146 (2021) MATHEMATICA BOHEMICA No. 4, 397–406

FUZZY DIFFERENTIAL SUBORDINATIONS CONNECTED WITH THE LINEAR OPERATOR

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Received November 15, 2019. Published online November 2, 2020.
Communicated by Dagmar Medková

Abstract. We obtain several fuzzy differential subordinations by using a linear operator $\mathcal{I}_{m,\gamma}^{n,\alpha} f(z) = z + \sum\limits_{k=2}^{\infty} (1+\gamma(k-1))^n m^{\alpha}(m+k)^{-\alpha} a_k z^k$. Using the linear operator $\mathcal{I}_{m,\gamma}^{n,\alpha}$, we also introduce a class of univalent analytic functions for which we give some properties.

Keywords: fuzzy differential subordination; fuzzy best dominant; linear operator

MSC 2020: 30C45

1. Introduction

Let $\mathcal{D} \subset \mathbb{C}$, $\mathcal{H}(\mathcal{D})$ be the class of holomorphic functions on $\mathcal{D}$ and denote by $\mathcal{H}_n(\mathcal{D})$ the class of holomorphic and univalent functions on $\mathcal{D}$. In this paper, we denote by $\mathcal{H}(\mathbb{U})$ the class of holomorphic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}\colon\ |z| < 1\}$ with $\partial\mathbb{U} = \{z \in \mathbb{C}\colon\ |z| = 1\}$ the boundary of the unit disc. For $a \in \mathbb{C}$ and $k \in \mathbb{N}$, we denote

$$\mathcal{H}[a,k] = \{f \in \mathcal{H}(\mathbb{U})\colon\ f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \ldots,\ z \in \mathbb{U}\},$$
$$\mathcal{A}_k = \{f \in \mathcal{H}(\mathbb{U})\colon\ f(z) = z + a_{k+1} z^{k+1} + \ldots,\ z \in \mathbb{U}\} \text{ with } \mathcal{A}_1 = \mathcal{A},$$

and

$$\mathcal{S} = \{f \in \mathcal{A}\colon\ f \text{ is a univalent function in } \mathbb{U}\}.$$

Denote by

$$\mathcal{K} = \Big\{f \in \mathcal{A} : \Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > 0,\ z \in \mathbb{U}\Big\}$$

the class of convex functions in $\mathbb{U}$.

DOI: 10.21136/MB.2020.0159-19

Definition 1.1 ([7] and [13]). Let f and g be analytic functions in $\mathbb{U}$. We say that f is *subordinate* to g, written $f \prec g$, if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In order to introduce the notion of fuzzy differential subordination, we use the following definitions and propositions:

Definition 1.2 ([10]). Assume that $\mathcal{Y}$ be a nonempty set. An application $\mathcal{F}\colon \mathcal{Y} \to [0,1]$ is called *fuzzy subset*. A pair $(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$, where $\mathcal{F}_{\mathcal{B}}\colon \mathcal{Y} \to [0,1]$ and

$$\mathcal{B} = \{x \in \mathcal{Y}\colon\ 0 < \mathcal{F}_{\mathcal{B}}(x) \leqslant 1\} = \sup(\mathcal{B}, \mathcal{F}_{\mathcal{B}}), \tag{1.1}$$

is called *fuzzy set*. The function $\mathcal{F}_{\mathcal{B}}$ is called *membership function* of the fuzzy set $(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$.

Proposition 1.1 ([14]).

(i) *If* $(\mathcal{B}, \mathcal{F}_{\mathcal{B}}) = (\mathcal{U}, \mathcal{F}_{\mathcal{U}})$, *then we have* $\mathcal{B} = \mathcal{U}$, *where* $\mathcal{B} = \sup(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$ *and* $\mathcal{U} = \sup(\mathcal{U}, \mathcal{F}_{\mathcal{U}})$.

(ii) *If* $(\mathcal{B}, \mathcal{F}_{\mathcal{B}}) \subseteq (\mathcal{U}, \mathcal{F}_{\mathcal{U}})$, *then we have* $\mathcal{B} \subseteq \mathcal{U}$, *where* $\mathcal{B} = \sup(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$ *and* $\mathcal{U} = \sup(\mathcal{U}, \mathcal{F}_{\mathcal{U}})$.

Let $f, g \in \mathcal{H}(\mathcal{D})$. We denote

$$f(\mathcal{D}) = \{f(z)\colon\ 0 < \mathcal{F}_{f(\mathcal{D})} f(z) \leqslant 1,\ z \in \mathcal{D}\} = \sup(f(\mathcal{D}), \mathcal{F}_{f(\mathcal{D})}) \tag{1.2}$$

and

$$g(\mathcal{D}) = \{g(z)\colon\ 0 < \mathcal{F}_{g(\mathcal{D})} g(z) \leqslant 1,\ z \in \mathcal{D}\} = \sup(g(\mathcal{D}), \mathcal{F}_{g(\mathcal{D})}). \tag{1.3}$$

Definition 1.3 ([14]). Let $z_0 \in \mathcal{D}$ and $f, g \in \mathcal{H}(\mathcal{D})$. The function f is said to be *fuzzy subordinate to* g, written $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$, when following conditions are satisfied:

(i) $f(z_0) = g(z_0)$,

(ii) $\mathcal{F}_{f(\mathcal{D})} f(z) \leqslant \mathcal{F}_{g(\mathcal{D})} g(z)$, $z \in \mathcal{D}$.

Proposition 1.2 ([14]). *Assume that* $z_0 \in \mathcal{D}$ *and* $f, g \in \mathcal{H}(\mathcal{D})$. *If* $f(z) \prec_{\mathcal{F}} g(z)$, $z \in \mathcal{D}$, *then*

(i) $f(z_0) = g(z_0)$,

(ii) $f(\mathcal{D}) \subseteq g(\mathcal{D})$, $\mathcal{F}_{f(\mathcal{D})} f(z) \leqslant \mathcal{F}_{g(\mathcal{D})} g(z)$, $z \in \mathcal{D}$, *where* $f(\mathcal{D})$ *and* $g(\mathcal{D})$ *are defined by* (1.2) *and* (1.3), *respectively.*

Definition 1.4 ([16]). Assume that $\Phi\colon \mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$ and $h\in\mathcal{S}$ with $\Phi(a,0,0;0)=h(0)=a$. If p is analytic in $\mathbb{U}$ with $p(0)=a$ and satisfies the second order fuzzy differential subordination

$$\mathcal{F}_{\Phi(\mathbb{C}^3\times\mathbb{U})}\Phi(p(z),zp'(z),z^2p''(z);z)\leqslant\mathcal{F}_{h(\mathbb{U})}h(z),\qquad\qquad (1.4)$$
$$\text{i.e. } \Phi(p(z),zp'(z),z^2p''(z);z)\prec_{\mathcal{F}} h(z),\quad z\in\mathbb{U},$$

then p is called a *fuzzy solution* of the fuzzy differential subordination. The univalent function q is called a *fuzzy dominant* of the fuzzy solutions for the fuzzy differential subordination if

$$\mathcal{F}_{p(\mathbb{U})}p(z)\leqslant\mathcal{F}_{q(\mathbb{U})}q(z),\quad \text{i.e. } p(z)\prec_{\mathcal{F}} q(z),\ z\in\mathbb{U}$$

for all p satisfying (1.4). A fuzzy dominant $\tilde{q}$ that satisfies

$$\mathcal{F}_{\tilde{q}(\mathbb{U})}\tilde{q}(z)\leqslant\mathcal{F}_{q(\mathbb{U})}q(z),\quad \text{i.e. } \tilde{q}(z)\prec_{\mathcal{F}} q(z),\ z\in\mathbb{U}$$

for all fuzzy dominants q of (1.4) is said to be the *fuzzy best dominant* of (1.4).

The *integral operator* $\mathcal{K}_m^\alpha\colon \mathcal{A}\to\mathcal{A}$ is defined for $\alpha>0$ and $m>-1$ as follows (see Komatu [12] with $p=1$):

$$\mathcal{K}_m^0 f(z)=f(z)=z+\sum_{k=1}^{\infty}a_{k+1}z^{k+1}$$

and

$$\mathcal{K}_m^\alpha f(z)=\frac{(m+1)^\alpha}{\Gamma(\alpha)z^m}\int_0^z t^{m-1}\Big(\log\frac{z}{t}\Big)^{\alpha-1}f(t)\,\mathrm{d}t.\qquad\qquad (1.5)$$

It can be easily verified that

$$\mathcal{K}_m^\alpha f(z)=z+\sum_{k=1}^{\infty}\Big(\frac{m+1}{m+k+1}\Big)^\alpha a_{k+1}z^{k+1}.\qquad\qquad (1.6)$$

El-Ashwah et al. (see [9] with $p=1$) introduced the *linear multiplier operator* $\mathcal{I}_{m,\gamma}^{n,\alpha}\colon \mathcal{A}\to\mathcal{A}$ given as:

$$\begin{aligned}
\mathcal{I}_{m,\gamma}^{0,0}f(z)&=f(z),\\
\mathcal{I}_{m,\gamma}^{1,\alpha}f(z)&=\mathcal{I}_{m,\gamma}^{\alpha}f(z)=(1-\gamma)\mathcal{K}_m^\alpha f(z)+\gamma z(\mathcal{K}_m^\alpha f(z))'\\
&=z+\sum_{k=1}^{\infty}(1+\gamma k)\Big(\frac{m+1}{m+k+1}\Big)^\alpha a_{k+1}z^{k+1},\\
\mathcal{I}_{m,\gamma}^{2,\alpha}f(z)&=\mathcal{I}_{m,\gamma}^{\alpha}(\mathcal{I}_{m,\gamma}^{1,\alpha}f(z))\\
&=z+\sum_{k=1}^{\infty}(1+\gamma k)^2\Big(\frac{m+1}{m+k+1}\Big)^\alpha a_{k+1}z^{k+1},
\end{aligned}$$

in general,

(1.7) $$\mathcal{I}_{m,\gamma}^{n,\alpha} f(z) = \mathcal{I}_{m,\gamma}^{\alpha}(\mathcal{I}_{m,\gamma}^{n-1,\alpha} f(z)) = z + \sum_{k=1}^{\infty} (1+\gamma k)^n \Big(\frac{m+1}{m+k+1}\Big)^{\alpha} a_{k+1} z^{k+1},$$
$$\alpha \geqslant 0,\ m > -1,\ \gamma \geqslant 0,\ n \in \mathbb{N}_0.$$

It is easily verified from (1.7) that

(1.8) $$z(\mathcal{I}_{m,\gamma}^{n,\alpha} f(z))' = (m+1)\mathcal{I}_{m,\gamma}^{n,\alpha-1} f(z) - m\mathcal{I}_{m,\gamma}^{n,\alpha} f(z)$$

and

(1.9) $$\gamma z(\mathcal{I}_{m,\gamma}^{n,\alpha} f(z))' = \mathcal{I}_{m,\gamma}^{n+1,\alpha} f(z) - (1-\gamma)\mathcal{I}_{m,\gamma}^{n,\alpha} f(z).$$

By specializing the parameters m, γ, α and n, we obtain the following operators:

$$\begin{aligned}
\mathcal{I}_{c,\gamma}^{0,\delta} f(z) &= \mathcal{K}_{c,1}^{\delta} f(z) && \text{(see [11]);}\\
\mathcal{I}_{a-1,\gamma}^{0,\alpha} f(z) &= L_a^{\alpha} f(z) && \text{(see [4] and [5]);}\\
\mathcal{I}_{c,0}^{1,\alpha} f(z) &= \mathcal{P}_c^{\alpha} f(z) && \text{(see [12] and [17]);}\\
\mathcal{I}_{c,0}^{1,1} f(z) &= \mathcal{L}_c f(z) && \text{(see [6]);}\\
\mathcal{I}_{1,0}^{1,\alpha} f(z) &= \mathcal{I}^{\alpha} f(z) && \text{(see [8]);}\\
\mathcal{I}_{m,\gamma}^{n,0} f(z) &= \mathcal{D}_{\gamma}^{n} f(z) && \text{(see [1]);}\\
\mathcal{I}_{m,1}^{n,0} f(z) &= \mathcal{D}^{n} f(z) && \text{(see [18]).}
\end{aligned}$$

Also, we note that

(1.10) $$\mathcal{I}_{m,\gamma}^{-n,\alpha} f(z) = z + \sum_{k=1}^{\infty} \Big(\frac{1}{1+\gamma k}\Big)^n \Big(\frac{m+1}{m+k+1}\Big)^{\alpha} a_{k+1} z^{k+1}.$$

From (1.7) and (1.10) we observe that the operator $\mathcal{I}_{m,\gamma}^{n,\alpha} f(z)$ is well defined from $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

By using the linear operator $\mathcal{I}_{m,\gamma}^{n,\alpha} f(z)$ defined by (1.7), we will derive several fuzzy diferential subordintions for this class.

Definition 1.5. A function $f \in \mathcal{A}$ *belongs to the class* $\mathcal{M}_{m,\gamma}^{F}(n, \alpha, \eta)$ for all $\eta \in [0,1)$, $n \in \mathbb{N}_0$, $m > -1$, $\gamma \geqslant 0$ and $\alpha \geqslant 0$ if it satisfies the inequality

$$F_{(\mathcal{I}_{m,\gamma}^{n,\alpha} f)'(\mathbb{U})}(\mathcal{I}_{m,\gamma}^{n,\alpha} f(z))' > \eta, \quad z \in \mathbb{U}.$$

2. Preliminary

To prove our results, we need the following lemmas.

Lemma 2.1 ([13]). *Let $\psi \in \mathcal{A}$ and $\mathcal{G}(z) = z^{-1}\int_0^z \psi(t)\,\mathrm{d}t$, $z \in \mathbb{U}$. If*

$$\Re(1 + z\psi''(z)/\psi'(z)) > -\frac{1}{2}, \quad z \in \mathbb{U},$$

then $\mathcal{G} \in \mathcal{K}$.

Lemma 2.2 ([15], Theorem 2.6). *Assume that h is a convex function with $h(0) = a$ and $\nu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with $\Re(\nu) \geqslant 0$. If $p \in \mathcal{H}[a, n]$ with $p(0) = a$, $\Phi\colon \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$, $\Phi(p(z), zp'(z); z) = p(z) + \nu^{-1}zp'(z)$ is analytic function in $\mathbb{U}$ and*

$$\mathcal{F}_{\Phi(\mathbb{C}^2\times\mathbb{U})}\Big(p(z) + \frac{1}{\nu}zp'(z)\Big) \leqslant \mathcal{F}_{h(\mathbb{U})}h(z), \quad \text{i.e. } p(z) + \frac{1}{\nu}zp'(z) \prec_{\mathcal{F}} h(z),\ z \in \mathbb{U},$$

then

$$\mathcal{F}_{p(\mathbb{U})}p(z) \leqslant \mathcal{F}_{q(\mathbb{U})}q(z) \leqslant \mathcal{F}_{h(\mathbb{U})}h(z), \quad \text{i.e. } p(z) \prec_{\mathcal{F}} q(z),\ z \in \mathbb{U},$$

where

$$q(z) = \frac{\nu}{nz^{\nu/n}}\int_0^z h(t)t^{-1+\nu/n}\,\mathrm{d}t, \quad z \in \mathbb{U}.$$

The function q is convex and it is the fuzzy best dominant.

Lemma 2.3 ([15], Theorem 2.7). *Let g be a convex function in $\mathbb{U}$ and let $\psi(z) = g(z) + n\gamma zg'(z)$, where $z \in \mathbb{U}$, $n \in \mathbb{N}$ and $\gamma > 0$. If*

$$p(z) = g(0) + p_n z^n + p_{n+1}z^{n+1} + \dots$$

is holomorphic in $\mathbb{U}$ and

$$\mathcal{F}_{p(\mathbb{U})}(p(z) + \gamma zp'(z)) \leqslant \mathcal{F}_{\psi(\mathbb{U})}\psi(z), \quad \text{i.e. } p(z) + \gamma zp'(z) \prec_{\mathcal{F}} \psi(z),\ z \in \mathbb{U},$$

then

$$\mathcal{F}_{p(\mathbb{U})}(p(z)) \leqslant \mathcal{F}_{g(\mathbb{U})}g(z), \quad \text{i.e. } p(z) \prec_{\mathcal{F}} g(z),\ z \in \mathbb{U};$$

this result is sharp.

For the general theory of fuzzy differential subordination and its applications, we refer the reader to [2], [12]–[14], [16].

The objective of the present section is to obtain several fuzzy differential subordinations associated with the integral operator $\mathcal{I}_{m,\gamma}^{n,\alpha}$ by using the method of fuzzy differential subordination.

3. Main results

Assume that $\eta \in [0,1)$, $n \in \mathbb{N}_0$, $m > 0$, $\alpha, \gamma \geqslant 0$ and $z \in \mathbb{U}$ are mentioned through this paper:

Theorem 3.1. *Let k be a convex function in $\mathbb{U}$ and suppose that $h(z) = k(z) + zk'(z)/(\lambda+2)$. If $f \in \mathcal{M}^F_{m,\gamma}(n,\alpha,\eta)$ and*

$$G(z) = I^{\lambda} f(z) = \frac{\lambda+2}{z^{\lambda+1}} \int_0^z t^{\lambda} f(t)\,\mathrm{d}t, \tag{3.1}$$

then

$$F_{(\mathcal{I}^{n,\alpha}_{m,\gamma} f)'(\mathbb{U})}(\mathcal{I}^{n,\alpha}_{m,\gamma} f(z))' \leqslant F_{h(\mathbb{U})} h(z), \quad \text{i.e. } (\mathcal{I}^{n,\alpha}_{m,\gamma} f(z))' \prec_{\mathcal{F}} h(z), \tag{3.2}$$

implies that

$$F_{(\mathcal{I}^{n,\alpha}_{m,\gamma} G)'(\mathbb{U})}(\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))' \leqslant F_{k(\mathbb{U})} k(z), \quad \text{i.e. } (\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))' \prec_{\mathcal{F}} k(z);$$

this result is sharp.

Proof. Since

$$z^{\lambda+1} G(z) = (\lambda+2) \int_0^z t^{\lambda} f(t)\,\mathrm{d}t,$$

by differentiating, we obtain

$$(\lambda+1) G(z) + zG'(z) = (\lambda+2) f(z)$$

and

$$(\lambda+1)\mathcal{I}^{n,\alpha}_{m,\gamma} G(z) + z(\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))' = (\lambda+2)\mathcal{I}^{n,\alpha}_{m,\gamma} f(z), \tag{3.3}$$

and also by differentiating (3.3) we obtain

$$(\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))' + \frac{1}{(\lambda+2)} z(\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))'' = (\mathcal{I}^{n,\alpha}_{m,\gamma} f(z))'. \tag{3.4}$$

By using (3.4), the fuzzy differential subordination (3.2) becomes

$$F_{(\mathcal{I}^{n,\alpha}_{m,\gamma} f)'(\mathbb{U})}\Big((\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))' + \frac{1}{(\lambda+2)} z(\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))''\Big) \leqslant F_{h(\mathbb{U})}\Big(k(z) + \frac{1}{(\lambda+2)} zk'(z)\Big). \tag{3.5}$$

We denote

$$q(z) = (\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))' \to q \in \mathcal{H}[1,n]. \tag{3.6}$$

From (3.6) in (3.5) we have

$$F_{(\mathcal{I}^{n,\alpha}_{m,\gamma} f)'(\mathbb{U})}\Big(q(z) + \frac{1}{(\lambda+2)} zq'(z)\Big) \leqslant F_{h(\mathbb{U})}\Big(k(z) + \frac{1}{(\lambda+2)} zk'(z)\Big), \tag{3.7}$$

by applying Lemma 2.3, we have

$$F_{q(\mathbb{U})} q(z) \leqslant F_{k(\mathbb{U})} k(z), \quad \text{i.e. } F_{(\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))'(\mathbb{U})} (\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))' \leqslant F_{k(\mathbb{U})} k(z),$$

then $(\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))' \prec_{\mathcal{F}} k(z)$, k is the fuzzy best dominant. □

Theorem 3.2. *Consider* $h(z) = (1 + (2\eta - 1)z)/(1 + z)$, $\eta \in [0, 1)$, $\lambda > 0$ *and* I^λ *is given by* (3.1). *Then*

$$I^\lambda(\mathcal{M}^F_{m,\gamma}(n, \alpha, \eta)) \subset \mathcal{M}^F_{m,\gamma}(n, \alpha, \zeta), \tag{3.8}$$

where

$$\zeta = 2\eta - 1 + (\lambda + 2)(2 - 2\eta) \int_0^1 \frac{t^{\lambda+2}}{t+1}\, \mathrm{d}t. \tag{3.9}$$

P r o o f. The function h is convex and using the same technique as in the proof of Theorem 3.1, we obtain from the hypothesis of Theorem 3.2 that

$$F_{q(\mathbb{U})}\Big(q(z) + \frac{1}{(\lambda+2)} zq'(z)\Big) \leqslant F_{h(\mathbb{U})} h(z),$$

where $q(z)$ is defined in (3.6). By using Lemma 2.2, we obtain that

$$F_{q(\mathbb{U})} q(z) \leqslant F_{k(\mathbb{U})} k(z) \leqslant F_{h(\mathbb{U})} h(z)$$

implies that

$$F_{(\mathcal{I}^{n,\alpha}_{m,\gamma} G)'(\mathbb{U})} (\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))' \leqslant F_{k(\mathbb{U})} k(z) \leqslant F_{h(\mathbb{U})} h(z),$$

where

$$k(z) = \frac{\lambda+2}{z^{\lambda+2}} \int_0^z t^{\lambda+1} \frac{1 + (2\eta - 1)t}{1+t}\, \mathrm{d}t = (2\eta - 1) + \frac{(\lambda+2)(2-2\eta)}{z^{\lambda+2}} \int_0^z \frac{t^{\lambda+1}}{1+t}\, \mathrm{d}t.$$

Since k is convex and $k(\mathbb{U})$ is symmetric with respect to the real axis, we conclude

$$F_{(\mathcal{I}^{n,\alpha}_{m,\gamma} G)'(\mathbb{U})} (\mathcal{I}^{n,\alpha}_{m,\gamma} G(z))' \geqslant \min_{|z|=1} F_{k(\mathbb{U})} k(z) = F_{k(\mathbb{U})} k(1), \tag{3.10}$$

and $\zeta = k(1) = 2\eta - 1 + (\lambda + 2)(2 - 2\eta) \int_0^1 t^{\lambda+2}/(t+1)\, \mathrm{d}t$. From (3.10), we conclude that we have the inclusion in relation (3.8) and hence the proof of the theorem is complete. □

Theorem 3.3. *Assume that k is a convex function in $\mathbb{U}$, $k(0)=1$, and let $h(z)=k(z)+zk'(z)$. If $f\in\mathcal{A}$ and satisfies the fuzzy differential subordination*

$$F_{(\mathcal{I}^{n,\alpha}_{m,\gamma}f)'(\mathbb{U})}(\mathcal{I}^{n,\alpha}_{m,\gamma}f(z))' \leqslant F_{h(\mathbb{U})}h(z), \quad \text{i.e. } (\mathcal{I}^{n,\alpha}_{m,\gamma}f(z))' \prec_{\mathcal{F}} h(z), \tag{3.11}$$

then

$$F_{\mathcal{I}^{n,\alpha}_{m,\gamma}f(\mathbb{U})}\frac{\mathcal{I}^{n,\alpha}_{m,\gamma}f(z)}{z} \leqslant F_{k(\mathbb{U})}k(z), \quad \text{i.e. } \frac{\mathcal{I}^{n,\alpha}_{m,\gamma}f(z)}{z} \prec_{\mathcal{F}} k(z), \tag{3.12}$$

and this result is sharp.

P r o o f. By denoting

$$\begin{aligned} q(z) &= \frac{1}{z}\mathcal{I}^{n,\alpha}_{m,\gamma}f(z) = \frac{1}{z}\Big(z+\sum_{k=2}^{\infty}(1+\gamma(k-1))^n\Big(\frac{m}{m+k}\Big)^{\alpha}a_kz^k\Big)\\ &= 1+\sum_{k=2}^{\infty}(1+\gamma(k-1))^n\Big(\frac{m}{m+k}\Big)^{\alpha}a_kz^{k-1}, \end{aligned}$$

we obtain that $q(z)+zq'(z)=(\mathcal{I}^{n,\alpha}_{m,\gamma}f(z))'$. Then

$$\begin{aligned} &F_{(\mathcal{I}^{n,\alpha}_{m,\gamma}f)'(\mathbb{U})}(\mathcal{I}^{n,\alpha}_{m,\gamma}f(z))' \leqslant F_{h(\mathbb{U})}h(z)\\ &\qquad\to F_{q(\mathbb{U})}(q(z)+zq'(z)) \leqslant F_{h(\mathbb{U})}h(z) = F_{k(\mathbb{U})}(k(z)+zk'(z)). \end{aligned}$$

Applying Lemma 2.3, we have

$$F_{q(\mathbb{U})}q(z) \leqslant F_{k(\mathbb{U})}k(z) \to F_{\mathcal{I}^{n,\alpha}_{m,\gamma}f(\mathbb{U})}\frac{\mathcal{I}^{n,\alpha}_{m,\gamma}f(z)}{z} \leqslant F_{k(\mathbb{U})}k(z),$$

we get

$$\frac{\mathcal{I}^{n,\alpha}_{m,\gamma}f(z)}{z} \prec_{\mathcal{F}} k(z),$$

and this result is sharp. □

Theorem 3.4. *For $h\in\mathcal{H}(\mathbb{U})$, $h(0)=1$, which satisfies $\Re(1+zh''(z)/h'(z))>-\frac{1}{2}$, if $f\in\mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{(\mathcal{I}^{n,\alpha}_{m,\gamma}f)'(\mathbb{U})}(\mathcal{I}^{n,\alpha}_{m,\gamma}f(z))' \leqslant F_{h(\mathbb{U})}h(z), \quad \text{i.e. } (\mathcal{I}^{n,\alpha}_{m,\gamma}f(z))' \prec_{\mathcal{F}} h(z), \tag{3.13}$$

then

$$F_{\mathcal{I}^{n,\alpha}_{m,\gamma}f(\mathbb{U})}\frac{\mathcal{I}^{n,\alpha}_{m,\gamma}f(z)}{z} \leqslant F_{k(\mathbb{U})}k(z), \quad \text{i.e. } \frac{\mathcal{I}^{n,\alpha}_{m,\gamma}f(z)}{z} \prec_{\mathcal{F}} k(z), \tag{3.14}$$

where

$$k(z)=\frac{1}{z}\int_0^z h(t)\,\mathrm{d}t.$$

The function k is convex and it is the fuzzy best dominant.

Proof. Let

$$q(z) = \frac{\mathcal{I}^{n,\alpha}_{m,\gamma} f(z)}{z} = 1 + \sum_{k=2}^{\infty} (1+\gamma(k-1))^n \Big(\frac{m}{m+k}\Big)^{\alpha} a_k z^{k-1}, \quad q \in \mathcal{H}[1,1],$$

where $\Re(1 + zh''(z)/h'(z)) > -\frac{1}{2}$. From Lemma 2.1, we have that

$$k(z) = \frac{1}{z} \int_0^z h(t)\,\mathrm{d}t$$

is a convex function which satisfies the fuzzy differential subordination (3.13). Since

$$k(z) + zk'(z) = h(z)$$

is the fuzzy best dominant, we have $q(z) + zq'(z) = (\mathcal{I}^{n,\alpha}_{m,\gamma} f(z))'$, then (3.13) becomes

$$F_{q(\mathbb{U})}(q(z) + zq'(z)) \leqslant F_{h(\mathbb{U})} h(z).$$

By applying Lemma 2.3, we have

$$F_{q(\mathbb{U})} q(z) \leqslant F_{k(\mathbb{U})} k(z) \to F_{\mathcal{I}^{n,\alpha}_{m,\gamma} f(\mathbb{U})} \frac{\mathcal{I}^{n,\alpha}_{m,\gamma} f(z)}{z} \leqslant F_{k(\mathbb{U})} k(z),$$

and we obtain that

$$\frac{\mathcal{I}^{n,\alpha}_{m,\gamma} f(z)}{z} \prec_{\mathcal{F}} k(z).$$

□

Putting $h(z) = (1 + (2\beta - 1)z)/(1 + z)$ in Theorem 3.4, we obtain the following corollary:

Corollary 3.1. *Let $h = (1 + (2\beta - 1)z)/(1 + z)$ be a convex function in $\mathbb{U}$ with $h(0) = 1$, $0 \leqslant \beta < 1$. If $f \in \mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{(\mathcal{I}^{n,\alpha}_{m,\gamma} f)'(\mathbb{U})} (\mathcal{I}^{n,\alpha}_{m,\gamma} f(z))' \leqslant F_{h(\mathbb{U})} h(z), \quad \text{i.e. } (\mathcal{I}^{n,\alpha}_{m,\gamma} f(z))' \prec_{\mathcal{F}} h(z),$$

then

$$k(z) = 2\beta - 1 + \frac{2(1-\beta)}{z} \ln(1+z),$$

and the function k is convex and it is the fuzzy best dominant.

4. Conclusion

All the above results give us information about fuzzy differential subordinations for a linear operator $\mathcal{I}^{n,\alpha}_{m,\gamma}$. We give some properties for the class $\mathcal{M}^F_{m,\gamma}(n,\alpha,\eta)$ of univalent analytic functions.

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