}_{12} - c_{11}(t) = 0, \\ z'_{12} + (b_1(t)z_{11} + b_2(t)z_{22} + \bar{a}_{11}(t) + a_{22}(t))z_{12} \\ \qquad\qquad + a_{12}(t)z_{11} + a_{21}(t)z_{22} - c_{12}(t) = 0, \\ z'_{22} + b_2(t)z_{22}^2 + 2\operatorname{Re} a_{22}(t)z_{22} + b_1(t)|z_{12}|^2 \\ \qquad\qquad + \bar{a}_{12}(t)z_{12} + a_{12}(t)\overline{z}_{12} - c_{22}(t) = 0, \end{cases} \quad t \geqslant t_0. \tag{2.5}$$

If $b_2(t) \neq 0$, $t \geqslant t_0$, then it is not difficult to verify that the first equation of the system (2.5) can be rewritten in the form

$$
\begin{aligned}
&z'_{11} + b_1(t)z_{11}^2 + 2\operatorname{Re} a_{11}(t)z_{11} \\
&\qquad + b_2(t)\Big|z_{12} + \frac{\bar{a}_{21}(t)}{b_2(t)}\Big|^2 - \frac{|a_{21}(t)|^2}{b_2(t)} - c_{11}(t) = 0, \quad t \geqslant t_0,
\end{aligned} \tag{2.6}
$$

and if, in addition, $\bar{a}_{21}(t)/b_2(t)$ is continuously differentiable on $[t_0, \infty)$, then by the substitution

$$
z_{12} = y - \frac{\bar{a}_{21}(t)}{b_2(t)}, \quad t \geqslant t_0, \tag{2.7}
$$

in the first and second equations of the system (2.5), we get the subsystem

$$
\begin{cases}
z'_{11} + b_1(t)z_{11}^2 + 2\operatorname{Re} a_{11}(t)z_{11} + b_2(t)|y|^2 - \dfrac{|a_{21}(t)|^2}{b_2(t)} - c_{11}(t) = 0, \\
y' + (b_1(t)z_{11} + b_2(t)z_{22} + \bar{a}_{11}(t) + a_{22}(t))y \\
\qquad + \Big(a_{12}(t) - \dfrac{b_1(t)}{b_2(t)}\bar{a}_{21}(t)\Big)z_{11} - \Big(\dfrac{\bar{a}_{21}(t)}{b_2(t)}\Big)' \\
\qquad - \dfrac{\bar{a}_{21}(t)}{b_2(t)}(\bar{a}_{11}(t) + a_{22}(t)) - c_{12}(t) = 0, \qquad t \geqslant t_0.
\end{cases} \tag{2.8}
$$

Analogously, if $b_1(t) \neq 0$, $t \geqslant t_0$, then the third equation of the system (2.5) can be rewritten in the form

$$
\begin{aligned}
&z'_{22} + b_2(t)z_{22}^2 + 2\operatorname{Re} a_{22}(t)z_{22} \\
&\qquad + b_1(t)\Big|z_{12} + \frac{a_{12}(t)}{b_1(t)}\Big|^2 - \frac{|a_{12}(t)|^2}{b_1(t)} - c_{22}(t) = 0, \quad t \geqslant t_0,
\end{aligned} \tag{2.9}
$$

and if, in addition, $a_{12}(t)/b_1(t)$ is continuously differentiable on $[t_0, \infty)$, then by the substitution

$$
z_{12} = v - \frac{a_{12}(t)}{b_1(t)}, \quad t \geqslant t_0, \tag{2.10}
$$

in the second and third equations of the system (2.5) we obtain the subsystem

$$
\begin{cases}
z'_{22} + b_2(t)z_{22}^2 + 2\operatorname{Re} a_{22}(t)z_{22} \\
\qquad + b_1(t)|v|^2 - \dfrac{|a_{12}(t)|^2}{b_1(t)} - c_{22}(t) = 0, \\
v' + (b_1(t)z_{11} + b_2(t)z_{22} + \bar{a}_{11}(t) + a_{22}(t))v \\
\qquad + \Big(\bar{a}_{21}(t) - \dfrac{b_2(t)}{b_1(t)}a_{12}(t)\Big)z_{22} - \Big(\dfrac{a_{12}(t)}{b_1(t)}\Big)' \\
\qquad - \dfrac{a_{12}(t)}{b_1(t)}(\bar{a}_{11}(t) + a_{22}(t)) - c_{12}(t) = 0, \qquad t \geqslant t_0.
\end{cases} \tag{2.11}
$$

If $(z_{11}(t), y(t))$ is a solution of the subsystem (2.8) on $[t_0, t_1)$ $(t_0 < t_1 \leqslant \infty)$ with $y(t_0) = 0$ and $(z_{22}(t), v(t))$ is a solution of the subsystem (2.11) on $[t_0, t_1)$ with $v(t_0) = 0$, then by Cauchy formula from the second equation of the subsystem (2.8) and from the second equation of the subsystem (2.11), we have, respectively,

$$\begin{aligned} y(t) = {} & -\exp\Big(-\int_{t_0}^{t} b_1(\tau) z_{11}(\tau)\,\mathrm{d}\tau\Big) \int_{t_0}^{t} \Big(\exp\Big(\int_{t_0}^{\tau} b_1(s) z_{11}(s)\,\mathrm{d}s\Big)\Big)' \\ & \times \Big(\frac{a_{12}(\tau)}{b_1(\tau)} - \frac{\bar{a}_{21}(\tau)}{b_2(\tau)}\Big) \exp\Big(-\int_{\tau}^{t} (b_2(s) z_{22}(s) + \bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big)\,\mathrm{d}\tau \\ & + \int_{t_0}^{t} \exp\Big(-\int_{\tau}^{t} (b_1(s) z_{11}(s) + b_2(s) z_{22}(s) + \bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big) \\ & \times \Big(\Big(\frac{\bar{a}_{21}(\tau)}{b_2(\tau)}\Big)' + \frac{\bar{a}_{21}(\tau)}{b_2(\tau)}(\bar{a}_{11}(\tau) + a_{22}(\tau)) + c_{12}(\tau)\Big)\,\mathrm{d}\tau, \\ v(t) = {} & -\exp\Big(-\int_{t_0}^{t} b_2(\tau) z_{22}(\tau)\,\mathrm{d}\tau\Big) \int_{t_0}^{t} \Big(\exp\Big(\int_{t_0}^{\tau} b_2(s) z_{22}(s)\,\mathrm{d}s\Big)\Big)' \\ & \times \Big(\frac{\bar{a}_{21}(\tau)}{b_2(\tau)} - \frac{a_{12}(\tau)}{b_1(\tau)}\Big) \exp\Big(-\int_{\tau}^{t} (b_1(s) z_{11}(s) + \bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big)\,\mathrm{d}\tau \\ & + \int_{t_0}^{t} \exp\Big(-\int_{\tau}^{t} (b_1(s) z_{11}(s) + b_2(s) z_{22}(s) + \bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big) \\ & \times \Big(\Big(\frac{a_{12}(\tau)}{b_1(\tau)}\Big)' + \frac{a_{12}(\tau)}{b_1(\tau)}(\bar{a}_{11}(\tau) + a_{22}(\tau)) + c_{12}(\tau)\Big)\,\mathrm{d}\tau, \quad t \in [t_0, t_1). \end{aligned}$$

From here it is easy to derive the following lemma.

Lemma 2.2. *Let $b_j(t) > 0$, $j = 1, 2$, let the functions $a_{12}(t)/b_1(t)$, $\bar{a}_{21}(t)/b_2(t)$ be continuously differentiable on $[t_0, t_1)$ $(t_0 < t_1 < \infty)$ and let $(z_{11}(t), y(t))$ and $(z_{22}(t), v(t))$ be solutions of the subsystems (2.8) and (2.11), respectively, on $[t_0, t_1)$ such that $z_{jj}(t) \geqslant 0$, $t \in [t_0, t_1)$, $j = 1, 2$, $y(t_0) = v(t_0) = 0$. Then*

$$\begin{aligned} |y(t)| \leqslant \mathfrak{M}(t) + \int_{t_0}^{t} \Big| & \exp\Big(-\int_{\tau}^{t} (\bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big) \\ & \times \Big(\Big(\frac{\bar{a}_{21}(\tau)}{b_2(\tau)}\Big)' + \frac{\bar{a}_{21}(\tau)}{b_2(\tau)}(\bar{a}_{11}(\tau) + a_{22}(\tau)) + c_{12}(\tau)\Big)\Big|\,\mathrm{d}\tau, \\ |v(t)| \leqslant \mathfrak{M}(t) + \int_{t_0}^{t} \Big| & \exp\Big(-\int_{\tau}^{t} (\bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big) \\ & \times \Big(\Big(\frac{a_{12}(\tau)}{b_1(\tau)}\Big)' + \frac{a_{12}(\tau)}{b_1(\tau)}(\bar{a}_{11}(\tau) + a_{22}(\tau)) + c_{12}(\tau)\Big)\Big|\,\mathrm{d}\tau, \quad t \in [t_0, t_1), \end{aligned}$$

where

$$\mathfrak{M}(t) \equiv \max_{\tau \in [t_0, t]} \Big|\exp\Big(-\int_{\tau}^{t} (\bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big)\Big(\frac{a_{12}(\tau)}{b_1(\tau)} - \frac{\bar{a}_{21}(\tau)}{b_2(\tau)}\Big)\Big|, \quad t \geqslant t_0.$$

Lemma 2.3. *For any two square matrices* $M_1 \equiv (m^1_{ij})^n_{ij=1}$, $M_2 \equiv (m^2_{ij})^n_{ij=1}$ *the equality*

$$\mathrm{tr}(M_1M_2) = \mathrm{tr}(M_2M_1)$$

is valid.

P r o o f. We have $\mathrm{tr}(M_1M_2) = \sum\limits_{j=1}^{n}\Big(\sum\limits_{k=1}^{n} m^1_{jk}m^2_{kj}\Big) = \sum\limits_{k=1}^{n}\Big(\sum\limits_{j=1}^{n} m^1_{jk}m^2_{kj}\Big) = \sum\limits_{k=1}^{n}\Big(\sum\limits_{j=1}^{n} m^2_{kj}m^1_{jk}\Big) = \mathrm{tr}(M_2M_1)$. The lemma is proved. □

3. Main results

Let $f_{jk}(t)$, $j,k = 1,2$, $t \geqslant t_0$, be real-valued continuous functions on $[t_0,\infty)$. Consider the linear system of equations

$$\begin{cases} \varphi_1' = f_{11}(t)\varphi_1 + f_{12}(t)\psi_1, \\ \psi_1' = f_{21}(t)\varphi_1 + f_{22}(t)\psi_1, \quad t \geqslant t_0, \end{cases} \tag{3.1}$$

and the Riccati equation

$$y' + f_{12}(t)y^2 + (f_{11}(t) - f_{22}(t))y - f_{12}(t) = 0, \quad t \geqslant t_0. \tag{3.2}$$

All solutions $y(t)$ of the last equation, existing on some interval $[t_1,t_2)$ ($t_0 \leqslant t_1 < t_2 \leqslant \infty$), are connected with solutions $(\varphi_1(t),\psi_1(t))$ of the system (3.1) by the following relations (see [8]):

$$\begin{aligned} &\varphi_1(t) = \varphi_1(t_1)\exp\bigg(\int_{t_1}^{t}(f_{12}(\tau)y(\tau) + f_{11}(\tau))\,\mathrm{d}\tau\bigg), \quad \varphi_1(t_1) \neq 0, \\ &\psi_1(t) = y(t)\varphi_1(t), \quad t \in [t_1,t_2). \end{aligned} \tag{3.3}$$

Definition 3.1. The system (3.1) is called *oscillatory* if for its every solution $(\varphi_1(t),\psi_1(t))$ the function $\varphi_1(t)$ has arbitrarily large zeros.

R e m a r k 3.1. Some explicit oscillatory criteria for the system (3.1) are proved in [10] and [11].

3.1. The case where $B(t)$ is a diagonal matrix.

In this subsection we will assume that $B(t) = \mathrm{diag}\{b_1(t),b_2(t)\}$. Denote:

$$\chi_j(t) \equiv \begin{cases} c_{jj}(t) & \text{if } b_{3-j}(t) = 0, \\ c_{jj}(t) + \dfrac{|a_{3-j,j}(t)|^2}{b_{3-j}(t)} & \text{if } b_{3-j}(t) \neq 0, \end{cases} \quad t \geqslant t_0,\ j = 1,2.$$

Theorem 3.1. *Assume $b_j(t) \geqslant 0$, $t \geqslant t_0$, and if $b_j(t) = 0$ then $a_{3-j,j}(t) = 0$, $j = 1,2$, $t \geqslant t_0$. Under these restrictions, the system* (1.1) *is oscillatory provided one of the systems*

$$(3.4_j) \qquad \begin{cases} \varphi_1' = 2\operatorname{Re}(a_{jj}(t))\varphi_1 + b_j(t)\psi_1, \\ \psi_1' = -\chi_j(t)\varphi_1, \qquad t \geqslant t_0, \end{cases}$$

$j = 1,2$, is oscillatory.

P r o o f. Suppose the system (1.1) is not oscillatory. Then for some conjoined solution $(\Phi(t), \Psi(t))$ of the system (1.2), there exists $t_1 \geqslant t_0$ such that $\det\Phi(t) \neq 0$, $t \geqslant t_1$. Due to (2.4), it follows that $Z(t) \equiv \Psi(t)\Phi^{-1}(t)$, $t \geqslant t_1$, is a Hermitian solution to equation (2.3) on $[t_1, \infty)$. Let $Z(t) = \begin{pmatrix} z_{11}(t) & z_{12}(t) \\ \overline{z}_{12}(t) & z_{22}(t) \end{pmatrix}$, $t \geqslant t_1$. Consider the Riccati equations

$$(3.5) \qquad y' + b_1(t)y^2 + 2(\operatorname{Re} a_{11}(t))y + b_2(t)|z_{12}(t)|^2 + a_{21}(t)z_{12}(t) + \bar{a}_{21}(t)\overline{z}_{12}(t) - c_{11}(t) = 0,$$

$$(3.6) \qquad y' + b_2(t)y^2 + 2(\operatorname{Re} a_{22}(t))y + b_1(t)|z_{12}(t)|^2 + \bar{a}_{12}(t)z_{12}(t) + a_{12}(t)\overline{z}_{12}(t) - c_{22}(t) = 0,$$

$$(3.7_j) \qquad y' + b_j(t)y^2 + 2(\operatorname{Re} a_{jj}(t)y + \chi_j(t) = 0, \quad j = 1,2, \ t \geqslant t_1.$$

By (2.6) and (2.9), from the conditions of the theorem it follows that

$$\chi_1(t) \leqslant b_2(t)|z_{12}(t)|^2 + a_{21}(t)z_{12}(t) + \bar{a}_{21}(t)\overline{z}_{12}(t) - c_{11}(t), \quad t \geqslant t_1,$$
$$\chi_2(t) \leqslant b_1(t)|z_{12}(t)|^2 + \bar{a}_{12}(t)z_{12}(t) + a_{12}(t)\overline{z}_{12}(t) - c_{22}(t), \quad t \geqslant t_1.$$

Using Theorem 2.1 to the pairs of equations (3.5), (3.7_1) and (3.6), (3.7_2) we conclude that the equations (3.7_j), $j = 1,2$, have solutions on $[t_1, \infty)$. By (3.1)–(3.3), it follows that the systems (3.4_j), $j = 1,2$, are not oscillatory, which contradicts the condition of the theorem. The obtained contradiction completes the proof of the theorem. □

Denote $I_j(\xi;t) \equiv \int_\xi^t \exp\big(-\int_\tau^t 2(\operatorname{Re} a_{jj}(s))\,\mathrm{d}s\big)\chi_j(\tau)\,\mathrm{d}\tau$, $t \geqslant \xi \geqslant t_0$, $j = 1,2$.

Theorem 3.2. *Assume $b_1(t) \geqslant 0$ ($\leqslant 0$), $b_2(t) \leqslant 0$ ($\geqslant 0$), and if $b_j(t) = 0$ then $a_{j,3-j}(t) = 0$, $j = 1,2$, $t \geqslant t_0$; in addition, assume there exist infinitely large sequences $\xi_{j,0} = t_0 < \xi_{j,1} < \ldots < \xi_{j,m} < \ldots$, $j = 1,2$, such that*

$$(1_j) \qquad (-1)^j \int_{\xi_{j,m}}^t \exp\left(\int_{\xi_{j,m}}^\tau (2\operatorname{Re} a_{jj}(s) - (-1)^j I_j(\xi_{j,m}, s))\,\mathrm{d}s\right)\chi_j(\tau)\,\mathrm{d}\tau \geqslant 0 \ (\leqslant 0),$$

$t \in [\xi_{j,m}, \xi_{j,m+1})$, $m = 1,2,3,\ldots$, $j = 1,2$. Then the system (1.1) *is non-oscillatory.*

P r o o f. Let us prove the theorem only for the case $b_1(t) \geqslant 0$, $b_2(t) \leqslant 0$, $t \geqslant t_0$. The case $b_1(t) \leqslant 0$, $b_2(t) \geqslant 0$, $t \geqslant t_0$, can be proved analogously. Let $(\Phi(t), \Psi(t))$ be a conjoined solution of the system (1.2) with $\Phi(t_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and let $[t_0, T)$ be the maximum interval such that $\det \Phi(t) \neq 0$, $t \in [t_0, T)$. Then by (2.4) the matrix function $Z(t) \equiv \Psi(t)\varphi^{-1}(t)$, $t \in [t_0, T)$, is a Hermitian solution to equation (2.3) on $[t_0, T)$. By (2.5), (2.7), (2.8), (2.10), (2.11) it follows that the subsystems (2.8) and (2.11) have solutions $(z_{11}(t), y(t))$ and $(z_{22}(t), v(t))$, respectively, on $[t_0, T)$ with $z_{11}(t_0) = 1$, $z_{22}(t_0) = -1$. We wish to show that

$$z_{11}(t) \geqslant 0, \quad t \in [t_0, T). \tag{3.8}$$

Consider the Riccati equations

$$z' + b_1(t)z^2 + 2(\operatorname{Re} a_{11}(t))z + b_2(t)|y(t)|^2 + \chi_1(t) = 0, \quad t \in [t_0, T), \tag{3.9}$$

$$z' + b_1(t)z^2 + 2(\operatorname{Re} a_{11}(t))z + \chi_1(t) = 0, \quad t \in [t_0, T). \tag{3.10}$$

By Theorem 2.2, it follows that the last equation has a nonnegative solution on $[t_0, T)$. Then using Theorem 2.1 to the pair of equations (3.9), (3.10) we conclude that equation (3.9) has a nonnegative solution $z_0(t)$ on $[t_0, T)$ with $z_0(t_0) = 0$. Then, since $z_{11}(t)$ is a solution to equation (3.9) on $[t_0, T)$ and $z_{11}(t_0) = 1$, we have (3.8). To show that

$$z_{22}(t) \leqslant 0, \quad t \in [t_0, T), \tag{3.11}$$

consider the Riccati equations

$$z' - b_2(t)z^2 + 2(\operatorname{Re} a_{22}(t))z - \chi_2(t) = 0, \quad t \in [t_0, T), \tag{3.12}$$

$$z' - b_2(t)z^2 + 2(\operatorname{Re} a_{22}(t))z - b_1(t)|v(t)|^2 - \chi_2(t) = 0, \quad t \in [t_0, T). \tag{3.13}$$

By Theorem 2.2 it follows that equation (3.12) has a nonnegative solution $z_1(t)$ on $[t_0, T)$ with $z_1(t_0) = 0$. Then using Theorem 2.1 to the pair of equations (3.12) and (3.13) we derive that equation (3.13) has a nonnegative solution $z_2(t)$ on $[t_0, T)$ with $z_2(t_0) = 0$. Hence, since obviously $-z_{22}(t)$ is a solution of equation (3.13) on $[t_0, T)$ and $-z_{11}(t_0) = 1$, we have (3.11). Since $b_1(t) \geqslant 0$, $b_2(t) \leqslant 0$, $t \in [t_0, T)$, from (3.8) and (3.11) it follows that

$$\int_{t_0}^{t} (b_1(\tau)z_{11}(\tau) + b_2(\tau)z_{22}(\tau))\,\mathrm{d}\tau \geqslant 0, \quad t \in [t_0, T). \tag{3.14}$$

To complete the proof of the theorem, it remains to show that $T=\infty$. Suppose $T<\infty$. Then, by virtue of Lemma 2.1, from (3.14) it follows that $[t_0,T)$ is not the maximum existence interval for $Z(t)$. By (2.4), it follows that $\det\Phi(t)\neq 0$, $t\in[t_0,T_1)$ for some $T_1>T$. We have obtained a contradiction, which completes the proof of the theorem. □

Remark 3.2. The conditions (1_j), $j=1,2$, are satisfied if in particular $(-1)^j\chi_j(t)\geqslant 0$ $(\leqslant 0)$, $t\geqslant t_0$.

Denote:

$$\begin{aligned}\chi_3(t)\equiv b_2(t)\biggl(\mathfrak{M}(t)+\int_{t_0}^t\biggl|\exp\biggl(-\int_\tau^t(\bar a_{11}(s)+a_{22}(s))\,\mathrm{d}s\biggr)\\ \times\biggl(\Bigl(\frac{\bar a_{21}(t)}{b_2(t)}\Bigr)'+\frac{\bar a_{21}(\tau)}{b_2(\tau)}(\bar a_{11}(\tau)+a_{22}(\tau))+c_{12}(\tau)\biggr)\biggr|\,\mathrm{d}\tau\biggr)^2\\ -\frac{|a_{21}(t)|^2}{b_2(t)}-c_{11}(t),\\ \chi_4(t)\equiv b_1(t)\biggl(\mathfrak{M}(t)+\int_{t_0}^t\biggl|\exp\biggl(-\int_\tau^t(\bar a_{11}(s)+a_{22}(s))\,\mathrm{d}s\biggr)\\ \times\biggl(\Bigl(\frac{a_{12}(t)}{b_1(t)}\Bigr)'+\frac{a_{12}(\tau)}{b_1(\tau)}(\bar a_{11}(\tau)+a_{22}(\tau)\Bigr)+c_{12}(\tau)\biggr)\biggr|\,\mathrm{d}\tau\biggr)^2\\ -\frac{|a_{12}(t)|^2}{b_1(t)}-c_{22}(t),\\ I_{j+2}(\xi;t)\equiv\int_\xi^t\exp\biggl(-\int_\tau^t 2(\operatorname{Re}a_{jj}(s))\,\mathrm{d}s\biggr)\chi_{j+2}(\tau)\,\mathrm{d}\tau,\quad t\geqslant\xi\geqslant t_0,\ j=1,2.\end{aligned}$$

Theorem 3.3. *Let the following conditions be satisfied:*

(1) $b_j(t)>0, t\geqslant t_0,\ j=1,2;$

(2) *the functions* $a_{12}(t)/b_1(t)$ *and* $\bar a_{21}(t)/b_2(t)$ *are continuously differentiable on* $[t_0,\infty)$*;*

(3) *there exist infinitely large sequences* $\xi_{j,0}=t_0<\xi_{j,1}<\ldots<\xi_{j,m}<\ldots$, $j=1,2$, *such that*

$$\int_{\xi_{j,m}}^t\exp\biggl(\int_{\xi_{j,m}}^\tau(2\operatorname{Re}a_{jj}(s)-I_{j+2}(\xi_{j,m},s))\,\mathrm{d}s\biggr)\chi_{j+2}(\tau)\,\mathrm{d}\tau\leqslant 0,\quad t\in[\xi_{j,m},\xi_{j,m+1}),$$

$m=1,2,3,\ldots$, $j=1,2$.

Then the system (1.1) *is non-oscillatory.*

P r o o f. Let $Z(t) \equiv \begin{pmatrix} z_{11}(t) & z_{12}(t) \\ \overline{z}_{12}(t) & z_{22}(t) \end{pmatrix}$ be the Hermitian solution of equation (2.3) on $[t_0, T)$ satisfying the initial condition $Z(t_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $[t_0, T)$ is the maximum existence interval for $Z(t)$. Due to (2.4), to prove the theorem it is enough to show that

$$T = \infty. \tag{3.15}$$

By (2.5), (2.7), (2.8), (2.10), (2.11), from conditions (1) and (2), it follows that $(z_{11}(t), z_{12}(t) + \bar{a}_{21}(t)/b_2(t))$ and $(z_{22}(t), z_{12}(t) + a_{12}(t)/b_1(t))$ are solutions of the subsystems (2.8) and (2.11), respectively, on $[t_0, T)$. To show that

$$z_{jj}(t) > 0, \quad t \in [t_0, T), \tag{3.16}$$

assume it is not so. Then there exists $T_1 \in (t_0, T)$ such that

$$z_{11}(t) z_{22}(t) > 0, \ t \in [t_0, T_1), \quad z_{11}(T_1) z_{22}(T_1) = 0. \tag{3.17}$$

Without loss of generality we may take that $a_{12}(t_0) = a_{21}(t_0) = 0$. Then by virtue of Lemma 2.2, from (3.17) it follows that

$$\begin{aligned}
&\Big| z_{12}(t) + \frac{\bar{a}_{21}(t)}{b_2(t)} \Big| \\
&\quad \leqslant \mathfrak{M}(t) + \int_{t_0}^{t} \Big| \exp\Big(- \int_{\tau}^{t} (\bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s \Big) \\
&\quad \times \Big(\Big(\frac{\bar{a}_{21}(\tau)}{b_2(\tau)} \Big)' + \frac{\bar{a}_{21}(\tau)}{b_2(\tau)} (\bar{a}_{11}(\tau) + a_{22}(\tau)) - c_{12}(\tau) \Big) \Big| \,\mathrm{d}\tau, \\
&\Big| z_{12}(t) + \frac{a_{12}(t)}{b_1(t)} \Big| \\
&\quad \leqslant \mathfrak{M}(t) + \int_{t_0}^{t} \Big| \exp\Big(- \int_{\tau}^{t} (\bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s \Big) \\
&\quad \times \Big(\Big(\frac{a_{12}(\tau)}{b_1(\tau)} \Big)' + \frac{a_{12}(\tau)}{b_1(\tau)} (\bar{a}_{11}(\tau) + a_{22}(\tau)) - c_{12}(\tau) \Big) \Big| \,\mathrm{d}\tau, \quad t \in [t_0, T_1).
\end{aligned}$$

Hence

$$\begin{aligned}
&b_2(t) \Big| z_{12}(t) + \frac{\bar{a}_{21}(t)}{b_2(t)} \Big| - \frac{|a_{21}(t)|^2}{b_2(t)} - c_{11}(t) \leqslant \chi_3(t), \\
&b_1(t) \Big| z_{12}(t) + \frac{a_{12}(t)}{b_1(t)} \Big|^2 - \frac{|a_{12}(t)|^2}{b_2(t)} - c_{22}(t) \leqslant \chi_4(t), \quad t \in [t_0, T_1),
\end{aligned}$$

By virtue of Theorem 2.1 and Theorem 2.2 and from condition (3), it follows that the Riccati equations

$$(3.18)\quad z' + b_1(t)z^2 + 2(\operatorname{Re} a_{11}(t))z + b_2(t)\Big|z_{12}(t) + \frac{\bar{a}_{21}(t)}{b_2(t)}\Big| - \frac{|a_{21}(t)|^2}{b_2(t)} - c_{11}(t) = 0,$$

$$(3.19)\quad z' + b_2(t)z^2 + 2(\operatorname{Re} a_{22}(t))z + b_1(t)\Big|z_{12}(t) + \frac{a_{12}(t)}{b_1(t)}\Big|^2 - \frac{|a_{12}(t)|^2}{b_2(t)} - c_{22}(t) = 0,$$

$t \in [t_0, T_1)$, have nonnegative solutions $z_1(t)$ and $z_2(t)$, respectively, on $[t_0, T_1)$ with $z_1(t_0) = z_2(t_0) = 0$. Obviously $z_{11}(t)$ and $z_{22}(t)$ are solutions of equation (3.18) and (3.19), respectively, on $[t_0, T_1]$. Therefore, since $z_{jj}(t_0) = 1 > z_j(t_0) = 0$, $j = 1, 2$, due to the uniqueness theorem $z_{jj}(t) > 0$, $t \in [t_0, T_1]$, $j = 1, 2$, which contradicts (3.17). The obtained contradiction proves (3.16). From (3.16) and condition (1) it follows that

$$(3.20)\qquad \int_{t_0}^{t} (b_1(\tau)z_{11}(\tau) + b_2(\tau)z_{22}(\tau))\,\mathrm{d}\tau \geqslant 0, \quad t \in [t_0, T).$$

Suppose $T < \infty$. Then by Lemma 2.1, from (3.20) it follows that $[t_0, T)$ is not the maximum existence interval for $Z(t)$, which contradicts our assumption. The obtained contradiction proves (3.15). The theorem is proved. □

R e m a r k 3.3. Condition (3) of Theorem 3.3 is satisfied if in particular $\chi_j(t) \leqslant 0$, $t \geqslant t_0$, $j = 1, 2$.

3.2. The case where $B(t)$ is nonnegative definite. In this subsection we will assume that $B(t)$ is nonnegative definite and $\sqrt{B(t)}$ is continuously differentiable on $[t_0, \infty)$. Consider the matrix equation

$$(3.21)\qquad \sqrt{B(t)}X\big(A(t)\sqrt{B(t)} - \sqrt{B(t)}'\big) = A(t)\sqrt{B(t)} - \sqrt{B(t)}', \quad t \geqslant t_0.$$

Obviously this equation has always a solution on $[a, b]$ $(\subset [t_0, \infty))$ when $B(t) > 0$, $t \in [a, b]$ $(X(t) = B^{-1}(t)$, $t \in [a, b])$. It may have also a solution on $[a, b]$ in some cases when $B(t) \geqslant 0$, $t \in [a, b]$ (e.g., $A(t) = \begin{pmatrix} a_1(t) & a_2(t) \\ 0 & 0 \end{pmatrix}$, $B(t) = \begin{pmatrix} b_1(t) & 0 \\ 0 & 0 \end{pmatrix}$, $b_1(t) > 0$, $t \in [a, b]$). In this subsection we also will assume that equation (3.21) has always a solution on $[t_0, \infty)$. Let $F(t)$ be a solution of equation (3.21) on $[t_0, \infty)$. Denote

$$(3.22)\qquad \begin{aligned} P(t) &\equiv F(t)\big(A(t)\sqrt{B(t)} - \sqrt{B(t)}'\big) = (p_{jk}(t))_{j,k=1}^2, \\ Q(t) &\equiv \sqrt{B(t)}C(t)\sqrt{B(t)} = (q_{jk}(t))_{j,k=1}^2, \\ \widetilde{\chi}_j(t) &\equiv q_{jj}(t) + |p_{3-j,j}(t)|^2, \quad j = 1, 2, \ t \geqslant t_0. \end{aligned}$$

Corollary 3.1. *The system* (1.1) *is oscillatory provided one of the equations*

$$\varphi_1'' + 2(\operatorname{Re} p_{jj}(t))\varphi_1' + \widetilde{\chi}_j(t)\varphi_1 = 0, \quad j = 1, 2,\ t \geqslant t_0 \tag{3.23$_j$}$$

is oscillatory.

Proof. Multiply equation (2.3) on the left and on the right by $\sqrt{B(t)}$. Taking into account the equality $\big(\sqrt{B(t)}Z\sqrt{B(t)}\big)' = \sqrt{B(t)}Z'\sqrt{B(t)} + \sqrt{B(t)}'Z\sqrt{B(t)} + \sqrt{B(t)}Z\sqrt{B(t)}'$, $t \geqslant t_0$, we obtain

$$V' + V^2 + P^*(t)V + VP(t) - Q(t) = 0, \quad t \geqslant t_0, \tag{3.24}$$

where $V \equiv \sqrt{B(t)}Z\sqrt{B(t)}$. This equation corresponds to the following matrix Hamiltonian system

$$\begin{cases} \Phi' = P(t)\Phi + \Psi, \\ \Psi' = Q(t)\Phi - P^*(t)\Psi, \quad t \geqslant t_0. \end{cases} \tag{3.25}$$

Suppose the system (1.1) is not oscillatory. Then by (2.4), equation (2.3) has a Hermitian solution $Z(t)$ on $[t_1, \infty)$ for some $t_1 \geqslant t_0$. Therefore, $V(t) \equiv \sqrt{B(t)}Z(t)\sqrt{B(t)}$, $t \geqslant t_1$, is a Hermitian solution of equation (3.24) on $[t_1, \infty)$ and hence the system (3.25) has a conjoined solution $(\Phi(t), \Psi(t))$ such that $\det \Phi(t) \neq 0$, $t \geqslant t$. It means that the Hamiltonian system

$$\begin{cases} \varphi' = P(t)\varphi + \psi, \\ \psi' = Q(t)\varphi - P^*(t)\psi, \quad t \geqslant t_0, \end{cases}$$

is not oscillatory. By Theorem 3.1, it follows that the scalar systems

$$\begin{cases} \varphi_1' = 2\operatorname{Re} p_{jj}(t)\varphi_1 + \psi_1, \\ \psi_1' = -\widetilde{\chi}_j(t)\varphi_1, \qquad\qquad t \geqslant t_0, \end{cases}$$

$j = 1, 2$, are not oscillatory. Therefore, the corresponding equations (3.23$_j$), $j = 1, 2$, are not oscillatory, which contradicts the conditions of the corollary. This completes the proof. □

Denote:

$$\widetilde{\mathfrak{M}}(t) \equiv \max_{\tau\in[t_0,t]}\left|\exp\left(-\int_\tau^t (\bar p_{11}(s)+p_{22}(s))\,\mathrm{d}s\right)(p_{12}(\tau)-\bar p_{21}(\tau))\right|;$$

$$\widetilde\chi_3(t) \equiv \left(\widetilde{\mathfrak{M}}(t) + \int_{t_0}^t \left|\exp\left(-\int_\tau^t (\bar p_{11}(s)+p_{22}(s))\,\mathrm{d}s\right)\right.\right.$$
$$\left.\left.\times\,(\overline{p}'_{21}+\bar p_{21}(\tau)(\bar p_{11}(\tau)+p_{22}(\tau))+q_{12}(\tau))\right|\,\mathrm{d}\tau\right)^2 - |p_{21}(t)|^2 - q_{11}(t);$$

$$\widetilde\chi_4(t) \equiv \left(\widetilde{\mathfrak{M}}(t) + \int_{t_0}^t \left|\exp\left(-\int_\tau^t \bar p_{11}(s)+p_{22}(s)\,\mathrm{d}s\right)\right.\right.$$
$$\left.\left.\times\,(p'_{12}(t)+p_{12}(\tau)(\bar p_{11}(\tau)+p_{22}(\tau))+q_{12}(\tau))\right|\,\mathrm{d}\tau\right)^2$$
$$-\,|p_{12}(t)|^2 - q_{22}(t),\quad t\geqslant t_0;$$

$$\tilde I_{j+2}(\xi,t)\equiv\int_\xi^t \exp\left(-\int_\tau^t 2(\operatorname{Re} p_{jj}(s))\,\mathrm{d}s\right)\widetilde\chi_{j+2}(\tau)\,\mathrm{d}\tau,\quad t\geqslant\xi\geqslant t_0,\ j=1,2.$$

Theorem 3.4. *Let the following conditions be satisfied:*

(1′) $B(t)\geqslant 0$, $t\geqslant t_0$;

(2′) *equation* (3.21) *has a solution* $F(t)$ *on* $[t_0,\infty)$;

(3′) *the functions* $p_{12}(t)$ *and* $p_{21}(t)$*, defined by* (3.22)*, are continuously differentiable on* $[t_0,\infty)$;

(4′) *there exist infinitely large sequences* $\xi_{j,0}=t_0<\xi_{j,1}<\ldots<\xi_{j,m}<\ldots$ *such that*

$$\int_{\xi_{j,m}}^t \exp\left(\int_{\xi_{j,m}}^\tau (2\operatorname{Re} a_{jj}(s)-\tilde I_{j+2}(\xi_{j,m},s))\,\mathrm{d}s\right)\widetilde\chi_{j+2}(\tau)\,\mathrm{d}\tau\leqslant 0,\quad t\in[\xi_{j,m},\ \xi_{j,m+1}),$$

$m=1,2,3,\ldots$, $j=1,2$.

Then the system (1.1) *is non-oscillatory.*

Proof. Let $Z(t)\equiv\begin{pmatrix} z_{11}(t) & z_{12}(t)\\ \overline{z}_{12}(t) & z_{22}(t)\end{pmatrix}$ be the Hermitian solution of equation (2.3) satisfying the initial condition $Z(t_0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, and let $[t_0,T)$ be the maximum existence interval for $Z(t)$. Then $V(t)\equiv\sqrt{B(t)}Z(t)\sqrt{B(t)}$ is a solution of equation (3.24) on $[t_0,T)$. Without loss of generality, we may assume that $B(t_0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. Then, $V(t_0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, and similarly to the proof of Theorem 3.3, we can show that

$$\int_{t_0}^t \operatorname{tr} V(\tau)\,\mathrm{d}\tau\geqslant 0,\quad t\in[t_0,T). \tag{3.26}$$

By virtue of Lemma 2.3 we have $\operatorname{tr} V(t) = \operatorname{tr}(B(t)Z(t))$, $t \in [t_0, T)$. From this and from (3.26) it follows that

$$\int_{t_0}^{t} \operatorname{tr}(B(\tau)Z(\tau))\,\mathrm{d}\tau \geqslant 0, \quad t \in [t_0, T). \tag{3.27}$$

To complete the proof of the theorem, it remains to show that $T = \infty$. Suppose $T < \infty$. Then, by virtue of Lemma 2.2, from (3.27) it follows that $[t_0, T)$ is not the maximum existence interval for $Z(t)$, which contradicts our assumption. This contradiction shows that $T = \infty$, and the theorem is proved. □

Example 3.1. Consider the second-order vector equation

$$\varphi'' + K(t)\varphi = 0, \quad t \geqslant t_0, \tag{3.28}$$

where $K(t) \equiv \begin{pmatrix} \mu(t) & 10\mathrm{i} \\ -10\mathrm{i} & -t^2 \end{pmatrix}$, $\mu(t) \equiv p_1 \sin(\lambda_1 t + \theta_1) + p_2 \sin(\lambda_2 t + \theta_2)$, $t \geqslant t_0$, p_j, $\lambda_j \neq 0$, θ_j, $j = 1, 2$, are real constants such that λ_1 and λ_2 are rational independent. This equation is equivalent to the system (1.1) with $A(t) \equiv 0$, $B(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C(t) = -K(t)$, $t \geqslant t_0$. Hence, by Theorem 3.1, equation (3.28) is oscillatory provided the scalar system

$$\begin{cases} \varphi_1' = \psi_1, \\ \psi_1' = -\mu(t)\varphi_1, \quad t \geqslant t_0, \end{cases}$$

is oscillatory. This system is equivalent to the second-order scalar equation

$$\varphi_1'' + \mu(t)\varphi_1 = 0, \quad t \geqslant t_0,$$

which is oscillatory (see [9]). Therefore, equation (3.28) is oscillatory. It is not difficult to verify that the results in works [2], [3], [5], [15], [17] are not applicable to equation (3.28).

Example 3.2. Let

$$B(t) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad t \geqslant t_0. \tag{3.29}$$

Then $\sqrt{B(t)} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\sqrt{B(t)}' \equiv 0$, $t \geqslant t_0$, and $F(t) = \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $t \geqslant t_0$, is a solution of equation (3.21) on $[t_0, \infty)$,

$$P(t) = \begin{pmatrix} a_{11}(t) + a_{12}(t) a_{11}(t) + a_{12}(t) \\ a_{21}(t) + a_{22}(t) a_{21}(t) + a_{22}(t) \end{pmatrix}, \tag{3.30}$$

$$Q(t) = (c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t))B(t), \quad t \geqslant t_0. \tag{3.31}$$

Assume

$$a_{11}(t) + a_{12}(t) = a_{21}(t) + a_{22}(t) \equiv 0, \quad t \geqslant t_0. \tag{3.32}$$

Then taking into account (3.30) and (3.31) we have $\widetilde{\chi}_1(t) = \widetilde{\chi}_2(t) = -c_{11}(t) - 2\operatorname{Re} c_{12}(t) - c_{22}(t)$, $t \geqslant t_0$. Therefore, by Corollary 3.1, (3.29), and (3.32), the system (1.1) is oscillatory provided the scalar equation

$$\varphi_1''(t) - (c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t))\varphi_1(t) = 0, \quad t \geqslant t_0,$$

is oscillatory.

Assume now:

$$\begin{aligned} a_{11}(t) + a_{12}(t) = a_{21}(t) + a_{22}(t) &= \frac{\alpha}{t}, \\ c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t) &= \frac{\alpha - \alpha^2}{t^2}, \end{aligned} \tag{3.33}$$

$0 \leqslant \alpha \leqslant 1$, $t \geqslant 1$. Then taking into account (3.30) and (3.31), it is not difficult to verify that $\widetilde{\chi}_3(t) = \widetilde{\chi}_4(t) = (\alpha^2 - \alpha)/t^2 \leqslant 0$, $t \geqslant 1$. Hence, by Theorem 3.4, (3.29) and (3.33) the system (1.1) is non-oscillatory.

Now assume:

(α_1) $a_{11}(t) + a_{12}(t) = a_{21}(t) + a_{22}(t) > 0$, $t \geqslant t_0$;

(α_2) $a_{11}(t) + a_{12}(t)$ is increasing and continuously differentiable on $[t_0, \infty)$;

(α_3) $|(a_{11}(t) + a_{12}(t))' + c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t)|/(a_{11}(t) + a_{12}(t)) \leqslant \lambda = \text{const.}$, $t \geqslant t_0$.

Then taking into account (3.30) and (3.31) it is not difficult to verify that $\widetilde{\chi}_3(t) \leqslant \lambda - (c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t))$, $\widetilde{\chi}_4(t) \leqslant \lambda - (c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t))$, $t \geqslant t_0$. Therefore by virtue of Theorem 3.4, (3.29), and conditions (α_1)–(α_3), the system (1.1) is non-oscillatory.

Remark 3.4. Under the restriction (3.29), $\det B(t) \equiv 0$, $t \geqslant t_0$, the results of works [1], [4], [12]–[14], [16], [18]–[20] are not applicable to the system (1.1) with (3.29).

References

[1] *K. I. Al-Dosary, H. K. Abdullah, D. Hussein*: Short note on oscillation of matrix Hamiltonian systems. Yokohama Math. J. *50* (2003), 23–30. zbl MR

[2] *G. J. Butler, L. H. Erbe, A. B. Mingarelli*: Riccati techniques and variational principles in oscillation theory for linear systems. Trans. Am. Math. Soc. *303* (1987), 263–282. zbl MR doi

[3] *R. Byers, B. J. Harris, M. K. Kwong*: Weighted means and oscillation conditions for second order matrix differential equations. J. Differ. Equations *61* (1986), 164–177. zbl MR doi

[4] *S. Chen, Z. Zheng*: Oscillation criteria of Yan type for linear Hamiltonian systems. Comput. Math. Appl. *46* (2003), 855–862. zbl MR doi

[5] *L. H. Erbe, Q. Kong, S. Ruan*: Kamenev type theorems for second order matrix differential systems. Proc. Am. Math. Soc. *117* (1993), 957–962. zbl MR doi

[6] *G. A. Grigoryan*: On two comparison tests for second-order linear ordinary differential equations. Differ. Equ. *47* (2011), 1237–1252; translation from Differ. Uravn. *47* (2011), 1225–1240. zbl MR doi

[7] *G. A. Grigoryan*: Two comparison criteria for scalar Riccati equations with applications. Russ. Math. *56* (2012), 17–30; translation from Izv. Vyssh. Uchebn. Zaved., Mat. *2012* (2012), 20–35. zbl MR doi

[8] *G. A. Grigoryan*: On the stability of systems of two first-order linear ordinary differential equations. Differ. Equ. *51* (2015), 283–292; translation from Differ. Uravn. *51* (2015), 283–292. zbl MR doi

[9] *G. A. Grigorian*: On one oscillatory criterion for the second order linear ordinary differential equations. Opusc. Math. *36* (2016), 589–601. zbl MR doi

[10] *G. A. Grigorian*: Oscillatory criteria for the systems of two first-order linear differential equations. Rocky Mt. J. Math. *47* (2017), 1497–1524. zbl MR doi

[11] *G. A. Grigorian*: Oscillatory and non oscillatory criteria for the systems of two linear first order two by two dimensional matrix ordinary differential equations. Arch. Math., Brno *54* (2018), 189–203. zbl MR doi

[12] *I. S. Kumary, S. Umamaheswaram*: Oscillation criteria for linear matrix Hamiltonian systems. J. Differ. Equations *165* (2000), 174–198. zbl MR doi

[13] *L. Li, F. Meng, Z. Zheng*: Oscillation results related to integral averaging technique for linear Hamiltonian systems. Dyn. Syst. Appl. *18* (2009), 725–736. zbl MR

[14] *F. Meng, A. B. Mingarelli*: Oscillation of linear Hamiltonian systems. Proc. Am. Math. Soc. *131* (2003), 897–904. zbl MR doi

[15] *A. B. Mingarelli*: On a conjecture for oscillation of second order ordinary differential systems. Proc. Am. Math. Soc. *82* (1981), 593–598. zbl MR doi

[16] *Y. G. Sun*: New oscillation criteria for linear matrix Hamiltonian systems. J. Math. Anal. Appl. *279* (2003), 651–658. zbl MR doi

[17] *Q. Wang*: Oscillation criteria for second-order matrix differential systems. Arch. Math. *76* (2001), 385–390. zbl MR doi

[18] *Q. Yang, R. Mathsen, S. Zhu*: Oscillation theorems for self-adjoint matrix Hamiltonian systems. J. Differ. Equations *190* (2003), 306–329. zbl MR doi

[19] *Z. Zheng*: Linear transformation and oscillation criteria for Hamiltonian systems. J. Math. Anal. Appl. *332* (2007), 236–245. zbl MR doi

[20] *Z. Zheng, S. Zhu*: Hartman type oscillation criteria for linear matrix Hamiltonian systems. Dyn. Syst. Appl. *17* (2008), 85–96. zbl MR

Author's address: *Gevorg A. Grigorian*, Institute of Mathematics National Academy of Science RA, 24/5 Marshal Baghramyan Avenue, Yerevan 0019, Armenia, `e-mail: mathphys2@instmath.sci.am`.