FINITE AND INFINITE ORDER OF GROWTH OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS NEAR A SINGULAR POINT

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Abstract. In this paper, we investigate the growth of solutions of a certain class of linear differential equation where the coefficients are analytic functions in the closed complex plane except at a finite singular point. For that, we will use the value distribution theory of meromorphic functions developed by Rolf Nevanlinna with adapted definitions.

Keywords: linear differential equation; growth of solution; finite singular point

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1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of a meromorphic function on the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C}\colon\ |z| < 1\}$ (see [7], [12], [17]). The importance of this theory has inspired many authors to find modifications and generalizations to different domains. Extensions of Nevanlinna theory to annuli have been made by [1], [8], [10], [11], [14]. In [4], Hamouda studied the growth of solutions of linear differential equations with analytic coefficients in the unit disc based on the behavior of the coefficients on a neighborhood of a point on the boundary of the unit disc. Recently in [2], [6], Fettouch and Hamouda investigated the growth of solutions of certain linear differential equations near a finite singular point. In this paper, we continue this investigation near a finite singular point to study other types of linear differential equations. First, we recall

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the appropriate definitions. Set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and suppose that $f(z)$ is meromorphic in $\overline{\mathbb{C}} \setminus \{z_0\}$ where $z_0 \in \mathbb{C}$. Define the counting function near z_0 by

$$(1.1) \qquad N_{z_0}(r, f) = -\int_{\infty}^{r} \frac{n(t, f) - n(\infty, f)}{t} \, \mathrm{d}t - n(\infty, f) \log r,$$

where $n(t, f)$ counts the number of poles of $f(z)$ in the region

$$\{z \in \mathbb{C}\colon \ t \leqslant |z - z_0|\} \cup \{\infty\}$$

each pole according to its multiplicity; and the proximity function by

$$(1.2) \qquad m_{z_0}(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(z_0 - r\mathrm{e}^{\mathrm{i}\varphi})| \, \mathrm{d}\varphi.$$

The characteristic function of f is defined in the usual manner by

$$(1.3) \qquad T_{z_0}(r, f) = m_{z_0}(r, f) + N_{z_0}(r, f).$$

In addition, the order of the meromorphic function $f(z)$ near z_0 is defined by

$$(1.4) \qquad \sigma_T(f, z_0) = \limsup_{r \to 0} \frac{\log^+ T_{z_0}(r, f)}{-\log r}.$$

For an analytic function $f(z)$ in $\overline{\mathbb{C}} \setminus \{z_0\}$, we have also the definition

$$(1.5) \qquad \sigma_M(f, z_0) = \limsup_{r \to 0} \frac{\log^+ \log^+ M_{z_0}(r, f)}{-\log r},$$

where $M_{z_0}(r, f) = \max\{|f(z)|\colon \ |z - z_0| = r\}$.

By the usual manner of the definition of the iterated order of a meromorphic function in the complex plane (see [9]), we define the n-iterated order near z_0 as follows:

$$(1.6) \qquad \sigma_{n,T}(f, z_0) = \limsup_{r \to 0} \frac{\log_n^+ T_{z_0}(r, f)}{-\log r},$$

and for an analytic function $f(z)$ in $\overline{\mathbb{C}} \setminus \{z_0\}$, we have also the definition

$$(1.7) \qquad \sigma_{n,M}(f, z_0) = \limsup_{r \to 0} \frac{\log_{n+1}^+ M_{z_0}(r, f)}{-\log r},$$

where $\log_{n+1}^+(x) = \ln^+ \log_n^+(x)$ ($n \geqslant 1$ is an integer) and $\ln^+(x) = \max(\ln x, 0)$.

R e m a r k 1.1. It is shown in [2] that if f is a non constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ and $g(w) = f(z_0 - 1/w)$, then $g(w)$ is meromorphic in $\mathbb{C}$ and we have

$$T(R, g) = T_{z_0}\Big(\frac{1}{R}, f\Big);$$

and so $\sigma(f, z_0) = \sigma(g)$. Also, if $f(z)$ is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, then, $g(w)$ is entire and thus $\sigma_T(f, z_0) = \sigma_M(f, z_0)$ and in general $\sigma_{n,T}(f, z_0) = \sigma_{n,M}(f, z_0) n \geqslant 1$. So, we can use the notation $\sigma_n(f, z_0)$ without any ambiguity.

We recall the following definitions.

Definition 1.1. The linear measure of a set $E \subset (0, \infty)$ is defined as $\int_0^\infty \chi_E(t)\,\mathrm{d}t$ and the logarithmic measure of E is defined by $\int_0^\infty \chi_E(t) t^{-1}\,\mathrm{d}t$ where $\chi_E(t)$ is the characteristic function of the set E.

In 2016, Fettouch and Hamouda proved the following result.

Theorem A ([2]). *Let $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfying $\max\{\sigma(A_j, z_0)\colon\ j \neq 0\} < \sigma(A_0, z_0)$. Then, every solution $f(z) \not\equiv 0$ of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \ldots + A_1(z) f' + A_0(z) f = 0$$

satisfies $\sigma(f, z_0) = \infty$ with $\sigma_2(f, z_0) = \sigma(A_0, z_0)$.

In the following two results, we will base our study on the domination of A_0 on only a curve tending to z_0. In this case, it may hapen that

$$\sigma(A_0, z_0) \leqslant \max\{\sigma(A_j, z_0)\colon\ j \neq 0\}.$$

Theorem 1.1. *Let $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$. If there exists a subset γ of a curve tending to z_0 such that the set $\gamma_0 = \{|z_0 - z|\colon\ z \in \gamma\} \cap (0, 1)$ is of infinite logarithmic measure, such that for $z \in \gamma$, $r = |z_0 - z| \in \gamma_0$ and for any fixed $\mu > 0$, we have*

$$\lim_{r \to 0} \frac{1}{|A_0(z)| r^{\mu}} \Bigg(\sum_{j=1}^{k-1} |A_j(z)| + 1\Bigg) = 0, \tag{1.8}$$

then every solution $f(z) \not\equiv 0$ of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \ldots + A_1(z) f + A_0(z) f = 0, \tag{1.9}$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ is of infinite order.

Corollary 1.1. *Let $P_j(z)$, $j = 1, 2, \ldots, k-1$ be polynomials and $P_0(z)$ be a transcendental entire function; let $A_j(z) = P_j(1/(z_0 - z))$; then every solution $f(z) \not\equiv 0$ of (1.9), that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, is of infinite order.*

Example 1.1. The differential equation

$$f''' + \frac{1}{z^3} f'' + \frac{1}{z^2} f' + \sum_{n=1}^{\infty} \frac{1}{n^{n^2} z^n} f = 0, \tag{1.10}$$

fulfills the assumptions of Theorem 1.1 as z tends to $z_0 = 0$ on the ray $\arg\theta = 0$. So, every solution $f(z) \not\equiv 0$ of (1.10) is of infinite order. We signal here that $\sigma(A_0, 0) = \sigma(A_1, 0) = \sigma(A_2, 0) = 0$.

Theorem 1.2. *Let $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$. If there exists a subset γ of a curve tending to z_0 such that the set $\gamma_0 = \{|z_0 - z|\colon z \in \gamma\} \cap (0,1)$ is of infinite logarithmic measure, such that for $z \in \gamma$ and $r = |z_0 - z| \in \gamma_0$, we have*

$$\lim_{r \to 0} \frac{1}{|A_0(z)|} \left(\sum_{j=1}^{k-1} |A_j(z)| + 1 \right) \exp_n \frac{\lambda}{r^{\mu}} = 0 \tag{1.11}$$

where $n \geqslant 1$ is an integer, $\lambda > 0$, $\mu > 0$ are real constants, then every solution $f(z) \not\equiv 0$ of (1.9), that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, satisfies $\sigma_n(f, z_0) = \infty$ and furthermore $\sigma_{n+1}(f, z_0) \geqslant \mu$.

Example 1.2. The differential equation

$$f''' + f'' \exp \frac{1}{z} + f' \exp_2 \frac{1}{z^3} + f \exp_2 \frac{1}{z^2} = 0, \tag{1.12}$$

fulfills the assumptions of Theorem 1.2 as z tends to $z_0 = 0$ on the ray $\arg\theta = \frac{1}{5}\pi$. So, every solution $f(z) \not\equiv 0$ of (1.12) is of infinite order with $\sigma_3(f, 0) \geqslant 2$.

Now, we will investigate the case when A_s, $s \neq 0$ dominates the other coefficients in a sector. Let $I(\varepsilon) = (\theta_1 + \varepsilon, \theta_2 - \varepsilon) \subset [0, 2\pi)$ and $S(\varepsilon)$ denote the sector $\{z\colon \arg(z_0 - z) \in I(\varepsilon)\}$, $\varepsilon \geqslant 0$.

Theorem 1.3. *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfying that there exist real constants $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ such that for any $\theta \in (\theta_1, \theta_2)$ there exists a set $\Gamma_\theta = \{r = |z - z_0|\colon \arg(z - z_0) = \theta\} \subset (0,1)$ of infinite logarithmic measure, and for every fixed $\mu > 0$, we have*

$$\lim_{z \to z_0} \frac{1}{|A_s(z)| r^{\mu}} \left(\sum_{j=0,\, j \neq s}^{k-1} |A_j(z)| + 1 \right) = 0, \quad s \neq 0 \tag{1.13}$$

where $\arg(z_0 - z) = \theta \in I(0)$ *and* $|z_0 - z| = r \in \Gamma_\theta$. *Given* $\varepsilon > 0$ *small enough, if* $f \not\equiv 0$ *is a solution of* (1.9) *that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and of finite order* $\sigma(f, z_0) < \infty$, *then the following statements hold.*

(i) *There exist* $j \in \{0, \ldots, s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to z_0$ *in the sector* $S(\varepsilon)$. *More precisely, for every fixed* $\mu > 0$ *we have*

$$\lim_{z \to z_0} \frac{|f^{(j)}(z) - b_j|}{r^\mu} = 0 \tag{1.14}$$

with $z \in S(\varepsilon)$ *and* $|z_0 - z| = r \in \Gamma_\theta$.

(ii) *For each integer* $m \geqslant j+1$, $f^{(m)}(z) \to 0$ *as* $z \to z_0$ *in* $S(\varepsilon)$. *More precisely, for every fixed* $\mu > 0$ *we have*

$$\lim_{z \to z_0} \frac{|f^{(m)}(z)|}{r^\mu} = 0 \tag{1.15}$$

with $z \in S(\varepsilon)$ *and* $|z_0 - z| = r \in \Gamma_\theta$.

Example 1.3. The function $f(z) = \mathrm{e}^{1/z} - 1$ satisfies the differential equation

$$f''' + \mathrm{e}^{-1/z} f'' + \Big(\frac{2}{z} - \frac{5}{z^2} - \frac{6}{z^3} - \frac{1}{z^4}\Big) f' + \Big(\frac{2}{z^3} + \frac{1}{z^4}\Big) f = 0. \tag{1.16}$$

The differential equation (1.16) fulfills the assumptions of Theorem 1.3 in any sector $(\theta_1, \theta_2) \subset (\frac{1}{2}\pi, \frac{3}{2}\pi)$ with $z_0 = 0$. In this example, $A_2(z) = \mathrm{e}^{-1/z}$ is the dominating coefficient, while we have $j = 0$ and $b_j = -1$.

Theorem 1.4. *Let* $A_0(z), \ldots, A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying that there exist real constants* $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ *such that for any* $\theta \in (\theta_1, \theta_2)$ *there exists a set* $\Gamma_\theta = \{r = |z - z_0| \colon \arg(z - z_0) = \theta\} \subset (0, 1)$ *of infinite logarithmic measure such that we have*

$$\lim_{z \to z_0} \frac{1}{|A_s(z)|} \Big(\sum_{j=0,\, j \neq s}^{k-1} |A_j(z)| + 1 \Big) \exp \frac{\lambda}{r^\alpha} = 0, \quad s \neq 0 \tag{1.17}$$

where $\arg(z_0 - z) = \theta \in I(0)$ *and* $|z_0 - z| = r \in \Gamma_\theta$, $\lambda > 0$, $\alpha > 0$ *are real constant. Given* $\varepsilon > 0$ *small enough, if* $f \not\equiv 0$ *is a solution of* (1.9), *analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and of finite order* $\sigma(f, z_0) < \infty$, *then the following statements hold.*

(i) *There exists* $j \in \{0, \ldots, s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to z_0$ *in the sector* $S(\varepsilon)$. *More precisely, for* $\lambda > \lambda' > 0$ *we have*

$$|f^{(j)}(z) - b_j| < \exp\Big(-\frac{\lambda'}{r^\alpha}\Big)$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.

(ii) *For each integer* $m \geqslant j+1$, $f^{(m)}(z) \to 0$ *as* $z \to z_0$ *in* $S(\varepsilon)$. *More precisely, for* $\lambda' > 0$ *we have*

$$|f^{(m)}(z)| < \exp\Big(-\frac{\lambda'}{r^{\alpha}}\Big)$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.

Corollary 1.2. *Let* $A_0(z), \dots, A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying that there exist real constants* $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ *such that for any* $\theta \in (\theta_1, \theta_2)$ *there exists a set* $\Gamma_\theta = \{r = |z - z_0| \colon \arg(z - z_0) = \theta\} \subset (0,1)$ *of infinite logarithmic measure, we have*

$$|A_s(z)| \geqslant \exp\frac{\alpha}{r^{\mu}}, \quad s \neq 0,$$
$$|A_j(z)| \leqslant \exp\frac{\beta}{r^{\mu}}$$

where $\arg(z_0 - z) = \theta \in (\theta_1, \theta_2)$ *and* $|z_0 - z| = r \in \Gamma_\theta$, $\alpha > \beta \geqslant 0$, $\mu > 0$ *are real constant. Given* $\varepsilon > 0$ *small enough, if* $f \not\equiv 0$ *is a solution of* (1.9) *that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and of finite order* $\sigma(f, z_0) < \infty$, *then the following statements hold.*

(i) *There exists* $j \in \{0, \dots, s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to z_0$ *in the sector* $S(\varepsilon)$. *More precisely, for* $\alpha - \beta > \lambda' > 0$ *we have*

$$|f^{(j)}(z) - b_j| < \exp\Big(-\frac{\lambda'}{r^{\mu}}\Big) \tag{1.18}$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.

(ii) *For each integer* $m \geqslant j+1$, $f^{(m)}(z) \to 0$ *as* $z \to z_0$ *in* $S(\varepsilon)$. *More precisely, for* $\alpha - \beta > \lambda' > 0$ *we have*

$$|f^{(m)}(z)| < \exp\Big(-\frac{\lambda'}{r^{\mu}}\Big) \tag{1.19}$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.

Indeed, by taking $\alpha - \beta > \lambda > 0$, the condition (1.17) holds; and then the assertions (1.18)–(1.19) hold by taking $\lambda > \lambda' > 0$. We can see similar results of these theorems in the complex plane and in the unit disc in [3], [5], [13].

2. Preliminary lemmas

To prove these results we need the following lemmas.

Lemma 2.1 ([2]). *Let f be a non constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$; let $\alpha > 0$, $\varepsilon > 0$ be given real constants and $j \in \mathbb{N}$; then*

(i) *there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure and a constant $A > 0$ that depends on α and j such that for all $r = |z - z_0|$ satisfying $r \in (0,1) \setminus E_1$, we have*

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \leqslant A\Big(\frac{1}{r^2} T_{z_0}(\alpha r, f) \log T_{z_0}(\alpha r, f)\Big)^j; \tag{2.1}$$

(ii) *there exists a set $E_2 \subset [0, 2\pi)$ that has a linear measure zero and a constant $A > 0$ that depends on α and j such that for all $\theta \in [0, 2\pi) \setminus E_2$ there exists a constant $r_0 = r_0(\theta) > 0$ such that* (2.1) *holds for all z satisfying* $\arg(z - z_0) \in [0, 2\pi) \setminus E_2$ *and $r = |z - z_0| < r_0$.*

Lemma 2.2 ([2]). *Let f be a non constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $\sigma(f, z_0) < \infty$; let $\varepsilon > 0$ be a given constant. Then,*

(i) *there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z - z_0| \in (0,1) \setminus E_1$, we have*

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \leqslant \frac{1}{r^{k(\sigma+2+\varepsilon)}}, \quad k \in \mathbb{N}; \tag{2.2}$$

(ii) *there exists a set $E_2 \subset [0, 2\pi)$ that has a linear measure zero such that for all $\theta \in [0, 2\pi) \setminus E_2$ there exists a constant $r_0 = r_0(\theta) > 0$ such that for all z satisfying* $\arg(z - z_0) \in [0, 2\pi) \setminus E_2$ *and $r = |z - z_0| < r_0$, the inequality* (2.2) *holds.*

Lemma 2.3. *Let f be a non constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $\sigma_n(f, z_0) = \sigma_n < \infty$ $(n \geqslant 1)$ and let $\varepsilon > 0$ be a given constant. Then, there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z - z_0| \in (0,1) \setminus E_1$, we have*

(i) *if $n = 1$,* (2.2) *holds,*

(ii) *and if $n \geqslant 2$*

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \leqslant \Big(\exp_{n-1} \frac{1}{r^{\sigma_n+\varepsilon}}\Big)^k, \quad k \in \mathbb{N}. \tag{2.3}$$

P r o o f. By the definition

$$\sigma_n(f,z_0)=\limsup_{r\to 0}\frac{\log_n T_{z_0}(r,f)}{-\log r}=\sigma_n,$$

for given $\varepsilon'>0$ there exists r_0 such that for $0<r<r_0$, we have

$$\frac{\log_n T_{z_0}(r,f)}{-\log r}<\sigma_n+\varepsilon';$$

which implies

$$T_{z_0}(r,f)<\exp_{n-1}\frac{1}{r^{\sigma_n+\varepsilon'}}. \tag{2.4}$$

Combining (2.4) with Lemma 2.1, for $\alpha>0$, there exists a set $E_1\subset(0,1)$ that has finite logarithmic measure and a constant $A>0$ that depends only on α such that for all $r=|z-z_0|$ satisfying $r\notin(0,1)\setminus E_1$, we have

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big|\leqslant A\Big(\frac{1}{r^2}\exp_{n-1}\Big(\frac{\alpha}{r}\Big)^{\sigma_n+\varepsilon'}\exp_{n-2}\Big(\frac{\alpha}{r}\Big)^{\sigma_n+\varepsilon'}\Big)^k.$$

Then, for $\varepsilon>\varepsilon'>0$ and r near enough to 0, we have

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big|\leqslant\Big(\exp_{n-1}\frac{1}{r^{\sigma_n+\varepsilon}}\Big)^k.$$

□

Lemma 2.4. *Let $f(z)$ be a non constant meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$. Then*

$$\sigma(f^{(j)},z_0)=\sigma(f,z_0),\quad j\in\mathbb{N}.$$

P r o o f. It is sufficient to prove that $\sigma(f',z_0)=\sigma(f,z_0)$. By Remark 1.1, $g(w)=f(z_0-1/w)$ is meromorphic in $\mathbb{C}$ and $\sigma(g)=\sigma(f,z_0)$. It is well known that for a meromorphic function in $\mathbb{C}$ we have $\sigma(g')=\sigma(g)$, (see [16], [15]). We have $f'(z)=g'(w)/w^2$. Set $h(w)=g'(w)/w^2$. Obviously, we have $\sigma(h)=\sigma(g')$. On the other hand, by Remark 1.1, we have $\sigma(h)=\sigma(f',z_0)$. So, we conclude that $\sigma(f',z_0)=\sigma(f,z_0)$. □

Lemma 2.5. *Let f be a non constant meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg(z_0-z)=\theta$. Then there exists an infinite sequence of points $z_m=z_0-r_m\mathrm{e}^{\mathrm{i}\theta}$, $m=1,2,\dots$, where $r_m\to 0$, such that $f^{(k)}(z_m)\to\infty$ and*

$$\Big|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\Big|\leqslant M,$$

where $M>0$ and $j\in(0,1,\dots,k-1)$.

P r o o f. Let $M(r,\theta,f^{(k)}) = \max|f^{(k)}(z)|$ where $z\in[z_0-r_1\mathrm{e}^{\mathrm{i}\theta}, z_0-r\mathrm{e}^{\mathrm{i}\theta}]$. Clearly, we may construct a sequence of points $z_m = z_0 - r_m\mathrm{e}^{\mathrm{i}\theta}$, $m\geqslant 1$, $r_m\to 0$, such that $M(r,\theta,f^{(k)}) = |f^{(k)}(z_m)|\to\infty$. For each m, by $(k-j)$-fold iteration integration along the line segment $[z_1,z_m]$ we have

$$\begin{aligned} f^{(j)}(z_m) = f^{(j)}(z_1) &+ f^{(j+1)}(z_1)(z_m-z_1) \\ &+\ldots+\frac{1}{(k-j-1)}f^{(k-1)}(z_1)(z_m-z_1)^{k-j-1} \\ &+\int_{z_1}^{z_m}\ldots\int_{z_1}^{y} f^{(k)}(x)\,\mathrm{d}x\,\mathrm{d}y\ldots\,\mathrm{d}t; \end{aligned}$$

and by an elementary triangle inequality estimate we obtain

$$\begin{aligned} |f^{(j)}(z_m)| \leqslant |f^{(j)}(z_1)| &+ |f^{(j+1)}(z_1)||(z_m-z_1)| \\ &+\ldots+\frac{1}{(k-j-1)}|f^{(k-1)}(z_1)||(z_m-z_1)|^{k-j-1} \\ &+\frac{1}{(k-j)}|f^{(k)}(z_m)||(z_m-z_1)|^{k-j}. \end{aligned} \tag{2.5}$$

From (2.5) and taking account that when $m\to\infty$, $f^{(k)}(z_m)\to\infty$, $z_m\to z_0$, we obtain

$$\Bigl|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\Bigr| \leqslant M, \quad M>0.$$

□

Lemma 2.6. *Let f be an analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$. Let $a\geqslant\frac{1}{2}$ and*

$$G = \Bigl\{z\colon\ |\arg(z_0-z)| < \frac{\pi}{2a}\Bigr\}.$$

Suppose that $\limsup\limits_{z\to\varsigma}|f(z)|\leqslant M$ for all $\varsigma\in\partial G$, where M is a fixed constant. Suppose further that there exist constants K, $b<a$ such that

$$|f(z)|\leqslant K\exp\frac{1}{r^b}\quad \text{as } r\to 0,$$

where $r=|z_0-z|$ and $z\in G$. Then, $|f(z)|\leqslant M$ for all $z\in G$.

P r o o f. The change of variable $w = 1/(z_0-z)$ maps G onto $H=\{w\colon\ |\arg(w)|<\pi/(2a)\}$ and the function $g(w)=f(z)$ is an entire function on $w\in\mathbb{C}$ and we have $|\arg(z_0-z)| = \pi/(2a) \Leftrightarrow |\arg(w)| = \pi/(2a)$ and $\limsup\limits_{w\to\xi}|g(w)| = \limsup\limits_{z\to\varsigma}|f(z)|\leqslant M$ for all $\xi\in\partial H$. Further, we have

$$|g(w)| = |f(z)|\leqslant K\exp\frac{1}{r^b} = K\exp R^b\quad\text{as } R\to\infty,$$

where $R=|w|=1/r$. Then, by Phragmen-Lindelöf theorem we get $|g(w)|\leqslant M$ for all $w\in H$. Therefore, $|f(z)|\leqslant M$ for all $z\in G$. □

Lemma 2.7. *If f is analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ such that for any $\mu > 0$, we have*

$$|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})| \leqslant r^{\mu} \quad \text{as } r \to 0$$

then $\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t$ converges and for every $\alpha > 0$, we have

$$\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t \leqslant r^{\alpha} \quad \text{as } r \to 0.$$

Proof. It is easy to show that $\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t$ converges; and we have

$$\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t \leqslant \int_0^r t^{\mu}\,\mathrm{d}t = \frac{r^{\mu+1}}{\mu+1}.$$

Let $\alpha > 0$. By taking $\mu + 1 > \alpha$, we have

$$\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t \leqslant \frac{r^{\mu+1}}{\mu+1} \leqslant r^{\alpha} \quad \text{as } r \to 0.$$

□

Lemma 2.8. *Let f be an analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$. The two following assertions are equivalent:*

(i) *for any $\mu > 0$, $|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})| \leqslant r^{\mu}$ as $r \to 0$,*
(ii) *for any $\alpha > 0$, $\lim\limits_{r\to 0} |f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})|/r^{\alpha} = 0$.*

Proof. (ii) $\Rightarrow$ (i). Suppose that for any $\alpha > 0$, $\lim\limits_{r\to 0} |f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})|/r^{\alpha} = 0$. For any $\alpha > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 < r < \delta$ we have $|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})| \leqslant \varepsilon r^{\alpha}$. By taking $\varepsilon = 1$ we get the assertion (i).

(i) $\Rightarrow$ (ii). Suppose that for any $\mu > 0$, $|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})| \leqslant r^{\mu}$ as $r \to 0$. Let $\alpha > 0$. We have

$$\frac{|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})|}{r^{\alpha}} \leqslant \frac{r^{\mu}}{r^{\alpha}}.$$

By taking $\mu > \alpha$, we obtain

$$\lim_{r\to 0} \frac{|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})|}{r^{\alpha}} = 0.$$

□

Lemma 2.9. *If f is analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ such that*

$$|f(z_0-te^{i\theta})|\leqslant \exp\Big(-\frac{\lambda}{t^{\alpha}}\Big),$$

where $\alpha>0$, $\lambda>0$, then $\int_0^r|f(z_0-te^{i\theta})|\,dt$ converges and we have

$$\int_0^r|f(z_0-te^{i\theta})|\,dt\leqslant \exp\Big(-\frac{\lambda}{r^{\alpha}}\Big)\quad \text{as } r\to 0.$$

Proof. It is easy to show that $\int_0^r|f(z_0-te^{i\theta})|\,dt$ converges; and we have

$$\begin{aligned}\int_0^r|f(z_0-te^{i\theta})|\,dt&\leqslant\int_0^r\exp\Big(-\frac{\lambda}{r^{\alpha}}\Big)\,dt\leqslant\exp\Big(-\frac{\lambda}{r^{\alpha}}\Big)\int_0^r dt\\ &\leqslant r\exp\Big(-\frac{\lambda}{r^{\alpha}}\Big)\leqslant\exp\Big(-\frac{\lambda}{r^{\alpha}}\Big)\quad\text{as } r\to 0.\end{aligned}$$

□

3. Proof of theorems

Proof of Theorem 1.1. Suppose that $f\not\equiv 0$ is a solution of (1.9) of finite order $\sigma(f,z_0)=\sigma<\infty$. By Lemma 2.3, for any given $\varepsilon>0$ there exists a set $E\subset(0,1)$ that has finite logarithmic measure such that for all $r=|z_0-z|\in(0,1)\setminus E$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big|\leqslant\frac{1}{r^{j(\sigma+2+\varepsilon)}},\quad j=1,\dots,k. \tag{3.1}$$

From (1.9) we can write

$$1\leqslant\frac{1}{|A_0(z)|}\Big|\frac{f^{(k)}}{f}\Big|+\frac{|A_{k-1}(z)|}{|A_0(z)|}\Big|\frac{f^{(k-1)}}{f}\Big|+\dots+\frac{|A_1(z)|}{|A_0(z)|}\Big|\frac{f'}{f}\Big|. \tag{3.2}$$

By the assumption (1.8), for $r\in F$ and any fixed $\mu>0$, we have

$$\lim_{r\to 0}\frac{|A_j(z)|}{|A_0(z)|r^{\mu}}=0,\quad j=1,\dots,k \tag{3.3}$$

and

$$\lim_{r\to 0}\frac{1}{|A_0(z)|r^{\mu}}=0. \tag{3.4}$$

Using (3.1), (3.3) and (3.4) in (3.2), a contradiction follows as $r\to 0$ with $r=|z_0-z|\in F\setminus E$. □

P r o o f of Theorem 1.2. Suppose that $f \not\equiv 0$ is a solution of (1.9) with $\sigma_n(f,z_0) = \sigma_n < \infty$, $n \geqslant 1$. If $n = 1$ we have (3.1) and if $n \geqslant 2$, by Lemma 2.3, for any given $\varepsilon > 0$ there exists a set $E \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z_0 - z| \in (0,1) \setminus E$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant \left(\exp_{n-1}\frac{1}{r^{\sigma_n+\varepsilon}}\right)^j, \quad j = 1,\dots,k. \tag{3.5}$$

By the assumption (1.11), for $r \in F$, we have

$$\lim_{r\to 0}\frac{|A_j(z)|}{|A_0(z)|}\exp_n\frac{\lambda}{r^\mu} = 0, \quad j = 1,\dots,k \tag{3.6}$$

and

$$\lim_{r\to 0}\frac{1}{|A_0(z)|}\exp_n\frac{\lambda}{r^\mu} = 0. \tag{3.7}$$

Using (3.1) or (3.5), (3.6) and (3.7) in (3.2), a contradiction follows as $r \to 0$ on γ with $r = |z_0 - z| \in F \setminus E$. So, $\sigma_n(f,z_0) = \infty$ for $n \geqslant 1$. Now, by Lemma 2.1, and since $\sigma_n(f,z_0) = \infty$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant A\left(\frac{1}{r}T_{z_0}(\alpha r, f)\right)^{2k}, \quad j = 1,\dots,k. \tag{3.8}$$

By the assumption (1.11), for $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, we have

$$\frac{|A_j(z)|}{|A_0(z)|} \leqslant \frac{\varepsilon_1}{\exp_n(\lambda/r^\mu)}, \quad j = 1,\dots,k \tag{3.9}$$

and

$$\frac{1}{|A_0(z)|} \leqslant \frac{\varepsilon_2}{\exp_n(\lambda/r^\mu)} \tag{3.10}$$

as $r \to 0$ on γ with $r = |z_0 - z| \in F$. Using (3.8)–(3.10) in (3.2), we obtain, for $r = |z_0 - z| \in F \setminus E$,

$$1 \leqslant \frac{M}{\exp_n(\lambda/r^\mu)}\left(\frac{1}{r}T_{z_0}(\alpha r, f)\right)^{2k}, \tag{3.11}$$

where $M > 0$ is a real constant. Set $R = \alpha r$. We signal here that E is of finite logarithmic measure if and only if αE is of finite logarithmic measure. So, from (3.11), we get

$$\exp_n\frac{\lambda\alpha^\mu}{R^\mu} \leqslant M\left(\frac{\alpha}{R}T_{z_0}(r,f)\right)^{2k}, \quad R \in F \setminus E. \tag{3.12}$$

From (3.12) we obtain

$$\sigma_{n+1}(f,z_0) = \limsup_{r\to 0}\frac{\log^+_{n+1} T_{z_0}(r,f)}{-\log R} \geqslant \mu.$$

□

P r o o f of Theorem 1.3. First, we have to prove that $f(z)$ is bounded in $S(\varepsilon)$, for $\varepsilon > 0$ small enough and for that we prove that $f^{(s)}(z)$ is also bounded in $S(\varepsilon)$. From Lemma 2.4 and Lemma 2.2, it follows that there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that for all $j \in \{s+1, \ldots, k\}$

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leqslant \frac{1}{r^{(j-s)(\sigma+2+\varepsilon)}}, \tag{3.13}$$

where $\arg(z_0 - z) \in I(0) \setminus E$ and $r = |z_0 - z| \in \Gamma_\theta$. If we suppose that $f^{(s)}(z)$ is unbounded on some ray $\arg(z_0 - z) = \varphi \in I(0) \setminus E$, then by Lemma 2.5 there exists an infinite sequence of points $z_m = z_0 - r_m e^{i\varphi}$, $m = 1, 2, \ldots$, with $r_m \to 0$, such that $f^{(k)}(z_m) \to \infty$ and

$$\left|\frac{f^{(q)}(z_m)}{f^{(s)}(z_m)}\right| \leqslant M_1, \tag{3.14}$$

where $M_1 > 0$, $q \in \{0, 1, \ldots, s-1\}$ and m large enough. From (1.9) we can write

$$\begin{aligned} 1 \leqslant \frac{1}{|A_s(z)|}\left|\frac{f^{(k)}}{f^{(s)}}\right| + \frac{|A_{k-1}(z)|}{|A_s(z)|}\left|\frac{f^{(k-1)}}{f^{(s)}}\right| + \ldots + \frac{|A_{s+1}(z)|}{|A_s(z)|}\left|\frac{f^{(s+1)}}{f^{(s)}}\right| \\ + \frac{|A_{s-1}(z)|}{|A_s(z)|}\left|\frac{f^{(s-1)}}{f^{(s)}}\right| + \ldots + \frac{|A_0(z)|}{|A_s(z)|}\left|\frac{f}{f^{(s)}}\right|. \end{aligned} \tag{3.15}$$

Combining now (1.13), (3.13)–(3.15) and letting $m \to \infty$ we obtain a contradiction. Therefore, $f^{(s)}(z)$ remains bounded on all rays $\arg(z_0 - z) = \varphi \in I(0) \setminus E$. By Lemma 2.6, we conclude that $f^{(s)}(z)$ is bounded, say $|f^{(s)}(z)| \leqslant M_2$, in the whole sector $S(\frac{1}{2}\varepsilon)$ for $\varepsilon > 0$ small enough.

By integrating s times along the line segment $[z_1, z]$ in $S(\frac{1}{2}\varepsilon)$, we have

$$\begin{aligned} f(z) = f(z_1) + f'(z_1)(z - z_1) + \ldots + \frac{1}{(s-1)!} f^{(s-1)}(z_1)(z - z_1)^{s-1} \\ + \int_{z_1}^{z} \ldots \int_{z_1}^{z} f^{(s)}(t)\,dt \ldots dt; \end{aligned}$$

and by an elementary triangle inequality estimate, we obtain

$$|f(z)| \leqslant |f(z_1)| + |f'(z_1)||z - z_1| + \ldots + \frac{1}{(s-1)!}|f^{(s-1)}(z_1)||z - z_1|^{s-1} + \frac{1}{(s)!}M|z - z_1|^s$$

and therefore, as $z \to z_0$, we get

$$|f(z)| \leqslant M_3 \tag{3.16}$$

for a certain constant $M_3 > 0$. Now, we begin to prove (1.15) for $m = s$. Using (1.9), we can write

$$|f^{(s)}(z)| \leqslant |f|\Big(\frac{1}{|A_s(z)|}\Big|\frac{f^{(k)}}{f}\Big| + \frac{|A_{k-1}(z)|}{|A_s(z)|}\Big|\frac{f^{(k-1)}}{f}\Big| + \ldots + \frac{|A_{s+1}(z)|}{|A_s(z)|}\Big|\frac{f^{(s+1)}}{f}\Big| + \frac{|A_{s-1}(z)|}{|A_s(z)|}\Big|\frac{f^{(s-1)}}{f}\Big| + \ldots + \frac{|A_1(z)|}{|A_s(z)|}\Big|\frac{f'}{f}\Big| + \frac{|A_0(z)|}{|A_s(z)|}\Big). \tag{3.17}$$

By the assumption (1.13), for any $\mu > 0$, for every $j \in \{0, 1, \ldots, s-1, s+1, \ldots, k-1\}$ and for $\varepsilon > 0$, there exists δ such that for $|z_0 - z| < \delta$ we have

$$\frac{|A_j(z)|}{|A_s(z)|} \leqslant \varepsilon |z_0 - z|^{\mu}, \tag{3.18}$$

$$\frac{1}{|A_s(z)|} \leqslant \varepsilon |z_0 - z|^{\mu}, \tag{3.19}$$

where $\arg(z_0 - z) = \theta \in I(0)$ and $|z_0 - z| = r \in \Gamma_\theta$. Substituting (3.13), (3.16), (3.18) and (3.19) into (3.17), we obtain that for any $\mu > 0$, we have

$$|f^{(s)}(z)| \leqslant M_4 \frac{|z_0 - z|^{\mu}}{r^{k(\sigma+2+\varepsilon)}} \quad \text{as } r \to 0.$$

We conclude that for any fixed $\alpha > 0$

$$\lim_{z \to z_0} \frac{|f^{(s)}(z)|}{r^{\alpha}} = 0, \tag{3.20}$$

with $r = |z_0 - z| \in \Gamma_\theta$ and $\arg(z_0 - z) = \varphi \in I(\frac{1}{2}\varepsilon) \setminus E$.

Proof of equation (1.15) for $m > s$. Consider $z = z_0 - re^{i\theta} \in S(\varepsilon)$ and $C(z)$ the circle centered at z of radius ϱ small enough such that $C(z)$ is contained in $S(\frac{1}{2}\varepsilon)$, we may take $\varrho = r \sin(\frac{1}{2}\varepsilon)$. By the Cauchy formula applied to the function $f^{(s)}(z)$ we have

$$f^{(m)}(z) = \frac{(m-s)!}{2\pi} \int_{C(z)} \frac{f^{(s)}(\zeta)}{(z - \zeta)^{m-s+1}} \, d\zeta, \tag{3.21}$$

and using (3.20), we get

$$|f^{(m)}(z)| \leqslant \frac{(m-s)!}{2\pi} \int_0^{2\pi} \frac{|z_0 - z|^{\mu}}{\varrho^{m-s+1}} \varrho \, d\theta \leqslant \frac{(m-s)!}{\sin^{m-s}(\frac{1}{2}\varepsilon)} \frac{|z_0 - z|^{\mu}}{r^{m-s}}.$$

We conclude that, for any fixed $\alpha > 0$ and $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$, we have

$$\lim_{z \to z_0} \frac{|f^{(m)}(z)|}{|z_0 - z|^{\alpha}} = 0.$$

Until now, we have proved the second assertion for $m \geqslant s$. We start to prove the first assertion for $j = s-1$. Set

$$a_s = \int_0^\infty f^{(s)}(z_0 - te^{i\theta})e^{i\theta}\, dt.$$

By (3.20), it is easy to see that $\int_0^\infty f^{(s)}(z_0 - te^{i\theta})e^{i\theta}\, dt$ converges. Moreover, a_s is independent of θ, because by (3.20), the integral of $f^{(s)}(\zeta)$ over the arc $z_0 - re^{i\theta}$, $\theta \in (\varphi, \varphi) \subset I(\frac{1}{2}\varepsilon)$, we get

$$\left| \int_\varphi^\varphi f^{(s)}(z_0 - re^{i\theta}) ire^{i\theta}\, d\theta \right| \leqslant Mr^{\alpha+1}|\varphi - \varphi| \to 0, \quad r \to 0,\ M > 0.$$

Define now $b_{s-1} = f^{(s-1)}(\infty) + a_s$, and suppose that $b_{s-1} \neq 0$. Let $z = z_0 - re^{i\theta}$ be an arbitrary point in $S(\varepsilon)$. Then, since

$$f^{(s-1)}(z) - b_{s-1} = \int_\infty^z f^{(s)}(\zeta)\, d\zeta - \int_0^\infty f^{(s)}(z_0 - te^{i\theta})e^{i\theta}\, dt,$$

we may apply (3.20) and Lemma 2.7, and we get

$$\begin{aligned} |f^{(s-1)}(z) - b_{s-1}| &= \left| \int_\infty^z f^{(s)}(\zeta)\, d\zeta - \int_0^\infty f^{(s)}(z_0 - te^{i\theta})e^{i\theta}\, dt \right| \\ &= \left| \int_r^\infty f^{(s)}(z_0 - te^{i\theta})e^{i\theta}\, dt + \int_\infty^0 f^{(s)}(z_0 - te^{i\theta})e^{i\theta}\, dt \right| \\ &= \left| \int_r^0 f^{(s)}(z_0 - te^{i\theta})e^{i\theta}\, dt \right| \\ &\leqslant \int_0^r |f^{(s)}(z_0 - te^{i\theta})|\, dt \leqslant r^\mu \quad \text{as } r \to 0 \end{aligned} \tag{3.22}$$

for any $\mu > 0$ and $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$. By Lemma 2.8, we have completed the proof in the case $b_{s-1} \neq 0$. If $b_{s-1} = 0$, we define $a_{s-1} = \int_0^\infty f^{(s-1)}(z_0 - te^{i\theta})e^{i\theta}\, dt$ and $b_{s-2} = f^{(s-2)}(\infty) + a_{s-1}$ and by applying Lemma 2.7 with (3.22) we obtain that, for every fixed $\mu > 0$,

$$|f^{(s-2)}(z) - b_{s-2}| \leqslant r^\mu \quad \text{as } r \to 0$$

for $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$. By the same method, if $b_{s-1} = b_{s-2} = \ldots = b_{j+1} = 0$ and $b_j \neq 0$, $j \in \{0, \ldots, s-1\}$, then for any fixed $\mu > 0$

$$|f^{(j)}(z) - b_j| \leqslant r^\mu \quad \text{as } r \to 0,$$

and

$$|f^{(m)}(z)| \leqslant r^\mu \quad \text{as } r \to 0 \text{ for all } m \geqslant j+1 \tag{3.23}$$

for $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$. Now it remains to show that the case $b_{s-1} = b_{s-2} = \ldots = b_0 = 0$ is not possible. In this case, we have, for any fixed $\mu > 0$

$$|f^{(m)}(z)| \leqslant r^\mu \quad \text{as } r \to 0 \tag{3.24}$$

for $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$, for every $m \geqslant 0$ and any $\mu > 0$, there exists $r_0(\mu, m) > 0$ such that if $|z_0 - z| = r < r_0$ then $|f^{(m)}(z)| \leqslant |z_0 - z|^\mu$. Now we take $z \in S(\varepsilon)$ such that $r = |z_0 - z| < r_1 = \min\limits_{m=0,\ldots,s} r_0(\mu, m)$; we remark here that if z is fixed then (3.24) is valid for only some $\mu > 0$ and not for all $\mu > 0$. From (1.9) we can write

$$\begin{aligned}\frac{|f^{(s)}(z)|}{|f(z)|} \leqslant \frac{1}{|A_s(z)|}\Big|\frac{f^{(k)}}{f}\Big| + \frac{|A_{k-1}(z)|}{|A_s(z)|}\Big|\frac{f^{(k-1)}}{f}\Big| + \ldots + \frac{|A_{s+1}(z)|}{|A_s(z)|}\Big|\frac{f^{(s+1)}}{f}\Big| \\ + \frac{|A_{s-1}(z)|}{|A_s(z)|}\Big|\frac{f^{(s-1)}}{f}\Big| + \ldots + \frac{|A_1(z)|}{|A_s(z)|}\Big|\frac{f'}{f}\Big| + \frac{|A_0(z)|}{|A_s(z)|},\end{aligned} \tag{3.25}$$

and by using (1.13) and Lemma 2.2 in (3.25), we obtain

$$\frac{|f^{(s)}(z)|}{|f(z)|} \leqslant |z_0 - z|^\mu, \tag{3.26}$$

and by (3.24) for $m = 0$ in (3.25), we get

$$|f^{(s)}(z)| \leqslant |z_0 - z|^{2\mu} \tag{3.27}$$

for $|z_0 - z| < r_1$ and $\arg(z_0 - z) \in I(\varepsilon) \setminus E$, hence in $S(\varepsilon + \frac{1}{2}\varepsilon)$ by Lemma 2.6. Repeating the reasoning of (3.22)–(3.24) with (3.27), we obtain

$$|f(z)| \leqslant |z_0 - z|^{2\mu},$$

and by combining with (3.26), we get

$$|f^{(s)}(z)| \leqslant |z_0 - z|^{3\mu},$$

in $S(\varepsilon + \frac{1}{2}\varepsilon + \frac{1}{2^2}\varepsilon)$. Inductively, by the same reasoning, after $(T-1)$ steps, we obtain

$$|f^{(s)}(z)| \leqslant |z_0 - z|^{T\mu} \tag{3.28}$$

in

$$S\Big(\varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \ldots + \frac{\varepsilon}{2^{T-1}}\Big) = S\Big(2\varepsilon\Big(1 - \frac{1}{2^{T-1}}\Big)\Big)$$

with $|z_0 - z| < r_1$. Thus, we have proved, in this special case $b_{s-1} = b_{s-2} = \ldots = b_0 = 0$, that (3.28) is valid in $S(2\varepsilon)$ for all $T \in \mathbb{N}$, provided $|z_0 - z| < r_1$. Fix now a finite line segment $L \subset S(2\varepsilon)$ with $|z_0 - z| < \min(1, r_1)$. By taking $T \to \infty$ in (3.28), $f^{(s)}(z)$ vanishes identically on such a line segment. Therefore, f must be a polynomial. Since f is analytic in $\overline{\mathbb{C}} - \{z_0\}$, f has to be a constant. It is easy to see that the only constant solution of (1.9) is $f \equiv 0$, a contradiction. □

P r o o f of Theorem 1.4. We will use the same method of the proof of Theorem 1.3. The assumption (1.17) implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $r = |z_0 - z| < \delta$, we have

$$\frac{|A_j(z)|}{|A_s(z)|} \leqslant \varepsilon \exp\Big(-\frac{\lambda}{r^\alpha}\Big), \tag{3.29}$$

$$\frac{1}{|A_s(z)|} \leqslant \varepsilon \exp\Big(-\frac{\lambda}{r^\alpha}\Big). \tag{3.30}$$

By the same steps (3.13)–(3.15) with (3.29) and (3.30), we can prove that $f^{(s)}(z)$ is bounded in $S(\varepsilon)$, say

$$|f^{(s)}(z)| \leqslant M_1,$$

in the whole sector $S(\frac{1}{2}\varepsilon)$ for some $\varepsilon > 0$ small enough. As above, we can prove also that

$$|f(z)| \leqslant M_2.$$

By using (3.29)–(3.30) in (3.17), for $r = |z_0 - z| \in \Gamma_\theta$ and $\arg(z_0 - z) = \varphi \in I(\frac{1}{2}\varepsilon)\setminus E$, we get

$$|f^{(s)}(z)| \leqslant \exp\frac{-\lambda + \tau}{r^\alpha},$$

where $0 < \tau < \lambda$. For $m > s$, as above, by (3.21) we obtain

$$|f^{(m)}(z)| \leqslant \exp\frac{-\lambda + \tau}{r^\alpha}$$

for all $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta,\ 0 < \tau < \lambda$. Puting a_s and b_{s-1} as above and by Lemma 2.9, we get

$$|f^{(s-1)}(z) - b_{s-1}| \leqslant \exp\frac{-\lambda + \tau}{r^\alpha}$$

as $r = |z_0 - z| \to 0$, where $0 < \tau < \lambda$. By the same method used in the proof of Theorem 1.3, we can prove the impossibility of the case $b_{s-1} = b_{s-2} = \ldots = b_0 = 0$. □

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