ON THE NECESSARY AND SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF THE DIFFERENCE SCHEMES FOR THE GENERAL BOUNDARY VALUE PROBLEM FOR THE LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We consider the numerical solvability of the general linear boundary value problem for the systems of linear ordinary differential equations. Along with the continuous boundary value problem we consider the sequence of the general discrete boundary value problems, i.e. the corresponding general difference schemes. We establish the effective necessary and sufficient (and effective sufficient) conditions for the convergence of the schemes. Moreover, we consider the stability of the solutions of general discrete linear boundary value problems, in other words, the continuous dependence of solutions on the small perturbation of the initial dates. In the direction, there are obtained the necessary and sufficient condition, as well. The proofs of the results are based on the concept that both the continuous and discrete boundary value problems can be considered as so called generalized ordinary differential equation in the sense of Kurzweil. Thus, our results follow from the corresponding well-posedness results for the linear boundary value problems for generalized differential equations.

Keywords: general linear boundary value problem; linear ordinary differential systems; numerical solvability; convergence of difference schemes; effective necessary and sufficient conditions; generalized ordinary differential equations in the Kurzweil sense

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1. Statement of the problem and basic notation

The work is dedicated to the investigation of the numerical solvability of the general linear boundary value problem for the system of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(t)x + q(t), \tag{1.1}$$

$$l(x) = c_0, \tag{1.2}$$

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where P and q are, respectively, real matrix valued and vector valued functions with Lebesque integrable components defined on a closed interval $[a, b]$, where $c_0 \in \mathbb{R}^n$ is a real vector and l is a linear bounded operator from the space of all continuous vector valued functions defined on $[a, b]$.

Throughout the paper, we will assume that the absolutely continuous vector function $x_0\colon\ [a, b] \to \mathbb{R}^n$ is the unique solution of problem (1.1), (1.2) (the conditions guaranteeing these can be found in [4], for example).

Along with problem (1.1), (1.2) we consider the difference scheme

$$
\begin{aligned}
(1.1_m) \qquad & \Delta y(k-1) = \frac{1}{m}(G_{1m}(k)y(k) + G_{2m}(k-1)\,y(k-1) \\
& \qquad + g_{1m}(k) + g_{2m}(k-1)), \quad k = 1, \dots, m, \\
(1.2_m) \qquad & \mathcal{L}_m(y) = \gamma_m,
\end{aligned}
$$

where $m \in \mathbb{N}$ and G_{jm} and g_{jm} $(j = 1, 2)$ are, respectively, mappings of the set $\mathbb{N}_m = \{1, \dots, m\}$ into $\mathbb{R}^{n\times n}$ and $\mathbb{R}^n$, $\gamma_m \in \mathbb{R}^n$. Furthermore, for a given $m \in \mathbb{N}_m$, $\mathcal{L}_m$ is a linear continuous mapping of the space of vector valued functions from $\mathbb{N}$ into $\mathbb{R}^n$ and with values in $\mathbb{R}^{n\times n}$.

In the paper, we want to present the effective necessary and sufficient (moreover, the effective sufficient) conditions for the convergence of the difference scheme (1.1_m), (1.2_m) to x_0. Moreover, a criterion is obtained for the stability of the difference scheme (1.1_m), (1.2_m).

The problem of numerical stability is a classical one. Up to now it has been considered by many authors, see e.g. [5], [6], [7], [8], [9], [11], [14] and references therein. Among them we can highlight the monograph [7], where a.o. the numerical solvability of the Cauchy-Nicoletti problem for a system of nonlinear functional-differential equations was treated. Let us note that both in this monograph as well as in the other above mentioned references, no necessary and the more so no necessary and sufficient conditions were found.

The problem analogous to the one considered in the paper is investigated in [5] for the initial problem.

Finally, we note that, like in [3], the second order difference linear problem can be reduced to some first order difference linear problem of the type (1.1_m), (1.2_m) and therefore we can obtain the necessary and sufficient conditions for the convergence of corresponding second order difference schemes. Analogously, we can consider the third order difference problem and so on.

The following notations and definitions will be used:

- ▷ $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are, respectively, the sets of all natural, integer and real numbers, $\widetilde{\mathbb{N}} = \{0\} \cup \mathbb{N}$, $\mathbb{R}_+ = [0, \infty[$, $[a, b]$ is a closed interval.

- ▷ $\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

- ▷ $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.
- ▷ $O_{n\times m}$ (or O) is the zero $n \times m$-matrix. I_n is an identity $n \times n$ matrix.
- ▷ 0_n is the zero n-vector.
- ▷ $\limsup\limits_{k\to\infty} x_k$ is the upper limit of the sequence $x_k \in \mathbb{R}$, $k = 1, 2, \dots$
- ▷ $X(t-)$ and $X(t+)$ are the left and the right limits of the matrix valued function $X\colon [a,b] \to \mathbb{R}^{n\times n}$ at the point t (we assume that $X(t) = X(a)$ for $t \leqslant a$ and $X(t) = X(b)$ for $t \geqslant b$, if necessary);

$$\mathrm{d}_1 X(t) = X(t) - X(t-), \quad \mathrm{d}_2 X(t) = X(t+) - X(t);$$
$$\|X\|_\infty = \sup\{\|X(t)\|\colon\ t \in [a,b]\}.$$

- ▷ $\det(X)$ is the determinant of the $n \times n$-matrix X.
- ▷ $\bigvee\limits_a^b(X)$ is the total variation of the matrix valued function $X\colon [a,b] \to \mathbb{R}^{n\times m}$, i.e. the sum of total variations of its components x_{ij}, $i = 1, \dots, n$; $j = 1, \dots, m$.
- ▷ $\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ is the space of all bounded variation matrix valued functions $X\colon [a,b] \to \mathbb{R}^{n\times m}$, i.e. such that $\bigvee\limits_a^b(X) < \infty$ with the norm $\|X\|_\infty$.
- ▷ $C([a,b];\mathbb{R}^{n\times m})$ is the space of all matrix valued functions $X\colon [a,b] \to \mathbb{R}^{n\times m}$ with continuous components on $[a,b]$ with the standard norm

$$\|X\|_c = \max\{\|X(t)\|\colon\ t \in [a,b]\}.$$

- ▷ $AC([a,b];\mathbb{R}^{n\times m})$ is the set of all matrix valued functions $X\colon [a,b] \to \mathbb{R}^{n\times m}$ with absolutely continuous components.
- ▷ $L([a,b];\mathbb{R}^{n\times m})$ is the set of all matrix valued functions $X\colon [a,b] \to \mathbb{R}^{n\times m}$ whose components are Lebesgue integrable.
- ▷ $|\!|\!|l|\!|\!|$ is the norm of a linear bounded vector valued functional l.
- ▷ s_1, s_2 and $s_c\colon \mathrm{BV}([a,b];\mathbb{R}) \to \mathrm{BV}([a,b];\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a) = 0 \qquad s_2(x)(a) = 0, \qquad s_c(x)(a) = x(a),$$
$$s_1(x)(t) = \sum_{a<\tau\leqslant t} \mathrm{d}_1 x(\tau), \quad s_2(x)(t) = \sum_{a\leqslant\tau<t} \mathrm{d}_2 x(\tau), \quad s_c(x)(t) = x(t) - \sum_{j=1}^{2} s_j(x)(t)$$

for $a < t \leqslant b$.

If $g \in \mathrm{BV}([a,b];\mathbb{R})$, $f\colon [a,b] \to \mathbb{R}$ and $a \leqslant s < t \leqslant b$, then we assume

$$\int_s^t x(\tau)\,\mathrm{d}g(\tau) = (L-S)\int_{]s,t[} x(\tau)\,\mathrm{d}g(\tau) + f(t)\,\mathrm{d}_1 g(t) + f(s)\,\mathrm{d}_2 g(s),$$

where $(L-S)\int_{]s,t[} f(\tau)\,\mathrm{d}g(\tau)$ is Lebesgue-Stieltjes integral over the open interval $]s,t[$. It is known (see [13]) that if this integral exists, then the right-hand side of the integral equality equals to Kurzeil-Stieltjes integral (K-S) $\int_s^t f(\tau)\,\mathrm{d}g(\tau)$ (see [10], [12], [15]) and therefore $\int_s^t x(\tau)\,\mathrm{d}g(\tau) = (\text{K-S})\int_{]s,t[} x(\tau)\,\mathrm{d}g(\tau)$.

If $a = b$, then we assume $\int_a^b x(t)\,\mathrm{d}g(t) = 0$.

If $G = (g_{ik})_{i,k=1}^{l,n} \in \mathrm{BV}([a,b];\mathbb{R}^{l\times n})$ and $F = (f_{kj})_{k,j=1}^{n,m}\colon [a,b] \to \mathbb{R}^{n\times m}$, then

$$S_c(G)(t) \equiv (s_c(g_{ik})(t))_{i,k=1}^{l,n}, \quad x_j(G)(t) \equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n}, \quad j = 1,2$$

and

$$\int_a^b \mathrm{d}G(\tau)\cdot F(\tau) = \biggl(\sum_{k=1}^n \int_a^b f_{kj}(\tau)\,\mathrm{d}g_{ik}(\tau)\biggr)_{i,j=1}^{l,m}.$$

For $X \in \mathrm{BV}([a,b];\mathbb{R}^{l\times n})$ and $Y \in \mathrm{BV}([a,b];\mathbb{R}^{n\times m})$, we define

$$\mathcal{B}(X,Y)(t) = X(t)Y(t) - X(a)Y(a) - \int_a^t \mathrm{d}X(\tau)\cdot Y(\tau) \quad \text{for } t \in [a,b],$$

$$\mathcal{I}(X,Y)(t) = \int_a^t \mathrm{d}(X(\tau) + \mathcal{B}(X,Y)(\tau))\cdot X^{-1}(\tau) \quad \text{for } t \in [a,b].$$

The operator $\mathcal{B}$ has the property

$$\mathcal{B}\Bigl(X, \int_a^{\cdot} \mathrm{d}Y(\tau)\cdot Z(\tau)\Bigr)(t) = \int_a^t \mathrm{d}\mathcal{B}(X,Y)(\tau)\cdot Z(\tau) \quad \text{for } t \in [a,b] \tag{1.3}$$

(see Lemma 2.1 from [2]).

Further, notice (cf. [15]) that the following relations hold for all $f, g \in \mathrm{BV}([a,b];\mathbb{R}^n)$

$$\int_a^b f(t)\,\mathrm{d}g(t) = \int_a^b f(t)\,\mathrm{d}g(t-) + f(b)\,\mathrm{d}_1 g(b) = \int_a^b f(t)\,\mathrm{d}g(t+) + f(a)\,\mathrm{d}_2 g(a), \tag{1.4}$$

$$\int_a^b f(t)\,\mathrm{d}g(t) + \int_a^b g(t)\,\mathrm{d}f(t) = f(b)g(b) - f(a)g(a) + \sum_{a<t\leqslant b} \mathrm{d}_1 f(t)\cdot \mathrm{d}_1 g(t) - \sum_{a\leqslant t<b} \mathrm{d}_2 f(t)\cdot \mathrm{d}_2 g(t) \tag{1.5}$$

(integration-by-parts formula),

$$\int_a^b f(t)\,\mathrm{d}s_1(g)(t) = \sum_{a<t\leqslant b} f(t)\,\mathrm{d}_1 g(t), \quad \int_a^b f(t)\,\mathrm{d}s_2(g)(t) = \sum_{a\leqslant t<b} f(t)\,\mathrm{d}_2 g(t), \tag{1.6}$$

and

$$\mathrm{d}_j\left(\int_a^t f(s)\,\mathrm{d}g(s)\right) = f(t)\,\mathrm{d}_j g(t) \quad \text{for } t \in I,\ j = 1,2. \tag{1.7}$$

For $m \in \mathbb{N}$, we will denote $\mathbb{N}_m = \{1,\dots,m\}$ and $\widetilde{\mathbb{N}}_m = \{0,1,\dots,m\}$. If $J \subset \mathbb{Z}$, then $\mathbb{E}(J;\mathbb{R}^{n\times m})$ is the space of all bounded matrix valued functions $Y\colon J \to \mathbb{R}^{n\times m}$ with the norm

$$\|Y\|_J = \max\{\|Y(k)\|\colon k \in J\}.$$

For $m \in \mathbb{N}$, $Y \in \mathbb{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times m})$ and $i \in \mathbb{N}_m$, we denote $\Delta Y(i-1) = Y(i) - Y(i-1)$. Further, $\tau_m = (b-a)/m$, $\tau_{0m} = a$, $\tau_{km} = a + k\tau_m$ and $I_{km} =]\tau_{k-1\,m}, \tau_{km}[$ for $m \in \mathbb{N}$ and $k \in \mathbb{N}_m$. Moreover, for $m \in \mathbb{N}$ we define the function ν_m by

$$\nu_m(t) = \left[\frac{t-a}{b-a}m\right] \quad \text{for } t \in [a,b],$$

where $[T]$ stands for the integer part of T. Obviously, $\nu_m(\tau_{km}) = k$ for all $m \in \mathbb{N}_m$ and $k \in \widetilde{\mathbb{N}}_m$.

Now, assume that $P \in L([a,b];\mathbb{R}^{n\times n})$, $q \in L([a,b];\mathbb{R}^n)$ and $l\colon C([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded vector valued functional. Let $G_{jm} \in \mathbb{E}(N_m;\mathbb{R}^{n\times n})$, $j = 1,2$, $g_{jm} \in \mathbb{E}(N_m;\mathbb{R}^n)$ and let $\mathcal{L}_m\colon \mathbb{E}(J;\mathbb{R}^{n\times m}) \to \mathbb{R}^n$ be a given linear bounded vector valued functional for $m \in \mathbb{N}$ and $j \in \{1,2\}$. In addition, assume

$$G_{1m}(0) = G_{2m}(m) = O_{n\times n} \quad \text{and} \quad g_{1m}(0) = g_{2m}(m) = 0_n \quad \text{for } m \in \mathbb{N}.$$

For all $m \in \mathbb{N}$, define the operators $p_m\colon \mathrm{BV}([a,b];\mathbb{R}^n) \to \mathbb{E}(\widetilde{N}_m;\mathbb{R}^n)$ and $q_m\colon \mathbb{E}(\widetilde{N}_m;\mathbb{R}^n) \to \mathrm{BV}([a,b];\mathbb{R}^n)$, respectively, by

$$p_m(x)(k) = x(\tau_{km}) \quad \text{for } x \in \mathrm{BV}([a,b];\mathbb{R}^n),\ k \in \widetilde{\mathbb{N}}_m$$

and

$$q_m(y)(t) = \begin{cases} y(k) & \text{if } t = \tau_{km} \text{ for some } k \in \widetilde{\mathbb{N}}_m, \\ y(k) - \dfrac{1}{m}G_{1m}(k)y(k) - \dfrac{1}{m}g_{1m}(k) & \text{if } t \in]\tau_{k-1\,m}, \tau_{km}[\text{ for some } k \in \widetilde{\mathbb{N}}_m, \\ & \text{for } y \in \mathbb{E}(\widetilde{N}_m;\mathbb{R}^n) \text{ and } t \in [a,b]. \end{cases}$$

2. Formulation of the main results

2.1. The convergence of difference schemes. We give the proofs of the results of this chapter below, in Chapter 4.

Definition 2.1. We say that a sequence $(G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m)$ $(m = 1, 2, \ldots)$ belongs to the set $\mathcal{CS}(P, q, l)$ if for every $c_0 \in \mathbb{R}^n$ and the sequence $\gamma_m \in \mathbb{R}^n$, $m = 1, 2, \ldots$ satisfying the condition

$$\lim_{m\to\infty} \gamma_m = c_0,$$

the difference problem (1.1_m), (1.2_m) has a unique solution $y_m \in \mathbb{E}(\widetilde{N}_m; \mathbb{R}^n)$ for any sufficiently large m and

$$\lim_{m\to\infty} \|y_m - p_m(x_0)\|_{\widetilde{N}_m} = 0.$$

Theorem 2.1. *Let the conditions*

$$\lim_{m\to\infty} \mathcal{L}_m(p_m(x)) = l(x) \quad \text{for } x \in \mathrm{BV}([a,b]; \mathbb{R}^n), \tag{2.1}$$

$$\limsup_{m\to\infty} |||\mathcal{L}_m||| < \infty \tag{2.2}$$

hold. Then

$$((G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m))_{m=1}^{\infty} \in \mathcal{CS}(P, q; l) \tag{2.3}$$

if and only if there exist a matrix valued function $H \in AC([a,b]; \mathbb{R}^{n\times n})$ and a sequence of matrix valued functions $H_{1m}, H_{2m} \in \mathbb{E}(\widetilde{\mathbb{N}}_m; \mathbb{R}^{n\times n})$, $m \in \mathbb{N}$, such that the conditions

$$\limsup_{m\to\infty} \sum_{k=1}^{m} \Big(\Big\| H_{2m}(k) - H_{1\,m}(k) + \frac{1}{m} H_{1m}(k) G_{1m}(k) \Big\| + \Big\| H_{1m}(k) - H_{2m}(k-1) + \frac{1}{m} H_{1m}(k)\, G_{2m}(k-1) \Big\| \Big) < \infty, \tag{2.4}$$

$$\inf\{|\det(H(t))| :\ t \in [a,b]\} > 0, \tag{2.5}$$

$$\lim_{m\to\infty} \max_{k\in\widetilde{\mathbb{N}}_m} \{\|H_{jm}(k) - H(\tau_{km})\|\} = 0, \quad j = 1, 2 \tag{2.6}$$

hold, and the conditions

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{\nu_m(t)} H_{1m}(k)\,(G_{1m}(k) + G_{2m}(k-1)) = \int_a^t H(\tau)P(\tau)\,\mathrm{d}\tau, \tag{2.7}$$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{\nu_m(t)} H_{1m}(k)\,(g_{1m}(k) + g_{2m}(k-1)) = \int_a^t H(\tau)q(\tau)\,\mathrm{d}\tau \tag{2.8}$$

are fulfilled uniformly on $[a,b]$.

R e m a r k 2.1. The limits equalities (2.7) and (2.8) are fulfilled uniformly on $[a,b]$ if and only if the conditions

$$\lim_{m\to\infty}\max_{l\in\mathbb{N}_m}\left\{\left|\frac{1}{m}\sum_{k=1}^{l}H_{1m}(k)\left(G_{1m}(k)+G_{2m}(k-1)\right)-\int_a^{\tau_{lm}}H(\tau)P(\tau)\,\mathrm{d}\tau\right|\right\}=O_{n\times n},$$

$$\lim_{m\to\infty}\max_{l\in\mathbb{N}_m}\left\{\left|\frac{1}{m}\sum_{k=1}^{l}H_{1m}(k)(g_{1m}(k)+g_{2m}(k-1))-\int_a^{\tau_{lm}}H(\tau)q(\tau)\,\mathrm{d}\tau\right|\right\}=0_n$$

hold, respectively.

Let X be the fundamental matrix of the system $\mathrm{d}x/\,\mathrm{d}t=P(t)x$ on $[a,b]$ such that $X(a)=I_n$, and for any $m\in\mathbb{N}$ let Y_m be the fundamental matrix of the system

$$\Delta y(k-1)=\frac{1}{m}(G_{1m}(k)\,y(k)+G_{2m}(k-1)\,y(k-1)),\quad k\in\mathbb{N}_m \tag{2.9}$$

such that $Y_m(0)=I_n$.

Theorem 2.2. *Let conditions* (2.1), (2.2) *and*

$$\det\Big(I_n+(-1)^j\frac{1}{m}G_{jm}(k)\Big)\neq 0,\quad j=1,2;\ k\in\mathbb{N}_m;\ m\in\mathbb{N} \tag{2.10}$$

hold. Then inclusion (2.3) *holds if and only if the conditions*

$$\lim_{m\to\infty}\max_{k\in\widetilde{\mathbb{N}}_m}\{\|Y_m^{-1}(k)-X^{-1}(\tau_{km})\|\}=0 \tag{2.11}$$

and

$$\lim_{m\to\infty}\max_{l\in\mathbb{N}_m}\left\{\left|\frac{1}{m}\sum_{k=1}^{l}Y_m^{-1}(k)(g_{1m}(k)+g_{2m}(k-1))-\int_a^{\tau_{lm}}X^{-1}(\tau)q(\tau)\,\mathrm{d}\tau\right|\right\}=0_n \tag{2.12}$$

hold.

R e m a r k 2.2.

(a) It is well known that if $P(t)\int_{t_0}^{t}P(\tau)\,\mathrm{d}\tau\equiv\int_{t_0}^{t}P(\tau)\,\mathrm{d}\tau\cdot P(t)$ for some $t_0\in[a,b]$, then $X(t)\equiv\exp\big(\int_{t_0}^{t}P(\tau)\,\mathrm{d}\tau\big)$;

(b) By (2.10) we conclude

$$Y_m(k)=\prod_{i=k}^{1}\Big(I_n-\frac{1}{m}G_{1m}(i)\Big)^{-1}\Big(I_n+\frac{1}{m}G_{2m}(i-1)\Big),\quad k\in\mathbb{N}_m \tag{2.13}$$

for every natural m;

(c) In Theorem 2.3, condition (2.4) automatically holds because Y_m is the fundamental matrix of the homogeneous system (2.9) for every natural m.

Now we give a method of constructing discrete real matrix valued and vector valued functions, respectively, G_{jm} and g_{jm} ($j = 1, 2$; $m \in \mathbb{N}$) for which the conditions of Theorem 2.3 hold. For the construction we use the inductive method. Let $\mathcal{E}_m \colon \widetilde{\mathbb{N}}_m \to \mathbb{R}^{n\times n}$ and $\xi_m \colon \widetilde{\mathbb{N}}_m \to \mathbb{R}^n$, $m \in \mathbb{N}$, be discrete matrix valued and vector valued functions, respectively, such that

$$\lim_{m\to\infty} \|\mathcal{E}_m\|_{\widetilde{\mathbb{N}}_m} = 0 \quad \text{and} \quad \lim_{m\to\infty} m\|\xi_m\|_{\widetilde{\mathbb{N}}_m} = 0.$$

Let

$$P_{lm} = X(\tau_{lm}) + \mathcal{E}_m(l) \quad \text{for } l \in \widetilde{\mathbb{N}}_m \text{ and } m \in \mathbb{N}.$$

Let m be an arbitrary natural number and let $G_{1m}(1)$ and $G_{2m}(0)$ be such that

$$Y_m(1) = P_{1m}.$$

According to (2.13) we get

$$\Big(I_n - \frac{1}{m}G_{1m}(1)\Big)^{-1}\Big(I_n + \frac{1}{m}G_{2m}(0)\Big) = P_{1m}.$$

Therefore $G_{1m}(1)$ and $G_{2m}(0)$ are arbitrary matrices such that

$$G_{1m}(1) = m(I_n - P_{1m}^{-1}) - G_{2m}(0)P_{1m}^{-1}.$$

Now, let $G_{1m}(k)$, $G_{2m}(k-1)$ and $Y_m(k)$, $k = 1, \ldots, l-1$, be constructed. For the construction of $G_{1m}(l)$ and $G_{2m}(l-1)$ we use the equalities

$$Y_m(l) = P_{lm}$$

and

$$Y_m(l) = \Big(I_n - \frac{1}{m}G_{1m}(l)\Big)^{-1}\Big(I_n + \frac{1}{m}G_{2m}(l-1)\Big)Y_m(l-1).$$

As above, we obtain the relation

$$G_{1m}(l) = m(I_n - P_{l-1\,m}P_{lm}^{-1}) - G_{2m}(l-1)\,P_{l-1\,m}\,P_{lm}^{-1}.$$

So $G_{1m}(l)$ and $G_{2m}(l-1)$ will be an arbitrary matrix satisfying the last equality.

Let us now construct the discrete vector valued functions g_{1m} and g_{2m}, $m \in \mathbb{N}$. As $g_{1m}(l)$ and $g_{2m}(l-1)$ we choose arbitrary vectors satisfying the equalities

$$\frac{1}{m}Y_m^{-1}(l)(g_{1m}(1) + g_{2m}(l-1)) = q_{lm}, \quad l \in \mathbb{N}_m,$$

where

$$q_{lm} = \xi_m(l) + \int_a^{\tau_{lm}} X^{-1}(\tau)q(\tau)\,\mathrm{d}\tau, \quad l \in \mathbb{N}_m$$

for every natural m. Therefore, we have the equalities

$$g_{1m}(l) + g_{2m}(l-1) = mY_m(l)q_{lm}, \quad l \in \mathbb{N}_m,\ m \in \mathbb{N}$$

for the definition of the vector valued functions g_{1m} and g_{2m}, $m \in \mathbb{N}$. It is evident that the constructed vector valued functions satisfy condition (2.12). We use the above constructed discrete matrix valued and vector valued functions in the following example.

Example 2.1. Let $X(t) \equiv \exp\left(\int_a^t P(\tau)\,\mathrm{d}\tau\right)$ be the fundamental matrix of system (1.1) and let $\mathcal{E}_m \equiv O_{n\times n}$ and $\xi_m \equiv 0_n$ for $m \in \mathbb{N}$. Then

$$P_{lm} = \exp\biggl(\int_a^{\tau_{lm}} P(\tau)\,\mathrm{d}\tau\biggr) \quad \text{for } l \in \widetilde{\mathbb{N}}_m \text{ and } m \in \mathbb{N}.$$

If we choose

$$G_{2m}(l-1) = P_{lm}P_{l-1\,m}^{-1} = \exp\biggl(\int_{\tau_{l-1\,m}}^{\tau_{lm}} P(\tau)\,\mathrm{d}\tau\biggr) \quad \text{for } l \in \mathbb{N}_m \text{ and } m \in \mathbb{N},$$

then

$$G_{1m}(l) = (m-1)I_n - m\exp\biggl(-\int_{\tau_{l-1\,m}}^{\tau_{lm}} P(\tau)\,\mathrm{d}\tau\biggr) \quad \text{for } l \in \mathbb{N}_m \text{ and } m \in \mathbb{N}.$$

For the definition of the discrete vector valued functions g_{1m} and g_{2m} we have the relations

$$g_{1m}(l) + g_{2m}(l-1) = m\int_a^{\tau_{lm}} C(\tau_{lm},\tau)q(\tau)\,\mathrm{d}\tau \quad \text{for } l \in \mathbb{N}_m \text{ and } m \in \mathbb{N},$$

where $C(t,\tau)$ is the Cauchy matrix of system (1.1).

In particular, we can take

$$g_{1m}(l) = \alpha m\int_a^{\tau_{lm}} C(\tau_{lm},\tau)q(\tau)\,\mathrm{d}\tau$$

and

$$g_{2m}(l-1) = (1-\alpha)m\int_a^{\tau_{lm}} C(\tau_{lm},\tau)q(\tau)\,\mathrm{d}\tau$$

for $l \in \mathbb{N}_m$ and $m \in \mathbb{N}$, where α is some number.

Moreover, we can choose these discrete vector valued functions for the connection with the Cauchy formulae for system (1.1).

Theorem 2.3. *Let conditions* (2.1), (2.2) *and*

$$\limsup_{m\to\infty}\sum_{k=1}^{m}\Big(\frac{1}{m}(\|G_{1m}(k)\|+\|G_{2m}(k-1)\|)\Big)<\infty$$

hold and let the conditions

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}(G_{1m}(k)+G_{2m}(k-1))=\int_a^t P(\tau)\,\mathrm{d}\tau, \tag{2.14}$$

and

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}(g_{1m}(k)+g_{2m}(k-1))=\int_a^t q(\tau)\,\mathrm{d}\tau \tag{2.15}$$

be fulfilled uniformly on I. Then inclusion (2.3) *holds.*

Proposition 2.1. *Let conditions* (2.1), (2.2), (2.4), (2.5), (2.6) *and*

$$\lim_{m\to\infty}\frac{1}{m}\max_{k\in\widetilde{\mathbb{N}}_m}\{\|G_{jm}(k)\|+\|g_{jm}(k)\|\}=0,\quad j=1,2 \tag{2.16}$$

hold and let conditions (2.7) *and* (2.8) *be fulfilled uniformly on* $[a,b]$, *where* $H\in AC([a,b];\mathbb{R}^{n\times n})$, $H_{1m},H_{2m}\in\mathbb{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$, $m\in\mathbb{N}$. *Let, moreover, either*

$$\limsup_{m\to\infty}\bigg(\frac{1}{m}\sum_{k=0}^{m}(\|G_{jm}(k)\|+\|g_{jm}(k)\|)\bigg)<\infty,\quad j=1,2$$

or

$$\limsup_{m\to\infty}\sum_{k=0}^{m}(\|H_{2m}(k)-H_{1m}(k)\|+\|H_{1m}(k)-H_{2m}(k-1)\|)<\infty.$$

Then inclusion (2.3) *holds.*

Theorem 2.4. *Let conditions* (2.1), (2.2), (2.4), (2.5), (2.6) *and* (2.16) *hold and let conditions* (2.14), (2.15),

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\,(G_{1m}(k)+G_{2m}(k-1))=\int_a^t P_*(\tau)\,\mathrm{d}\tau$$

and

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)(g_{1m}(k)+g_{2m}(k-1))=\int_a^t q_*(\tau)\,\mathrm{d}\tau$$

be fulfilled uniformly on $[a,b]$, *where* $P_* \in L([a,b];\mathbb{R}^{n\times n})$, $q_* \in L([a,b];\mathbb{R}^n)$, $H \in AC([a,b];\mathbb{R}^{n\times n})$, $H_{1m}, H_{2m} \in \mathbb{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$, $m\in\mathbb{N}$. *Let moreover the system*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (P(t)-P_*(t))x + q(t) - q_*(t)$$

have a unique solution under the boundary value condition (1.2). *Then*

$$((G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m))_{m=1}^{\infty} \in \mathcal{CS}(P-P_*, q-q_*; l).$$

Corollary 2.1. *Let conditions* (2.1) *and* (2.2) *hold and there exist a natural* μ *and matrix valued functions* $B_{jl} \in \mathbb{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$, $B_{jl}(a) = O_{n\times n}$ $(j=1,2;\ l=0,\dots,\mu-1)$ *such that*

$$\limsup_{m\to\infty}\sum_{k=1}^{m}\Big(\Big\|H_{2m\mu}(k) - H_{1m\mu}(k) + \frac{1}{m}H_{1m\mu}(k)\,G_{1m\mu}(k)\Big\| + \Big\|H_{1m\mu}(k) - H_{2m\mu}(k-1) + \frac{1}{m}H_{1m\mu}(k)\,G_{2m\mu}(k-1)\Big\|\Big) < \infty,$$

$$\lim_{m\to\infty}\max_{k\in\widetilde{\mathbb{N}}_m}\{\|H_{jm\mu}(k) - I_n\|\} = 0, \quad j=1,2$$

and let the conditions

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}(G_{1m\mu}(k) + G_{2m\mu}(k-1)) = \int_a^t P(\tau)\,\mathrm{d}\tau,$$

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}(g_{1m\mu}(k) + g_{2m\mu}(k-1)) = \int_a^t q(\tau)\,\mathrm{d}\tau$$

be fulfilled uniformly on $[a,b]$, *where*

$$H_{1m0}(k) = H_{2m0}(k) \equiv I_n,$$
$$H_{1m\,l+1}(k) \equiv \Big(\frac{1}{m}H_{1ml}(k)\,G_{1m}(k) + \mathcal{Q}_1(H_{1ml}, G_{1m}, G_{2m})(k) + B_{1\,l+1}(k)\Big)H_{1ml}(k),$$
$$H_{2m\,l+1}(k) \equiv (\mathcal{Q}_2(H_{1ml}, G_{1m}, G_{2m})(k) + B_{2\,l+1}(k))H_{2ml}(k),$$
$$G_{1m\,l+1}(k) \equiv H_{1ml}(k)G_{1m}(k), G_{2m\,l+1}(k) \equiv H_{1ml}(k+1)G_{2m}(k),$$
$$g_{1m\,l+1}(k) \equiv H_{ml}(k)g_{1m}(k), g_{2m\,l+1}(k) \equiv H_{ml}(k+1)g_{2m}(k),$$
$$\mathcal{Q}_j(H_{1ml}, G_{1m}, G_{2m})(k) \equiv 2I_n - H_{jml}(k) - \frac{1}{m}\sum_{i=1}^{k}H_{1ml}(i)\,(G_{1m}(i) + G_{2m}(i-1))$$
$$j=1,2;\ l=0,\dots,\mu-1;\ m=1,2,\dots$$

Then inclusion (2.3) *holds.*

If $\mu = 1$ and $B_{j0}(t) \equiv O_{n\times n}$, $j = 1, 2$, then Corollary 2.1 has the form of Theorem 2.3.

Remark 2.3. In Theorems 2.1, 2.4, Proposition 2.1 and Corollary 2.1, if condition (2.10) holds, we can assume that $H_m(t) \equiv Y_m^{-1}(t)$, where Y_m is the fundamental matrix of the homogeneous system (2.9), defined by (2.13), for every natural m. Moreover, condition (2.4) and analogous conditions automatically hold everywhere in the results circumscribed above, as well.

2.2. The stability of difference schemes. Consider now the question of the stability of a solution of the difference linear boundary value problem

$$\begin{aligned}(2.17)\qquad \Delta y(k-1) &= G_1(k)\,y(k) + G_2(k-1)y(k-1)\\ &\quad + g_1(k) + g_2(k-1), \quad k \in \mathbb{N}_{m_0},\end{aligned}$$

$$(2.18)\qquad \mathcal{L}(y) \equiv \sum_{k=0}^{m_0} B(k)y(k) = \gamma_0,$$

where $m_0 \geqslant 2$ is a fixed natural number, $G_j \in \mathbb{E}(N_{m_0};\mathbb{R}^{n\times n})$, $j = 1, 2$, $\gamma_0 \in \mathbb{R}^n$, $g \in \mathbb{E}(N_{m_0};\mathbb{R}^n)$, and $B \in \mathbb{E}(N_{m_0};\mathbb{R}^n)$.

Along with problem (2.17), (2.18) consider the sequence of the problems

$$\begin{aligned}(2.17_m)\qquad \Delta y(k-1) &= G_{1m}(k)\,y(k) + G_{2m}(k-1)\,y(k-1)\\ &\quad + g_{1m}(k) + g_{2m}(k-1), \quad k \in \mathbb{N}_{m_0},\end{aligned}$$

$$(2.18_m)\qquad \mathcal{L}_m(y) \equiv \sum_{k=0}^{m_0} B_m(k)y(k) = \gamma_m, \quad m \in \mathbb{N},$$

where $G_{jm} \in \mathbb{E}(N_{m_0};\mathbb{R}^{n\times n})$, $j = 1, 2$, $g_m \in \mathbb{E}(N_{m_0};\mathbb{R}^n)$, $B_m \in \mathbb{E}(N_{m_0};\mathbb{R}^n)$, and $\gamma_m \in \mathbb{R}^n$ for every natural m. As above, we assume that

$$\begin{aligned}G_1(0) = G_{1m}(0) = O_{n\times n}, &\qquad g_1(0) = g_{1m}(0) = 0_n, \qquad m \in \mathbb{N},\\ G_2(m_0) = G_{2m}(m_0) = O_{n\times n}, &\quad g_2(m_0) = g_{2m}(m_0) = 0_n, \quad m \in \mathbb{N}\end{aligned}$$

and problem (2.17), (2.18) has the unique solution $y^0 \in \mathbb{E}(\widetilde{N}_{m_0};\mathbb{R}^n)$ (the necessary and sufficient conditions are given in [3], for example).

Definition 2.2. We say that a sequence $(G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m)$, $m = 1, 2, \ldots$, belongs to the set $\mathcal{S}(G_1, G_2, g_1, g_2; \mathcal{L})$ if for every $\gamma_0 \in \mathbb{R}^n$ and the sequence $\gamma_m \in \mathbb{R}^n$, $m = 1, 2, \ldots$, satisfying the condition

$$\lim_{m\to\infty} \gamma_m = \gamma_0,$$

the difference boundary value problem (2.17_m), (2.18_m) has a unique solution $y_m \in \mathbb{E}(\widetilde{N}_{m_0}; \mathbb{R}^n)$ for any sufficiently large m and

$$\lim_{m\to\infty} \|y_m - y_0\|_{\widetilde{N}_{m_0}} = 0.$$

Theorem 2.5. *Let*

$$\det(I_n + (-1)^j G_j(k)) \neq 0 \quad \text{for } k \in \widetilde{N}_{m_0},\ j = 1, 2 \tag{2.19}$$

and

$$\lim_{m\to\infty} B_m(k) = B(k) \quad \text{for } k \in \widetilde{N}_{m_0}. \tag{2.20}$$

Then

$$((G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m))_{m=1}^{\infty} \in \mathcal{S}(G_1, G_2, g_1, g_2; \mathcal{L}) \tag{2.21}$$

if and only if

$$\lim_{m\to\infty} (G_{1m}(k) + G_{2m}(k-1)) = G_1(k) + G_2(k-1) \quad \text{for } k \in N_{m_0} \tag{2.22}$$

and

$$\lim_{m\to\infty} (g_{1m}(k) + g_{2m}(k-1)) = g_1(k) + g_2(k-1) \quad \text{for } k \in N_{m_0}. \tag{2.23}$$

Proposition 2.2. *Let conditions* (2.19), (2.20),

$$\lim_{m\to\infty} G_{jm}(k) = G_j(k) \quad \text{for } k \in \widetilde{N}_{m_0},\ j = 1, 2$$

and

$$\lim_{m\to\infty} g_{jm}(k) = g_j(k) \quad \text{for } k \in \widetilde{N}_{m_0},\ j = 1, 2$$

hold. Then inclusion (2.21) *holds.*

Corollary 2.2. *Let conditions* (2.19) *and* (2.20) *hold and there exist a natural* μ *and matrix valued functions* $B_{jl} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}; \mathbb{R}^{n\times n})$, $B_{jl}(a) = O_{n\times n}$ $(j = 1, 2;$

$l = 0, \dots, \mu - 1$) *such that the conditions*

$$\limsup_{m\to\infty} \sum_{k=0}^{m_0} (\|H_{2m\mu}(k) - H_{1m\mu}(k) + H_{1m\mu}(k)\, G_{1m\mu}(i)\| \\ + \|H_{1m\mu}(k) - H_{2m\mu}(k-1) + H_{1m\mu}(k) G_{2m\mu}(k-1)\|) < \infty,$$
$$\lim_{m\to\infty} H_{jm\mu}(k) = I_n \quad \text{for } k \in \widetilde{N}_{m_0},\ j = 1, 2,$$
$$\lim_{m\to\infty} (G_{1m\mu}(k) + G_{2m\mu}(k-1)) = G_1(k) + G_2(k-1) \quad \text{for } k \in N_{m_0}$$

and

$$\lim_{m\to\infty} (g_{1m\mu}(k) + g_{2m\mu}(k-1)) = g_1(k) + g_2(k-1) \quad \text{for } k \in N_{m_0}$$

hold, where

$$H_{1m0}(k) = H_{2m0}(k) \equiv I_n,$$
$$H_{1m\,l+1}(k) \equiv \big(H_{1ml}(k) G_{1m}(k) + \mathcal{Q}_1(H_{1ml}, G_{1m}, G_{2m})(k) + B_{1\,l+1}(k)\big) H_{1ml}(k),$$
$$H_{2m\,l+1}(k) \equiv \big(\mathcal{Q}_2(H_{1ml}, G_{1m}, G_{2m})(k) + B_{2\,l+1}(k)\big) H_{2ml}(k),$$
$$G_{1m\,l+1}(k) \equiv H_{1ml}(k) G_{1m}(k), G_{2m\,l+1}(k) \equiv H_{1ml}(k+1) G_{2m}(k),$$
$$g_{1m\,l+1}(k) \equiv H_{ml}(k) g_{1m}(k), g_{2m\,l+1}(k) \equiv H_{ml}(k+1) g_{2m}(k),$$
$$\mathcal{Q}_j(H_{1ml}, G_{1m}, G_{2m})(k) \equiv 2I_n - H_{jml}(k) - \sum_{i=1}^{k} H_{1ml}(i)\, (G_{1m}(i) + G_{2m}(i-1)) \\ j = 1, 2;\ l = 0, \dots, \mu - 1;\ m = 1, 2, \dots$$

Then inclusion (2.21) *holds.*

If $\mu = 1$ and $B_{j0}(t) = O_{n\times n}$, $j = 1, 2$, then Corollary 2.2 coincides with the necessary conditions of Theorem 2.5.

3. Generalized ordinary differential equations

The proofs of the results given above are based on the following concept. We rewrite both problems (1.1), (1.2) and (1.1_m), (1.2_m) ($m \in \mathbb{N}$) as a linear boundary value problem for systems of so called generalized ordinary differential equations in the sense of Kurzweil ([1]–[5], [10], [12], [15]). So the continuous system (1.1) as well as discrete systems (1.1_m) ($m \in \mathbb{N}$) are, really, the same types of equations. Therefore, the convergence of differential scheme (1.1_m), (1.2_m) ($m \in \mathbb{N}$) to the solution of problem (1.1), (1.2) is equivalent to the well-possed question for the boundary value problem for the last systems. So, using the results of papers [1], [2] we established the present results.

We rewrite the boundary value problem (1.1), (1.2)) as a boundary value problem for the linear system of generalized ordinary differential equations (in the sense of Kurzweil), i.e. in the form

$$\mathrm{d}x = \mathrm{d}A(t)\cdot x + \mathrm{d}f(t), \tag{3.1}$$

$$l(x) = c_0, \tag{3.2}$$

where $A \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, $f \in \mathrm{BV}([a,b];\mathbb{R}^n)$, $l\colon \mathrm{BV}([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded operator and $c_0 \in \mathbb{R}^n$ is a constant vector.

Under a solution of system (3.1) we understand a vector valued function $x \in \mathrm{BV}([a,b];\mathbb{R}^n)$ such that

$$x(t) = x(s) + \int_s^t \mathrm{d}A(\tau)\cdot x(\tau) + f(t) - f(s) \quad \text{for } a \leqslant s \leqslant t \leqslant b.$$

Along with problem (1.1), (1.2) we consider the sequence of the problems

$$\mathrm{d}x = \mathrm{d}A_m(t)\cdot x + \mathrm{d}f_m(t), \tag{3.1_m}$$

$$l_m(x) = c_m, \quad m = 1,2,\ldots, \tag{3.2_m}$$

where $A_m \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, $f_m \in \mathrm{BV}([a,b];\mathbb{R}^n)$, $l_m\colon \mathrm{BV}([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded operator, and $c_m \in \mathbb{R}^n$ is a constant vector for every natural m.

We use the following.

Definition 3.1. We say that a sequence $(A_m, f_m; l_m)$, $m = 1,2,\ldots$ belongs to the set $\mathcal{S}(A,f;l)$ if for every $c_0 \in \mathbb{R}^n$ and the sequence $c_m \in \mathbb{R}^n$, $m = 1,2,\ldots$ satisfying the condition

$$\lim_{m\to\infty} c_m = c_0, \tag{3.3}$$

the boundary value problem (3.1_m), (3.2_m) has a unique solution $x_m \in \mathrm{BV}([a,b];\mathbb{R}^n)$ for any sufficiently large m and

$$\lim_{m\to\infty} \|x_m - x_0\|_\infty = 0. \tag{3.4}$$

Along with systems (3.1) and (3.1_m) ($m \in \mathbb{N}$) we consider, respectively, the corresponding homogeneous systems

$$\mathrm{d}x(t) = \mathrm{d}A(t)\cdot x(t) \tag{3.1_0}$$

and

$$\mathrm{d}x(t) = \mathrm{d}A_m(t)\cdot x(t). \tag{3.1_{m0}}$$

We give some results from [2] concerning inclusion

$$((A_m, f_m; l_m))_{m=1}^{\infty} \in \mathcal{S}(A, f; l) \tag{3.5}$$

to be used to prove the main results.

Theorem 3.1. *Let the conditions*

$$\lim_{m\to\infty} l_m(x) = l(x) \quad \text{for } x \in \mathrm{BV}([a,b]; \mathbb{R}^n), \tag{3.6}$$

$$\limsup_{m\to\infty} \|l_m\| < \infty \tag{3.7}$$

and

$$\det(I_n + (-1)^j \mathrm{d}_j A(t)) \neq 0 \quad \text{for } t \in [a,b],\ j = 1,2 \tag{3.8}$$

hold. Then inclusion (3.5) *holds if and only if there exists a sequence of matrix valued functions* $H, H_m \in \mathrm{BV}([a,b]; \mathbb{R}^{n\times n})$ $(m = 1, 2, \ldots)$ *such that conditions* (2.5) *and*

$$\limsup_{m\to\infty} \bigvee_a^b (H_m + \mathcal{B}(H_m, A_m)) < \infty \tag{3.9}$$

hold, and the conditions

$$\lim_{m\to\infty} H_m(t) = H(t), \tag{3.10}$$

$$\lim_{m\to\infty} \mathcal{B}(H_m, A_m)(t) = \mathcal{B}(H, A)(t), \tag{3.11}$$

$$\lim_{m\to\infty} \mathcal{B}(H_m, f_m)(t) = \mathcal{B}(H, f)(t) \tag{3.12}$$

are fulfilled uniformly on $[a,b]$.

Theorem 3.2. *Let conditions* (3.6), (3.7) *and*

$$\det(I_n + (-1)^j \mathrm{d}_j A_m(t)) \neq 0 \quad \text{for } t \in [a,b],\ m \in \widetilde{\mathbb{N}},\ j = 1,2 \tag{3.13}$$

hold. Then inclusion (3.5) *holds if and only if the conditions*

$$\lim_{m\to\infty} X_m^{-1}(t) = X_0^{-1}(t) \tag{3.14}$$

and

$$\lim_{m\to\infty} \mathcal{B}(X_m^{-1}, f_m)(t) = \mathcal{B}(X_0^{-1}, f)(t)$$

are fulfilled uniformly on $[a,b]$, *where* X_m *is the fundamental matrix of the homogeneous system* (3.1_{m0}) *for every* $m \in \widetilde{\mathbb{N}}$.

Theorem 3.3. *Let conditions* (3.6), (3.7), (3.8) *and*

$$\limsup_{m\to\infty} \bigvee_a^b (A_m) < \infty \tag{3.15}$$

hold and let the conditions

$$\lim_{m\to\infty} (A_m(t) - A_m(a)) = A(t) - A(a) \tag{3.16}$$

and

$$\lim_{m\to\infty} (f_m(t) - f_m(a)) = f(t) - f(a) \tag{3.17}$$

be fulfilled uniformly on I. Then inclusion (3.5) *holds.*

Corollary 3.1. *Let conditions* (2.5), (3.6), (3.7), (3.8) *hold and let conditions* (3.10),

$$\lim_{m\to\infty} \int_a^t H_m(s)\mathrm{d}A_m(s) = \int_a^t H(s)\mathrm{d}A(s), \tag{3.18}$$

$$\lim_{m\to\infty} \int_a^t H_m(s)\mathrm{d}f_m(s) = \int_a^t H(s)\mathrm{d}f(s), \tag{3.19}$$

$$\lim_{m\to\infty} \mathrm{d}_j A_m(t) = \mathrm{d}_j A(t) \quad \text{and} \quad \lim_{m\to\infty} \mathrm{d}_j f_m(t) = \mathrm{d}_j f(t), \quad j = 1, 2 \tag{3.20}$$

be fulfilled uniformly on $[a, b]$, where $H, H_m \in \mathrm{BV}([a, b]; \mathbb{R}^{n\times n})$ $(m \in \mathbb{N})$. Let moreover either

$$\limsup_{m\to\infty} \sum_{a\leqslant t\leqslant b} (\|\mathrm{d}_j A_m(t)\| + \|\mathrm{d}_j f_m(t)\|) < \infty, \quad j = 1, 2 \tag{3.21}$$

or

$$\limsup_{m\to\infty} \sum_{a\leqslant t\leqslant b} \|\mathrm{d}_j H_m(t)\| < \infty, \quad j = 1, 2. \tag{3.22}$$

Then inclusion (3.5) *holds.*

Theorem 3.4. *Let conditions* (2.5), (3.6), (3.7), (3.8) *hold and let conditions* (3.10), (3.16), (3.17),

$$\lim_{m\to\infty} \int_a^t \mathrm{d}(H^{-1}(s)H_m(s)) \cdot A_m(s) = A_*(t)$$

and

$$\lim_{m\to\infty} \int_a^t \mathrm{d}(H^{-1}(s)\, H_m(s)) \cdot f_m(s) = f_*(t)$$

be fulfilled uniformly on $[a,b]$, *where* $A_*,\ H, H_m \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, $m\in\mathbb{N}$, $f_*, f_m \in \mathrm{BV}([a,b];\mathbb{R}^n)$, $m\in\mathbb{N}$. *Let moreover the system*

$$\mathrm{d}x = \mathrm{d}(A(t) - A_*(t))\cdot x + \mathrm{d}(f(t) - f_*(t))$$

have a unique solution under condition (1.2). *Then*

$$((A_m, f_m; l_m))_{m=1}^{\infty} \in \mathcal{S}(A - A_*, f - f_*; l).$$

Corollary 3.2. *Let conditions* (3.6)–(3.8) *hold and there exist a natural* μ *and matrix valued and vector valued functions* $B_l \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, $B_l(a) = O_{n\times n}$ $(l = 0,\dots,\mu-1)$ *such that*

$$\limsup_{m\to\infty} \bigvee_a^b (A_{m\mu}) < \infty,$$

and let the conditions

$$\begin{aligned}
&\lim_{m\to\infty} H_{m\,\mu-1}(t) = I_n,\\
&\lim_{m\to\infty} (A_{m\mu}(t) - A_{m\mu}(a)) = A(t) - A(a),\\
&\lim_{m\to\infty} (f_{m\mu}(t) - f_{m\mu}(a)) = f(t) - f(a)
\end{aligned}$$

be fulfilled uniformly on $[a,b]$, *where*

$$\begin{aligned}
H_{m0}(t) &\equiv I_n,\\
H_{m\,l+1}(t) &\equiv (I_n - A_{ml+1}(t) + A_{m\,l}(a) + B_{l+1}(t))H_{ml}(t),\\
A_{m\,l+1}(t) &\equiv H_{ml}(t) + \mathcal{B}(H_{ml}, A_m)(t),\\
f_{ml+1}(t) &\equiv \mathcal{B}(H_{ml}, f_m)(t), \quad l = 0,\dots,\mu-1;\ m = 1,2,\dots
\end{aligned}$$

Then inclusion (3.5) *holds.*

If $\mu = 1$ and $B_0(t) \equiv O_{n\times n}$, then Corollary 3.2 has the form of Theorem 3.3. For completeness, we give the proofs of the results presented and used in the section in brief (the full version one can be found in [1], [2]).

Below, in the proofs, we will assume that $A_0(t) \equiv A(t)$, $f_0(t) \equiv f(t)$, $l_0(x) \equiv l(x)$ and $H_0(t) \equiv H(t)$.

P r o o f of Theorem 3.3. Let us show that

$$\det(I_n + (-1)^j \, \mathrm{d}_j A_m(t)) \neq 0 \quad \text{for } t \in [a,b],\ j = 1,2 \tag{3.23}$$

for any sufficiently large m. By (3.16)

$$\lim_{m\to\infty} \mathrm{d}_j A_m(t) = \mathrm{d}_j A(t), \quad j = 1,2 \tag{3.24}$$

uniformly on $[a,b]$. Since $\bigvee_a^b(A) < \infty$, the series $\sum_{t\in[a,b]} \|\mathrm{d}_j A(t)\|$ $(j = 1,2)$ converge. Thus, for any $j \in \{1,2\}$ the inequality $\|\mathrm{d}_j A(t)\| \geqslant \frac{1}{2}$ may hold only for some finite number of points $t_{j1}, \ldots, t_{j\,m_j}$ in $[a,b]$. Therefore

$$\|\mathrm{d}_j A(t)\| < \frac{1}{2} \quad \text{for } t \in [a,b],\ t \neq t_{ji},\ i = 1,\ldots,m_j. \tag{3.25}$$

It follows from (3.8), (3.24) and (3.25) that for any sufficiently large m and $j \in \{1,2\}$

$$\det(I_n + (-1)^j \, \mathrm{d}_j A_m(t_{ji})) \neq 0, \quad i = 1,\ldots,m_j \tag{3.26}$$

and

$$\|\mathrm{d}_j A_k(t)\| < \frac{1}{2} \quad \text{for } t \in [a,b],\ t \neq t_{ji},\ i = 1,\ldots,m_j. \tag{3.27}$$

The latter inequality implies that the matrices $I_n + (-1)^j \mathrm{d}_j A_m(t)$, $j = 1,2$, are invertible for $t \in [a,b]$, $t \neq t_{ji}$ $(i = 1,\ldots,m_j)$ too. Therefore (3.23) is proved.

Besides, by (3.26) and (3.27) there exists a positive number r_0 such that for any sufficiently large m

$$\|(I_n + (-1)^j \, \mathrm{d}_j A_m(t))^{-1}\| \leqslant r_0 \quad \text{for } t \in [a,b],\ j = 1,2. \tag{3.28}$$

Let m be a sufficiently large natural number. In view of (3.8) and (3.23) there exist (see [15], Theorem III.2.10) fundamental matrices X and X_m of the homogeneous systems (3.1_0) and (3.1_{m0}), respectively, satisfying $X(a) = X_m(a) = I_n$. Moreover, $X, X_m^{-1} \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, $m \in \mathbb{N}$.

Let us show that

$$\lim_{m\to\infty} \|X_m - X\|_\infty = 0. \tag{3.29}$$

We set $Z_m(t) = X_m(t) - X(t)$ and $B_m(t) = A_m(t-)$ for $t \in [a,b]$, $m \in \mathbb{N}$. Due to (1.4), for every $t \in [a,b]$, we have $\mathrm{d}_1(B_m(t) - A_m(t)) = -\mathrm{d}_2(B_m(t) - A_m(t)) = -\mathrm{d}_1 A_m(t)$ and

$$\int_a^t \mathrm{d}(B_m(\tau) - A_m(\tau)) \cdot Z_m(\tau) = -\mathrm{d}_1 A_m(t) \cdot Z_m(t).$$

Consequently,

$$Z_m(t) \equiv (I_n - \text{d}_1 A_m(t))^{-1}\biggl(\int_a^t \text{d}(A_m(\tau) - A(\tau)) \cdot X(\tau) + \int_a^t \text{d}B_m(\tau) \cdot Z_m(\tau)\biggr).$$

From this and (3.28) we get

$$\|Z_m(t)\| \leqslant r_0\biggl(\varepsilon_m + \int_a^t \text{d}\|V(B_m)(\tau)\| \cdot \|Z_m(\tau)\|\biggr) \quad \text{for } t \in [a,b],$$

where

$$\varepsilon_m = \sup\biggl\{\biggl\|\int_a^t \text{d}(A_m(\tau) - A(\tau)) \cdot X(\tau)\biggr\| \colon t \in [a,b]\biggr\}.$$

Hence, according to the Gronwall inequality (see [15], Theorem I.4.30),

$$\|Z_m(t)\| \leqslant r_0\varepsilon_m \exp\Bigl(r_0 \bigvee_a^b (B_m)\Bigr) \leqslant r_0\varepsilon_m \exp\Bigl(r_0 \bigvee_a^b (A_m)\Bigr) \quad \text{for } t \in [a,b].$$

By (3.15), (3.16) and Lemma 2 from [1], this inequality implies (3.29).

It is known (see [15], Theorem III.2.13) that if x_m is the solution of (3.1_m), then

$$x_m(t) \equiv X_m(t)x_m(a) + f_m(t) - f_m(a) - X_m(t)\int_a^t \text{d}X_m^{-1}(\tau) \cdot (f_m(\tau) - f_m(a)).$$

Thus, problem (3.1_m), (3.2_m) has a unique solution if and only if

$$\det(l_m(X_m)) \neq 0. \tag{3.30}$$

Since problem (3.1), (3.2) has the unique solution x_0, we have

$$\det(l(X)) \neq 0. \tag{3.31}$$

Besides, by (3.6), (3.7) and (3.29) we find

$$\lim_{m\to\infty} l_m(X_m) = l(X).$$

Therefore, in view of (3.31), there exists a natural number m_0 such that condition (3.30) holds for every $m \geqslant m_0$. Thus, problem (3.1_m), (3.2_m) has the unique solution x_m for $m \geqslant m_0$ and

$$x_m(t) \equiv X_m(t)(l_m(X_m))^{-1}(c_m - l_m(F_m(f_m))) + F_m(f_m)(t), \tag{3.32}$$

where

$$F_m(f_m)(t) = f_m(t) - f_m(a) - X_m(t) \int_a^t \mathrm{d}X_m^{-1}(\tau) \cdot (f_m(\tau) - f_m(a)).$$

According to Lemma 2 from [1] we conclude that

$$\lim_{m \to \infty} \|X_m^{-1} - X^{-1}\|_\infty = 0 \tag{3.33}$$

and

$$\varrho = \sup\{\|X_m^{-1}(t)\| + \|X_m(t)\| \colon\ t \in [a,b],\ m \geqslant m_0\} < \infty. \tag{3.34}$$

The equality

$$X_m^{-1}(t) - X_m^{-1}(s) = X_m^{-1}(s) \int_t^s \mathrm{d}A_m(\tau) \cdot X_m(\tau) X_m^{-1}(t)$$

implies

$$\|X_m^{-1}(t) - X_m^{-1}(s)\| \leqslant \varrho^3 \bigvee_s^t (A_m) \quad \text{for } a \leqslant s \leqslant t \leqslant b,\ m \geqslant m_0.$$

This inequality, together with (3.15) and (3.34), yields

$$\limsup_{m \to \infty} \bigvee_a^b (X_m^{-1}) < \infty.$$

By this, (3.17) and (3.29), it follows from [1], Lemma 1 that

$$\lim_{m \to \infty} \int_a^t \mathrm{d}X_m^{-1}(\tau) \cdot (f_m(\tau) - f_m(a)) = \int_a^t \mathrm{d}X^{-1}(\tau) \cdot (f(\tau) - f(a)) \tag{3.35}$$

uniformly on $[a,b]$.

Using (3.3), (3.6), (3.7), (3.17), (3.29), (3.30), (3.31) and (3.35), from (3.32) we get

$$\lim_{m \to \infty} \|x_k - z\|_\infty = 0,$$

where

$$z(t) = X(t)(l(X))^{-1}(c_0 - l(F(f))) + F(f)(t),$$
$$F(f)(t) = f(t) - f(a) - X(t) \int_a^t \mathrm{d}X^{-1}(\tau) \cdot (f(\tau) - f(a)).$$

It is easy to verify that the vector valued function $z\colon\ [a,b] \to \mathbb{R}$ is the solution of problem (3.1), (3.2). Therefore $x_0(t) = z(t)$ for $t \in [a,b]$. □

P r o o f of Theorem 3.1. First we prove the sufficiency. Due to (2.5) and (3.10) we can assume without loss of generality that

$$\lim_{m\to\infty} \|H_m^{-1} - H^{-1}\|_\infty = 0. \tag{3.36}$$

By this and Lemma 2.2 from [2], we conclude for every $m \in \widetilde{\mathbb{N}}$ that the function $x \in \mathrm{BV}([a,b];\mathbb{R}^n)$ is a solution of problem (3.1_m), (3.2_m) if and only if the function $y(t) \equiv H_m(t)\,x(t)$ is a solution of the problem

$$\mathrm{d}y = \mathrm{d}A_m^*(t)\cdot y + \mathrm{d}f_m^*(t), \quad l_m^*(y) = c_m,$$

where

$$A_m^*(t) \equiv \mathcal{I}(H_m, A_m)(t), \quad f_m^*(t) \equiv \mathcal{B}(H_m, f_m)(t), \quad l_m^*(y) \equiv l_m(H_m^{-1}y), \quad m \in \widetilde{\mathbb{N}}.$$

In addition, inclusion (3.5) holds if and only if

$$((A_m^*, f_m^*; l_m^*))_{m=1}^\infty \in \mathcal{S}(A_0^*, f_0^*; l_0^*).$$

Moreover, the conditions of the theorem concerning the sufficient case, coincide with the ones of Theorem 3.3 for the introduced problems. Thus, the sufficiency follows from Theorem 3.3.

Let us show the necessity. Let inclusion (3.5) hold. Let $c_m \in \mathbb{R}^n$ $(m = 0,1,\ldots)$ be an arbitrary sequence of constant vectors satisfying condition (3.3) and let $e_j = (\delta_{ij})_{i=1}^n$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$, $i,j = 1,\ldots,n$ (Kronecker symbol).

In view of (3.5) we may assume that problem (3.1_m), (3.2_m) has a unique solution x_m for every natural m without loss of generality.

For any $m \in \widetilde{\mathbb{N}}$ and $j \in \{1,\ldots,n\}$ let us denote $z_{mj}(t) \equiv x_m(t) - x_{mj}(t)$, where x_{mj} is the unique solution of system (3.1_m) under the boundary value condition $l_m(x) = c_m - e_j$. Moreover, let $X_m(t)$ be the matrix valued function whose columns are $z_{m1}(t),\ldots,z_{mn}(t)$. It is evident that

$$l_m(z_{mj}) = e_j, \quad j = 1,\ldots,n;\ m = 0,1,\ldots \tag{3.37}$$

So, if $\sum\limits_{j=1}^n \alpha_j z_{mj}(t) \equiv 0$ for some $m \in \widetilde{\mathbb{N}}$ and $\alpha_1,\ldots,\alpha_n \in \mathbb{R}$, then using (3.37) we have $\sum\limits_{j=1}^n \alpha_j e_j = 0$ and therefore $\alpha_1 = \ldots = \alpha_n = 0$, i.e. X_m $(X_0(t) \equiv X(t))$ is the fundamental matrix of the homogeneous system (3.1_{m0}).

We may assume without loss of generality that $X_m(a) = I_n$, $m \in \widetilde{\mathbb{N}}$. Due to (3.5), condition (3.29) holds and therefore using Lemma 2 from [1] we get that (3.33) holds.

Let us assume that $H_m(t) \equiv X_m^{-1}(t)$ $(m \in \widetilde{\mathbb{N}})$ and verify conditions (3.9)–(3.12) of the theorem. Condition (2.5) is evident because $H(t) = H_0(t) \equiv X_0^{-1}(t)$, and X_0 is the fundamental matrix of system (3.1_0). By (3.33), condition (3.10) holds uniformly on $[a,b]$. According to Proposition III.2.15, from [15] we have

$$X_m^{-1}(t) \equiv I_n - \mathcal{B}(X_m^{-1}, A_m)(t), \quad m \in \widetilde{\mathbb{N}}. \tag{3.38}$$

Therefore,

$$H_m(t) + \mathcal{B}(H_m, A_m)(t) \equiv I_n, \quad m \in \widetilde{\mathbb{N}}. \tag{3.39}$$

So condition (3.9) holds.

Due to (3.33), condition (3.38) implies that (3.11) is fulfilled uniformly on $[a,b]$. On the other hand, by (1.3), (3.38) and the definition of the solution of system (3.1_m) we find

$$\begin{aligned}
\mathcal{B}(H_m, f_m)(t) &= \mathcal{B}\bigg(H_m, x_m - \int_a^{\cdot} \mathrm{d}A_m(s)\cdot x_m(s)\bigg)(t)\\
&= \mathcal{B}(H_m, x_k)(t) - \mathcal{B}\bigg(H_m, \int_a^{\cdot} \mathrm{d}A_m(s)\cdot x_m(s)\bigg)(t)\\
&= \mathcal{B}(X_m^{-1}, x_k)(t) - \int_a^t \mathrm{d}\mathcal{B}(X_m^{-1}, A_m)(s)\cdot x_m(s)\\
&= X_m^{-1}(t)x_m(t) - x_m(a) - \int_a^t \mathrm{d}Y_m^{-1}(s)\cdot x_k(s)\\
&\quad - \int_a^t \mathrm{d}(I_n - X_m^{-1}(s))\cdot x_k(s)\\
&= X_m^{-1}(t)x_m(t) - x_m(a) \quad \text{for } t \in [a,b],\ m \in \widetilde{\mathbb{N}}.
\end{aligned}$$

Hence

$$\mathcal{B}(H_m, f_m)(t) \equiv H_m(t)\, x_m(t) - x_m(a), \quad m \in \widetilde{\mathbb{N}}.$$

By this and (3.33), if we take into account that due to the necessity of the theorem, condition (3.4) holds, we conclude that condition (3.12) holds uniformly on I, as well. The theorem is proved. □

P r o o f of Theorem 3.2. The theorem immediately follows from the proof of the necessity of Theorem 3.1. □

P r o o f of Corollary 3.1. By (3.26) and (3.27) (or (3.28)) we have

$$\lim_{m\to\infty}\sum_{a<s\leqslant t}(\mathrm{d}_1H_m(s)\cdot\mathrm{d}_1A_m(s)-\mathrm{d}_1H(s)\cdot\mathrm{d}_1A(s))=O_{n\times n},$$

$$\lim_{m\to\infty}\sum_{a<s\leqslant t}(\mathrm{d}_1H_m(s)\cdot\mathrm{d}_1f_m(s)-\mathrm{d}_1H(s)\cdot\mathrm{d}_1f(s))=0_n,$$

$$\lim_{m\to\infty}\sum_{a\leqslant s<t}(\mathrm{d}_2H_m(s)\cdot\mathrm{d}_2A_m(s)-\mathrm{d}_2H(s)\cdot\mathrm{d}_2A(s))=O_{n\times n}$$

and

$$\lim_{m\to\infty}\sum_{a\leqslant s<t}(\mathrm{d}_2H_m(s)\cdot\mathrm{d}_2f_m(s)-\mathrm{d}_2H(s)\cdot\mathrm{d}_2f(s))=0_n$$

uniformly on $[a,b]$. From these, integration-by-parts formula (1.5), (3.24) and (3.25), we get that conditions (3.11) and (3.12) are fulfilled uniformly on $[a,b]$. So, the corollary follows from Theorem 3.1. □

The proofs of Theorem 3.4 and Corollary 3.2 can be found in paper [2]—they coincide, respectively, with Corollary 1.4 and Corollary 1.5 from this paper.

R e m a r k 3.1. In Theorem 3.2, equality (3.11) from Theorem 3.1 has the form

$$\lim_{m\to\infty}\mathcal{B}(X_m^{-1},A_m)(t)=\mathcal{B}(X_0^{-1},A)(t),$$

which evidently holds due to equalities (3.14) and (3.38) for every $m\in\widetilde{\mathbb{N}}$. Moreover, by (3.38) condition (3.9) is valid.

R e m a r k 3.2. Using equality (1.7), from (3.1_{m0}) by the definition of the solution of the homogeneous system, we conclude

$$\mathrm{d}_jX_m(t)\equiv\mathrm{d}_jA(t)\cdot X_m(t),\quad j=1,2;\ m=0,1,\dots$$

and therefore by (3.13)

$$\mathrm{d}_jX_m^{-1}(t)\equiv -X_m^{-1}(t)(I_n+(-1)^j\mathrm{d}_jA(t))^{-1}\mathrm{d}_jA(t),\quad j=1,2;\ m=0,1,\dots$$

If we take into account these equalities, due to integration-by-parts formula (1.5) and (1.6) we obtain

$$\mathcal{B}(X_m^{-1},A_m)(t)\equiv\int_a^t X_m^{-1}(\tau)\mathrm{d}\mathcal{D}_m(A_m)(\tau),\quad m\in\widetilde{\mathbb{N}}$$

and

$$\mathcal{B}(X_m^{-1},f_m)(t)\equiv\int_a^t X_m^{-1}(\tau)\mathrm{d}\mathcal{D}_m(f_m)(\tau),\quad m\in\widetilde{\mathbb{N}},$$

where $\mathcal{D}_m(X)\colon \operatorname{BV}([a,b]:\mathbb{R}^{n\times l}) \to \mathbb{R}^{n\times l}$, $m\in\widetilde{\mathbb{N}}$, are the operators defined by

$$\begin{aligned}\mathcal{D}_m(X)(t) = X(t) + \sum_{a<\tau\leqslant t} (I_n - \mathrm{d}_1 A_m(\tau))^{-1}\mathrm{d}_1 A_m(\tau) X(\tau)\\ - \sum_{a\leqslant\tau<t} (I_n + \mathrm{d}_2 A_m(\tau))^{-1}\mathrm{d}_2 A_m(\tau)\, X(\tau) \quad \text{for } t\in[a,b],\ m\in\widetilde{\mathbb{N}}.\end{aligned}$$

R e m a r k 3.3. In all theorems and corollaries, we can assume without loss of generality that $H(t)\equiv I_n$.

R e m a r k 3.4. In Theorems 3.1, 3.3 and 3.4 and Corollaries 3.1, 3.2, if condition (3.13) holds, we can assume that $H_m(t)\equiv Y_m^{-1}(t)$, where Y_m is the fundamental matrix of the homogeneous system (3.1_{m0}) for every $m\in\widetilde{\mathbb{N}}$. In addition, condition (3.9) automatically holds, because by (3.39) its left-hand side equals to 1. Analogous condition, i.e. the condition concerning the upper limits, is automatically held everywhere, as well.

4. Proofs of the main results

4.1. Proofs of Theorems 2.1–2.4. Due to the definition of the solutions of the generalized system (3.1) we conclude that the vector valued function $x\in AC([a,b];\mathbb{R}^n)$ is a solution of system (1.1) if and only if it is a solution of system (3.1), where

$$A(t)\equiv\int_a^t P(\tau)\,\mathrm{d}\tau,\quad f(t)\equiv\int_a^t q(\tau)\,\mathrm{d}\tau.$$

Moreover, by the Hahn-Banach theorem there exists a bounded vector valued functional $l_*\colon \operatorname{BV}([a,b];\mathbb{R}^n)\to\mathbb{R}^n$ such that

$$l_*(x) = l(x) \quad \text{for } x\in C([a,b];\mathbb{R}^n)$$

and the norm of the operator l_* on $\operatorname{BV}([a,b];\mathbb{R}^n)$ equals to the norm of the operator l on $C([a,b];\mathbb{R}^n)$, i.e. $|||l_*||| = |||l|||$.

So we can assume that $l_*(x)\equiv l(x)$ without loss of generality. Therefore problem (1.1), (1.2) is equivalent to problem (3.1), (3.2).

Consider now the difference boundary value problem (1.1_m), (1.2_m), where $m\in\mathbb{N}$. For every natural m we define the matrix valued and vector valued functions $A_m\in \operatorname{BV}([a,b];\mathbb{R}^{n\times n})$ and $f_m\in\operatorname{BV}([a,b];\mathbb{R}^n)$ and the bounded vector valued functional

$l_m\colon \mathrm{BV}([a,b];\mathbb{R}^n)\to\mathbb{R}^n$, respectively, by the equalities

$$(4.1)\quad A_m(a)=A_m(\tau_{0m})=O_{n\times n},\quad A_m(\tau_{km})=\frac{1}{m}\Big(\sum_{i=0}^{k}G_{1m}(i)+\sum_{i=1}^{k}G_{2m}(i-1)\Big),$$

$$A_m(t)=\frac{1}{m}\Big(\sum_{i=0}^{k-1}G_{1m}(i)+\sum_{i=1}^{k}G_{2m}(i-1)\Big)\quad\text{for } t\in\,]\tau_{k-1\,m},\tau_{k\,m}[,\ k\in\mathbb{N}_m;$$

$$(4.2)\quad f_m(a)=f(\tau_{0m})=0_n,\quad f_m(\tau_{km})=\frac{1}{m}\Big(\sum_{i=0}^{k}g_{1m}(i)+\sum_{i=1}^{k}g_{2m}(i-1)\Big),$$

$$f_m(t)=\frac{1}{m}\Big(\sum_{i=0}^{k-1}g_{1m}(i)+\sum_{i=1}^{k}g_{2m}(i-1)\Big)\quad\text{for } t\in\,]\tau_{k-1\,m},\tau_{k\,m}[,\ k\in\mathbb{N}_m;$$

$$(4.3)\quad l_m(x)=\mathcal{L}_m(p_m(x))\quad\text{for } x\in\mathrm{BV}([a,b];\mathbb{R}^n),\ c_m=\gamma_m.$$

It is not difficult to verify that the defined matrix valued and vector valued functions have the following properties:

$$(4.4)\quad \mathrm{d}_1A_m(\tau_{k\,m})=\frac{1}{m}G_{1m}(k),\quad \mathrm{d}_2A_m(\tau_{km})=\frac{1}{m}G_{2m}(k),\quad k=1,\dots,m,$$
$$\mathrm{d}_jA_m(t)=O_{n\times n}\quad\text{for } t\in[a,b]\setminus\{\tau_{1m},\dots,\tau_{km}\},\ j=1,2;$$

$$(4.5)\quad \mathrm{d}_1f_m(\tau_{k\,m})=\frac{1}{m}g_{1m}(k),\quad \mathrm{d}_2f_m(\tau_{km})=\frac{1}{m}g_{2m}(k),\quad k=1,\dots,m,$$
$$\mathrm{d}_jf_m(t)=0_n\quad\text{for } t\in[a,b]\setminus\{\tau_{1m},\dots,\tau_{km}\},\ j=1,2$$

for every $m\in\mathbb{N}$.

Lemma 4.1. *Let m be an arbitrary natural number. Then the vector valued function $y\in\mathbb{E}(\widetilde{N}_m;\mathbb{R}^n)$ is a solution of the difference problem* (1.1_m), (1.2_m) *if and only if the vector valued function $x=q_m(y)\in\mathrm{BV}([a,b];\mathbb{R}^n)$ is a solution of the generalized problem* (3.1_m), (3.2_m), *where the matrix valued and vector valued functions $A_m\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and $f_m\in\mathrm{BV}([a,b];\mathbb{R}^n)$ and the bounded vector valued functional l_m are defined by* (4.1)–(4.3), *respectively.*

P r o o f of Lemma 4.1. Let $y\in\mathbb{E}(\widetilde{N}_m;\mathbb{R}^n)$ be a solution of system (1.1_m), $m\in\mathbb{N}$. Then by (1.6), (1.7) and the equality $x(\tau_{km})=q_m(y)(\tau_{km})=y(k)$, $k\in\widetilde{\mathbb{N}}_m$, we get

$$\begin{aligned}&\int_{\tau_{k-1\,m}}^{\tau_{km}}\mathrm{d}A_m(\tau)x_m(\tau)+f(\tau_{km})-f(\tau_{k-1\,m})\\
&\quad=\frac{1}{m}G_{1m}(k)x_m(\tau_{km})+\frac{1}{m}G_{2m}(k-1)x_m(\tau_{k-1\,m})+\frac{1}{m}g_{1m}(k)+\frac{1}{m}g_{2m}(k-1)\\
&\quad=\frac{1}{m}G_{1m}(k)y(k)+\frac{1}{m}G_{2m}(k-1)y(k-1)+\frac{1}{m}g_{1m}(k)+\frac{1}{m}g_{2m}(k-1)\\
&\quad=\Delta y(k-1)=x_m(\tau_{km})-x_m(\tau_{k-1\,m})\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}_1 x_m(\tau_{km}) &= x_m(\tau_{km}) - x_m(\tau_{km}-) = \frac{1}{m} G_{1m}(k) y(k) + \frac{1}{m} g_{1m}(k) \\
&= \mathrm{d}_1 A_m(\tau_{km}) + \mathrm{d}_1 f_m(\tau_{km}), \quad k \in \mathbb{N}_m; \\
\mathrm{d}_2 x_m(\tau_{k-1\,m}) &= x_m(\tau_{k-1\,m}+) - x_m(\tau_{k-1\,m}) \\
&= y(k) - y(k-1) - \frac{1}{m} G_{1m}(k) y(k) - \frac{1}{m} g_{1m}(k) \\
&= \frac{1}{m} G_{2m}(k-1) y(k-1) + \frac{1}{m} g_{2m}(k-1) \\
&= \mathrm{d}_2 A_m(\tau_{k-1\,m}) + \mathrm{d}_2 f_m(\tau_{k-1\,m})
\end{aligned}$$

for every $m \in \mathbb{N}$ and $k \in \mathbb{N}_m$.

Analogously, we show that if the vector valued function $x \in \mathrm{BV}([a,b];\mathbb{R}^n)$ is a solution of the generalized problem (3.1_m), (3.2_m) defined above, than the vector valued function $y(k) = p_m(x)(k)$ $(k = 1,\ldots,m)$ will be a solution of the difference problem (1.1_m), (1.2_m) for every natural m. □

So, we show that the convergence of the difference schemes (3.1_m), (3.2_m), $m \in \mathbb{N}$, is equivalent to the well-possed question for the corresponding linear generalized boundary value problem (3.1), (3.2).

In view of Definitions 2.1 and 3.1, the following lemma is true.

Lemma 4.2. *Inclusion* (2.3) *holds if and only if inclusion* (3.5) *holds, where the $n \times n$-matrix valued functions A, A_m, n-vector valued functions f, f_m and n-vector valued functionals l, l_m, $m = 1, 2, \ldots$, are defined as above, by* (4.1)–(4.3), *respectively.*

In order to use Theorems (3.1)–(3.4) and Corollaries 3.1, 3.2, we need to establish the forms of the operators applying in those results for particular case which correspond to the matrix valued and vector valued functions and vector valued functional defined by (4.1)–(4.3).

Let H, H_m, $m \in \mathbb{N}$, be the matrix valued functions appearing in Theorem 3.1. It follows from the proof of this theorem that the matrix valued functions H_m $(m \in \mathbb{N})$ appearing in the proof have the property analogous to matrix valued functions A_m, $m \in \mathbb{N}$. In particular, we can assume that $H_m(t) = I_n$ for $t \in \,]\tau_{k-1\,m}, \tau_{k\,m}[$, $k \in \widetilde{\mathbb{N}}_m$, $m \in \mathbb{N}$. So we have

$$H_m(\tau_{k-1\,m}+) = H_m(\tau_{km}-), \quad k \in \widetilde{\mathbb{N}}_m,\ m \in \mathbb{N}. \tag{4.6}$$

Due to the definition of the operator $\mathcal{B}$, integration-by-parts formula (1.5) and equalities (1.6) we have

$$\begin{aligned}\mathcal{B}(H_m,A_m)(t) &= \int_a^t H_m(\tau)\,\mathrm{d}A_m(\tau) - \sum_{a<\tau\leqslant t} \mathrm{d}_1H_m(\tau)\cdot \mathrm{d}_1A_m(\tau)\\ &\quad + \sum_{a\leqslant\tau<t} \mathrm{d}_2H_m(\tau)\cdot \mathrm{d}_2A_m(\tau)\\ &= \sum_{a<\tau\leqslant t} H_m(\tau)\mathrm{d}_1A_m(\tau) + \sum_{a\leqslant\tau<t} H_m(\tau)\mathrm{d}_2A_m(\tau)\\ &\quad - \sum_{a<\tau\leqslant t} \mathrm{d}_1H_m(\tau)\cdot \mathrm{d}_1A_m(\tau) + \sum_{a\leqslant\tau<t} \mathrm{d}_2H_m(\tau)\cdot \mathrm{d}_2A_m(\tau)\end{aligned}$$

for $t\in[a,b]$, $m\in\mathbb{N}$ and therefore

$$\begin{aligned}\mathcal{B}(H_m,A_m)(t) &= \sum_{k=1}^{\nu_m(t)} H_m(\tau_{km}-)\mathrm{d}_1A_m(\tau_{km})\\ &\quad + \sum_{k=0}^{\nu_m(t)-1} H_m(\tau_{km}+)\mathrm{d}_2A_m(\tau_{km}) \quad \text{for } t\in[a,b],\ m\in\mathbb{N}.\end{aligned} \tag{4.7}$$

Analogously, we show that

$$\begin{aligned}\mathcal{B}(H_m,f_m)(t) &= \sum_{k=1}^{\nu_m(t)} H_m(\tau_{km}-)\mathrm{d}_1f_m(\tau_{km})\\ &\quad + \sum_{k=0}^{\nu_m(t)-1} H_m(\tau_{km}+)\mathrm{d}_2f_m(\tau_{km}) \quad \text{for } t\in[a,b],\ m\in\mathbb{N}.\end{aligned} \tag{4.8}$$

Let

$$H_{1m}(k) = H_m(\tau_{km}-) \quad \text{and} \quad H_{2m}(k) = H_m(\tau_{km}), \quad k\in\widetilde{\mathbb{N}}_m,\ m\in\mathbb{N}.$$

Then due to (4.6) we get

$$H_m(\tau_{k-1\,m}+) = H_{1m}(k), \quad k\in\widetilde{\mathbb{N}}_m,\ m\in\mathbb{N}.$$

From this and equalities (4.7) and (4.8), using equalities (4.4) and (4.5), for every natural m we obtain

$$\mathcal{B}(H_m,A_m)(t) = \frac{1}{m}\sum_{k=1}^{\nu_m(t)} H_{1m}(k)(G_{1m}(k)+G_{2m}(k-1)) \tag{4.9}$$

and

$$\mathcal{B}(H_m,f_m)(t) = \frac{1}{m}\sum_{k=1}^{\nu_m(t)} H_{1m}(k)(g_{1m}(k)+g_{2m}(k-1)) \quad \text{for } t\in[a,b]. \tag{4.10}$$

Moreover, for every natural m we have the equalities

$$\mathrm{d}_j H_m(t) = \mathrm{d}_j \mathcal{B}(H_m, A_m)(t) = O_{n\times n}, \tag{4.11}$$

$$\mathrm{d}_j \mathcal{B}(H_m, f_m)(t) = 0_n \quad \text{for } t \in [a,b] \setminus \{\tau_0, \dots, \tau_m\},\ j = 1,2,$$

$$\mathrm{d}_j H_m(\tau_{km}) = (-1)^j (H_{1m}(k+j-1) - H_{2m}(k)), \tag{4.12}$$

$$\mathrm{d}_j \mathcal{B}(H_m, A_m)(\tau_{km}) = \frac{1}{m} H_{1m}(k+j-1) G_{jm}(k) \quad \text{for } k \in \widetilde{\mathbb{N}}_m,\ j = 1,2.$$

Hence, by (4.9)–(4.12) we conclude

$$\bigvee_a^b (H_m + \mathcal{B}(H_m, A_m)) = \sum_{k=1}^m \Big(\Big\| H_{2m}(k) - H_{1m}(k) + \frac{1}{m} H_{1m}(k) G_{1m}(k) \Big\| + \Big\| H_{1m}(k) - H_{2m}(k-1) + \frac{1}{m} H_{1m}(k)\, G_{2m}(k-1) \Big\| \Big), \quad m \in \mathbb{N}. \tag{4.13}$$

Thanks to Lemmas 4.1, 4.2 and equalities (4.9)–(4.13), we conclude that Theorems 3.1–3.4 have the forms of Theorems 2.1–2.4, respectively, and Corollaries 3.1, 3.2 have the forms of Proposition 2.1 and Corollary 2.1, respectively, in the considered case.

4.2. Proof of Theorem 2.5. As above we show that problems (2.17), (2.18) and (2.17_m), (2.18_m), $m \in \mathbb{N}$, are equivalent to the generalized boundary value problems (3.1), (3.2) and (3.1_m), (3.2_m), $m \in \mathbb{N}$, respectively, where $x = q_{m_0}(y)$,

$$A_m(a) = A_m(\tau_{0m_0}) = O_{n\times n}, \quad A_m(\tau_{km_0}) = \sum_{i=0}^k G_{1m}(i) + \sum_{i=1}^k G_{2m}(i-1)$$

and

$$A_m(t) = \sum_{i=0}^{k-1} G_{1m}(i) + \sum_{i=1}^k G_{2m}(i-1) \quad \text{for } t \in \,]\tau_{k-1\,m}, \tau_{k\,m}[,\ k \in \mathbb{N}_{m_0};$$

$$f_m(a) = f_m(\tau_{0m_0}) = 0_n, \quad f_m(\tau_{km_0}) = \sum_{i=0}^k g_{1m}(i) + \sum_{i=1}^k g_{2m}(i-1)$$

and

$$f_m(t) = \sum_{i=0}^{k-1} g_{1m}(i) + \sum_{i=1}^k g_{2m}(i-1) \quad \text{for } t \in \,]\tau_{k-1\,m}, \tau_{k\,m}[,\ k \in \mathbb{N}_{m_0};$$

$$l_m(x) = \mathcal{L}_m(p_{m_0}(x)) \quad \text{for } x \in \mathrm{BV}([a,b];\mathbb{R}^n),\ c_m = \gamma_m$$

for every $m \in \widetilde{\mathbb{N}}$. Here we assume that $A_0(t) \equiv A(t)$, $f_0(t) \equiv f(t)$, $l_0(x) \equiv l(x)$; $G_{j0}(k) \equiv G_j(k)$, $g_{j0}(k) \equiv g_j(k)$, $j = 1,2$; $\mathcal{L}_0(y) \equiv \mathcal{L}_m(y)$.

In addition, Definition 2.2 is equivalent to Definition 3.1. So, in this case, Theorem 3.1 has the form of Theorem 2.5, Corollary 3.2 has the form of Corollary 2.2.

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