STARLIKE AND CONVEX FUNCTIONS OF COMPLEX ORDER INVOLVING GENERALIZED MULTIPLIER TRANSFORMATIONS

Adela Osman Mostafa, Mohamed Kamal Aouf, Mansoura,
Fatma Z. El-Emam, Talkha

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Abstract. We investigate the starlike, convex and close-to-convex functions of complex order involving generalized multiplier transformations by means of the Hadamard product.

Keywords: starlike; convex; close-to-convex; complex order; Hadamard product; generalized multiplier transformations

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in $\mathbb{U} = \{z \in \mathbb{C}\colon |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be *starlike of complex order* b ($b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$) if $z^{-1}f(z) \neq 0$ and

$$\operatorname{Re}\Big\{1 + \frac{1}{b}\Big(\frac{zf'(z)}{f(z)} - 1\Big)\Big\} > 0, \tag{1.2}$$

and is said to be *convex of complex order* b ($b \in \mathbb{C}^*$) if $f'(z) \neq 0$ ($z \in \mathbb{U}$) and

$$\operatorname{Re}\Big\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\Big\} > 0. \tag{1.3}$$

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We denote by $S_0^*(b)$ and $K_0(b)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of complex order b and the subclass of $\mathcal{A}$ consisting of functions which are convex of complex order b, respectively. Furthermore, let $S_1^*(b)$ and $K_1(b)$ denote the classes of functions $f \in \mathcal{A}$ satisfying

$$\Big|\frac{zf'(z)}{f(z)} - 1\Big| < |b|, \quad b \in \mathbb{C}^*, \tag{1.4}$$

and

$$\Big|\frac{zf''(z)}{f'(z)}\Big| < |b|, \quad b \in \mathbb{C}^*, \tag{1.5}$$

respectively.

We note that $S_1^*(b) \subset S_0^*(b)$ and $K_1(b) \subset K_0(b)$ (see [6]),

$$f \in K_0(b) \Leftrightarrow zf' \in S_0^*(b), \quad b \in \mathbb{C}^* \tag{1.6}$$

and

$$f \in K_1(b) \Leftrightarrow zf' \in S_1^*(b), \quad b \in \mathbb{C}^*. \tag{1.7}$$

The class $S_0^*(b)$ was introduced and studied by Nasr and Aouf (see [7] and [8]), the class $K_0(b)$ was introduced by Wiatrowski (see [13]) and the classes $S_1^*(b)$ and $K_1(b)$ were introduced by Choi (see [6]).

Remark 1.1. Putting $b = 1 - \alpha$, $0 \leqslant \alpha < 1$, we have the known class $S_0^*(1-\alpha) = S^*(\alpha)$ $(K_0(1-\alpha) = K(\alpha))$, where $S^*(\alpha)$ $(K(\alpha))$ denotes the usual class of starlike (convex) functions of order α (see [9]).

In [6], Choi introduced the class $C_0(b,d)$ of complex order b $(b \in \mathbb{C}^*)$ and complex type d $(d \in \mathbb{C}^*)$ defined as follows.

A function $f \in \mathcal{A}$ is said to be *in the class* $C_0(b,d)$ $(b, d \in \mathbb{C}^*)$ if there exists a function $h(z) \in S_0^*(d)$ $(d \in \mathbb{C}^*)$ satisfying the condition

$$\operatorname{Re}\Big\{1 + \frac{1}{b}\Big(\frac{zf'(z)}{h(z)} - 1\Big)\Big\} > 0, \quad z \in \mathbb{U}. \tag{1.8}$$

Remark 1.2. We note that $C_0(b,1) = C(b)$ is the class of close-to-convex functions of complex order b $(b \in \mathbb{C}^*)$ which was introduced by Al-Amiri and Fernando (see [1]), $C_0(1-\alpha, 1-\beta) = C(\alpha,\beta)$ $(0 \leqslant \alpha, \beta < 1)$ the class of close-to-convex functions of order α and type β studied by Aouf (see [3]), and $C_0(1,1) = C$ the class of close-to-convex functions.

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.9}$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.10}$$

Cătaş et al. (see [4]) motivated the multiplier transformation by the operator $I^n(\lambda, l)\colon \mathcal{A} \to \mathcal{A}$ ($n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$, $\lambda \geqslant 0$, $l \geqslant 0$) of the infinite series

$$I^n(\lambda, l) f(z) = z + \sum_{k=2}^{\infty} \Big(\frac{1 + l + \lambda(k-1)}{1+l}\Big)^n a_k z^k. \tag{1.11}$$

It follows from (1.11) that $I^0(\lambda, l) f(z) = f(z)$,

$$I^{n_1}(\lambda, l)(I^{n_2}(\lambda, l) f(z)) = I^{n_2}(\lambda, l)(I^{n_1}(\lambda, l) f(z))$$

for all integers n_1 and n_2.

For different values of l, n and λ, the operator $I^n(\lambda, l)$ generalizes many others, see cf. [2], [5], [11] and [12].

If f is given by (1.1), then we have

$$I^n(\lambda, l) f(z) = (\varphi^n_{\lambda,l} * f)(z), \tag{1.12}$$

where

$$\varphi^n_{\lambda,l}(z) = z + \sum_{k=2}^{\infty} \Big(\frac{1 + l + \lambda(k-1)}{1+l}\Big)^n z^k. \tag{1.13}$$

In this paper, we investigate the starlike, convex and close-to-convex functions of complex order involving generalized multiplier transformations by means of the Hadamard product.

To prove our main results, we need the following lemmas.

Lemma 1.1 ([10]). *Let $\phi(z)$ and $g(z)$ be analytic in $\mathbb{U}$ with $\phi(0) = g(0) = 0$, $\phi'(0) \neq 0$ and $g'(0) \neq 0$. Further, let for every σ ($|\sigma| = 1$) and ϱ ($|\varrho| = 1$)*

$$\phi(z) * \Big(\frac{1 + \varrho\sigma z}{1 - \sigma z}\Big) g(z) \neq 0, \quad z \in \mathbb{U}^* = \mathbb{U} \setminus \{0\}.$$

Then for each function $F(z)$ analytic in $\mathbb{U}$ and satisfying the inequality $\operatorname{Re}\{F(z)\} > 0$, $z \in \mathbb{U}$, we get

$$\operatorname{Re}\Big\{\frac{(\phi * Fg)(z)}{(\phi * g)(z)}\Big\} > 0, \quad z \in \mathbb{U}.$$

Lemma 1.2 ([10]). *If $\phi(z)$ is convex and $g(z)$ is starlike in $\mathbb{U}$ then for every function $F(z)$ analytic in the unit disc $\mathbb{U}$ and satisfying the inequality* $\operatorname{Re}\{F(z)\} > 0$, $z \in \mathbb{U}$, *we get*

$$\operatorname{Re}\Big\{\frac{(\phi * Fg)(z)}{(\phi * g)(z)}\Big\} > 0, \quad z \in \mathbb{U}.$$

2. Main results

We assume in the reminder of this paper that $b \in \mathbb{C}^*$, $n \in \mathbb{N}_0$, $\lambda \geqslant 0$, $l \geqslant 0$, $z \in \mathbb{U}^*$, $h(z) \in S_0^*(b)$ and$f(z)$ is defined by (1.1).

Theorem 2.1. *Let $f(z) \in S_0^*(b)$ and let*

$$\varphi^n_{\lambda,l}(z) * \Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big) bf(z) \neq 0. \tag{2.1}$$

Then

$$I^n(\lambda, l)f(z) \in S_0^*(b)$$

for every σ ($|\sigma| = 1$) and ϱ ($|\varrho| = 1$).

P r o o f. It is sufficient to show that for every σ ($|\sigma| = 1$) and ϱ ($|\varrho| = 1$),

$$\operatorname{Re}\Big\{1 + \frac{1}{b}\Big(\frac{z(I^n(\lambda,l)f(z))'}{I^n(\lambda,l)f(z)} - 1\Big)\Big\} > 0, \quad z \in \mathbb{U}. \tag{2.2}$$

Since

$$\begin{aligned} 1 + \frac{1}{b}\Big(\frac{z(I^n(\lambda,l)f(z))'}{I^n(\lambda,l)f(z)} - 1\Big) &= 1 + \frac{1}{b}\Big(\frac{I^n(\lambda,l)(zf'(z))}{I^n(\lambda,l)f(z)} - 1\Big) \\ &= \frac{\varphi^n_{\lambda,l}(z) * ((b-1)f(z) + zf'(z))}{\varphi^n_{\lambda,l}(z) * bf(z)}, \end{aligned} \tag{2.3}$$

putting $\phi(z) = \varphi^n_{\lambda,l}(z)$, $g(z) = bf(z)$ and

$$F(z) = 1 + \frac{1}{b}\Big(\frac{zf'(z)}{f(z)} - 1\Big)$$

in Lemma 1.1, we see that

$$\operatorname{Re}\Big\{1 + \frac{1}{b}\Big(\frac{z(I^n(\lambda,l)f(z))'}{I^n(\lambda,l)f(z)} - 1\Big)\Big\} > 0,$$

which completes the proof of Theorem 2.1. □

Putting $l = 0$ in Theorem 2.1, we get

Corollary 2.1. *Let* $f(z) \in S_0^*(b)$ *and*

$$D_\lambda^n \Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big) bf(z) \neq 0.$$

Then

$$D_\lambda^n f(z) \in S_0^*(b)$$

for every σ $(|\sigma| = 1)$ *and* ϱ $(|\varrho| = 1)$.

Putting $l = 0$ and $\lambda = 1$ in Theorem 2.1, we get

Corollary 2.2. *Let* $f(z) \in S_0^*(b)$ *and*

$$D^n \Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big) bf(z) \neq 0.$$

Then

$$D^n f(z) \in S_0^*(b)$$

for every σ $(|\sigma| = 1)$ *and* ϱ $(|\varrho| = 1)$.

Putting $\lambda = 1$ in Theorem 2.1, we get

Corollary 2.3. *Let* $f(z) \in S_0^*(b)$ *and*

$$I_l^n \Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big) bf(z) \neq 0.$$

Then

$$I_l^n f(z) \in S_0^*(b)$$

for every σ $(|\sigma| = 1)$ *and* ϱ $(|\varrho| = 1)$.

Putting $l = \lambda = 1$ in Theorem 2.1, we get

Corollary 2.4. *Let* $f(z) \in S_0^*(b)$ *and*

$$I_n \Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big) bf(z) \neq 0.$$

Then

$$I_n f(z) \in S_0^*(b)$$

for every σ $(|\sigma| = 1)$ *and* ϱ $(|\varrho| = 1)$.

Corollary 2.5. *Let $\varphi^n_{\lambda,l}(z)$ be convex and let $f(z) \in S^*_1(b)$, $|b| < 1$, where $\varphi^n_{\lambda,l}(z)$ is given by* (1.13). *Then $I^n(\lambda, l)f(z) \in S^*_0(b)$.*

P r o o f. From the hypothesis, we have

$$f(z) \in S^*_1(b) \subset S^*(0) = S^*, \quad |b| < 1,$$

where S^* denotes the class of all functions in $\mathcal{A}$ which are starlike (with respect to the origin) in $\mathbb{U}$. By applying Lemma 1.2 and in view of Theorem 2.1, we have the desired result immediately. □

Theorem 2.2. *Let $f(z) \in K_0(b)$ and*

$$I^n(\lambda, l)\Big(\frac{1 + \varrho\sigma z}{1 - \sigma z}\Big)bzf'(z) \neq 0.$$

Then

$$I^n(\lambda, l)f(z) \in K_0(b)$$

for every σ ($|\sigma| = 1$) and ϱ ($|\varrho| = 1$).

P r o o f. Applying (1.6) and Theorem 2.1, we observe that

$$f(z) \in K_0(b) \Leftrightarrow zf'(z) \in S^*_0(b) \Rightarrow I^n(\lambda, l)zf'(z) \in S^*_0(b) \Rightarrow z(I^n(\lambda, l)f(z))' \in S^*_0(b)$$
$$\Leftrightarrow I^n(\lambda, l)f(z) \in K_0(b),$$

which evidently proves Theorem 2.2. □

Taking $l = 0$ in Theorem 2.2, we get

Corollary 2.6. *Let $f(z) \in K_0(b)$ and*

$$D^n_\lambda\Big(\frac{1 + \varrho\sigma z}{1 - \sigma z}\Big)bzf'(z) \neq 0.$$

Then

$$D^n_\lambda f(z) \in K_0(b)$$

for every σ ($|\sigma| = 1$) and ϱ ($|\varrho| = 1$).

Taking $l = 0$ and $\lambda = 1$ in Theorem 2.2, we get

Corollary 2.7. *Let $f(z) \in K_0(b)$ and*

$$D^n\Big(\frac{1 + \varrho\sigma z}{1 - \sigma z}\Big)bzf'(z) \neq 0.$$

Then

$$D^n f(z) \in K_0(b)$$

for every σ ($|\sigma| = 1$) and ϱ ($|\varrho| = 1$).

Taking $\lambda = 1$ in Theorem 2.2, we get

Corollary 2.8. *Let $f(z) \in K_0(b)$ and*

$$I_l^n\Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big)bzf'(z) \neq 0.$$

Then

$$I_l^n f(z) \in K_0(b)$$

for every σ ($|\sigma| = 1$) and ϱ ($|\varrho| = 1$).

Taking $l = \lambda = 1$ in Theorem 2.2, we get

Corollary 2.9. *Let $f(z) \in K_0(b)$ and*

$$I_n\Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big)bzf'(z) \neq 0.$$

Then

$$I_n f(z) \in K_0(b)$$

for every σ ($|\sigma| = 1$) and ϱ ($|\varrho| = 1$).

Corollary 2.10. *Let $\varphi^n_{\lambda,l}(z)$ be convex and let $f(z) \in K_1(b)$, $|b| < 1$, where $\varphi^n_{\lambda,l}(z)$ is given by (1.13). Then $I^n(\lambda, l)f(z) \in K_0(b)$.*

Theorem 2.3. *Let $f(z) \in C_0(b, b)$ and*

$$\varphi^n_{\lambda,l}(z) * \Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big)bh(z) \neq 0.$$

Then

$$I^n(\lambda, l)f(z) \in C_0(b, b)$$

for every σ ($|\sigma| = 1$) and ϱ ($|\varrho| = 1$).

Proof. By Theorem 2.1, we have that if $h(z) \in S_0^*(b)$, then $I^n(\lambda, l)h(z) \in S_0^*(b)$. It is sufficient to show that

$$\operatorname{Re}\Big\{1 + \frac{1}{b}\Big(\frac{z(I^n(\lambda, l)f(z))'}{I^n(\lambda, l)h(z)} - 1\Big)\Big\} > 0, \quad z \in \mathbb{U}.$$

Since

$$1 + \frac{1}{b}\Big(\frac{z(I^n(\lambda, l)f(z))'}{I^n(\lambda, l)h(z)} - 1\Big) = 1 + \frac{1}{b}\Big(\frac{I^n(\lambda, l)(zf'(z))}{I^n(\lambda, l)h(z)} - 1\Big)$$
$$= \frac{\varphi^n_{\lambda,l}(z) * ((b-1)h(z) + zf'(z))}{\varphi^n_{\lambda,l}(z) * bh(z)},$$

putting $\phi(z) = \varphi^n_{\lambda,l}(z)$, $g(z) = bh(z)$ and

$$F(z) = 1 + \frac{1}{b}\Big(\frac{zf'(z)}{h(z)} - 1\Big)$$

in Lemma 1.1, we see that

$$\mathrm{Re}\Big\{1 + \frac{1}{b}\Big(\frac{z(I^n(\lambda,l)f(z))'}{I^n(\lambda,l)h(z)} - 1\Big)\Big\} > 0,$$

which completes the proof of Theorem 2.3. □

Taking $l = 0$ in Theorem 2.3, we get

Corollary 2.11. *Let* $f(z) \in C_0(b,b)$ *and*

$$D^n_\lambda\Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big)bh(z) \neq 0.$$

Then

$$D^n_\lambda f(z) \in C_0(b,b)$$

for every σ $(|\sigma| = 1)$ *and* ϱ $(|\varrho| = 1)$.

Taking $l = 0$ and $\lambda = 1$ in Theorem 2.3, we get

Corollary 2.12. *Let* $f(z) \in C_0(b,b)$ *and*

$$D^n\Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big)bh(z) \neq 0.$$

Then

$$D^n f(z) \in C_0(b,b)$$

for every σ $(|\sigma| = 1)$ *and* ϱ $(|\varrho| = 1)$.

Taking $\lambda = 1$ in Theorem 2.3, we get

Corollary 2.13. *Let* $f(z) \in C_0(b,b)$ *and*

$$I^n_l\Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big)bh(z) \neq 0.$$

Then

$$I^n_l f(z) \in C_0(b,b)$$

for every σ $(|\sigma| = 1)$ *and* ϱ $(|\varrho| = 1)$.

Taking $l = \lambda = 1$ in Theorem 2.3, we get

Corollary 2.14. *Let $f(z) \in C_0(b,b)$ and*

$$I_n\Big(\frac{1+\varrho\sigma z}{1-\sigma z}\Big) bh(z) \neq 0.$$

Then

$$I_n f(z) \in C_0(b,b)$$

for every σ ($|\sigma| = 1$) and ϱ ($|\varrho| = 1$).

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Authors' addresses: *Adela Osman Mostafa*, *Mohamed Kamal Aouf*, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, e-mail: adelaeg254@yahoo.com, mkaouf127@yahoo.com; *Fatma Z. El-Emam*, Delta Higher Institute for Engineering & Technology, Talkha Sherbeen St, Talkha City, Talkha, Dakahlia Governorate, Egypt, e-mail: fatma_elemam@yahoo.com.