ON UNIT GROUP OF FINITE SEMISIMPLE GROUP ALGEBRAS OF NON-METABELIAN GROUPS UP TO ORDER 72

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Abstract. We characterize the unit group of semisimple group algebras $\mathbb{F}_qG$ of some non-metabelian groups, where F_q is a field with $q = p^k$ elements for p prime and a positive integer k. In particular, we consider all 6 non-metabelian groups of order 48, the only non-metabelian group $((C_3 \times C_3) \rtimes C_3) \rtimes C_2$ of order 54, and 7 non-metabelian groups of order 72. This completes the study of unit groups of semisimple group algebras for groups upto order 72.

Keywords: unit group; finite field; Wedderburn decomposition

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1. Introduction

Let $\mathbb{F}_q$ denote a finite field with $q = p^k$ elements for an odd prime p, G a finite group and let $\mathbb{F}_qG$ be the group algebra. We refer to [18] for elementary definitions and results related to the group algebras and [2], [17] for the abelian group algebras. One of the most important research problems in the theory of group algebras is the determination of their unit groups, which are very important from the application point of view; for instance, in the exploration of Lie properties of group algebras, the isomorphism problem etc., see [1]. Hurley in [7] suggested the construction of convolutional codes from units in group algebra as an important application of units.

Considering some of the existing literature, we refer to [3], [6], [13], [15] for the unit group $U(\mathbb{F}G)$ of dihedral groups G and [5], [6], [9], [12], [14], [15], [19]–[21] for some non abelian groups other than the dihedral groups. The unit group of finite semisimple group algebras of metabelian groups (groups in which there exists a normal subgroup N of G such that both N and G/N are abelian) has been well studied. From [16], it can be seen that all groups up to order 23 are metabelian.

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The only non-metabelian groups of order 24 are S_4 and $SL(2,3)$, and their unit group algebras have been discussed in [9], [12]. Further, [16] also implies that there are non-metabelian groups of order 48, 54, 60 and 72. It can be verified that A_5 is the only non-metabelian group of order 60 and the unit group of its group algebra, i.e. $U(\mathbb{F}_q A_5)$ can be easily deduced from [14] for $p \geqslant 5$.

The main motive of this paper is to characterize the unit groups of $\mathbb{F}_q G$, where first we consider G to be a non-metabelian group of order 48. There are 6 such groups up to isomorphism. After that we consider the only non-metabelian group of order 54. Finally, we consider all the non-metabelian groups of order 72. In all, we cover the unit groups of 14 semisimple group algebras of non-metabelian groups. The rest of the paper is organized in the following manner: we recall all the basic definitions and results to be used later on in Section 2. Our main results for the characterization of the unit groups are presented in the third, fourth and fifth sections. Some remarks are discussed in the last section.

2. Preliminaries

Let e denote the exponent of G, ζ be a primitive eth root of unity and $\mathbb{F}$ be an arbitrary finite field. On the lines of [4], we define

$$I_{\mathbb{F}} = \{n\colon\ \zeta \mapsto \zeta^n \text{ is an automorphism of } \mathbb{F}(\zeta) \text{ over } \mathbb{F}\}.$$

Since, the Galois group $\mathrm{Gal}(\mathbb{F}(\zeta), \mathbb{F})$ is a cyclic group and for any $\tau \in \mathrm{Gal}(\mathbb{F}(\zeta), \mathbb{F})$, there exists a positive integer s which is invertible modulo e such that $\tau(\zeta) = \zeta^s$. In other words, $I_{\mathbb{F}}$ is a subgroup of the multiplicative group $\mathbb{Z}_e^*$. For any p-regular element $g \in G$, i.e. an element whose order is not divisible by p, let the sum of all conjugates of g be denoted by γ_g, and the cyclotomic $\mathbb{F}$-class of γ_g be denoted by $S(\gamma_g) = \{\gamma_{g^n}\colon\ n \in I_{\mathbb{F}}\}$. The cardinality of $S(\gamma_g)$ and the number of cyclotomic $\mathbb{F}$-classes will be incorporated later on for the characterization of the unit groups. Now, we recall the following two results related to the cyclotomic $\mathbb{F}$-classes.

Theorem 2.1 ([4]). *The number of simple components of $\mathbb{F}G/J(\mathbb{F}G)$ and the number of cyclotomic $\mathbb{F}$-classes in G are equal.*

Theorem 2.2 ([4]). *Let j be the number of cyclotomic $\mathbb{F}$-classes in G. If K_i, $1 \leqslant i \leqslant j$, are the simple components of center of $\mathbb{F}G/J(\mathbb{F}G)$ and S_i, $1 \leqslant i \leqslant j$, are the cyclotomic $\mathbb{F}$-classes in G, then $|S_i| = [K_i : \mathbb{F}]$ for each i after suitable ordering of the indices.*

For determining the structure of the unit group $U(\mathbb{F}G)$, we need Wedderburn decomposition of the group algebra $\mathbb{F}G$. In other words, we want to determine the simple components of $\mathbb{F}G$. Based on the existing literature, we can always claim that $\mathbb{F}$ is one of the simple components in the decomposition of $\mathbb{F}G/J(\mathbb{F}G)$. The simple proof is given here for completeness.

Lemma 2.1. *Let A_1 and A_2 denote the finite dimensional algebras over $\mathbb{F}$. Further, let A_2 be semisimple and g be an onto map between A_1 and A_2, then we must have $A_1/J(A_1) \cong A_3 + A_2$, where A_3 is some semisimple $\mathbb{F}$-algebra.*

Proof. From [8], we have $J(A_1) \subseteq \operatorname{Ker}(g)$. This means there exists an $\mathbb{F}$-algebra homomorphism g_1 from $A_1/J(A_1)$ to A_2 which is also onto. In other words, we have $g_1\colon A_1/J(A_1) \mapsto A_2$ defined by $g_1(a + J(A_1)) = g(a)$, $a \in A_1$. As $A_1/J(A_1)$ is semisimple, there exists an ideal I of $A_1/J(A_1)$ such that $A_1/J(A_1) = \ker(g_1) \oplus I$. Our claim is that $I \cong A_2$. To prove this, note that any element $a \in A_1/J(A_1)$ can be uniquely written as $a = a_1 + a_2$, where $a_1 \in \ker(g_1)$, $a_2 \in I$. So, define $g_2\colon A_1/J(A_1) \mapsto \ker(g_1) \oplus A_2$ by $g_2(a) = (a_1, g_1(a_2))$. Since $\ker(g_1)$ is a semisimple algebra over $\mathbb{F}$, the result holds. $\square$

The above lemma concludes that $\mathbb{F}$ is one of the simple components of $\mathbb{F}G$, provided $J(\mathbb{F}G) = 0$. Now we characterize the set $I_{\mathbb{F}}$ defined in the beginning of this section.

Theorem 2.3 ([11])**.** *Let $\mathbb{F}$ be a finite field with prime power order q. If e is such that $\gcd(e, q) = 1$, ζ is the primitive eth root of unity and $|q|$ is the order of q modulo e, then $I_{\mathbb{F}} = \{1, q, q^2, \dots, q^{|q|-1}\}$.*

The next two results are Propositions 3.6.11 and 3.6.7, respectively, from [18] and are quite useful in our work.

Theorem 2.4. *If RG is a semisimple group algebra, then $RG \cong R(G/G') \oplus \Delta(G, G')$, where G' is the commutator subgroup of G, $R(G/G')$ is the sum of all commutative simple components of RG, and $\Delta(G, G')$ is the sum of all others.*

Theorem 2.5. *Let RG be a semisimple group algebra and H be a normal subgroup of G. Then $RG \cong R(G/H) \oplus \Delta(G, H)$, where $\Delta(G, H)$ is the left ideal of RG generated by the set $\{h - 1\colon\ h \in H\}$.*

3. Unit group of $\mathbb{F}_qG$ for non-metabelian groups of order 48

The main objective of this section is to characterize the unit groups of $\mathbb{F}_qG$, where G is a non-metabelian group of order 48. Up to isomorphism, there are 6 non-metabelian groups of order 48, namely $G_1 = C_2 \cdot S_4$, $G_2 = GL(2,3)$, $G_3 = A_4 \rtimes C_4$, $G_4 = C_2 \times SL(2,3)$, $G_5 = ((C_4 \times C_2) \rtimes C_2) \rtimes C_3$ and $G_6 = C_2 \times S_4$. Here $C_2 \cdot S_4$ represents the non-split extension of S_4 by C_2. We consider each of these groups one by one and discuss the unit groups of their respective group algebras along with the WD's in the subsequent subsections (here WD means Wedderburn decomposition and from now onwards we use this notation).

3.1. The group $G_1 = C_2 \cdot S_4$. Group G_1 has the following presentation:

$$\begin{aligned}\langle x, y, z, w, t \mid\ & x^2t^{-1}, z^{-1}x^{-1}zxt^{-1}w^{-1}z^{-1}, y^{-1}x^{-1}yxy^{-1}, w^{-1}x^{-1}wxt^{-1}w^{-1}z^{-1},\\ & y^3, t^{-1}x^{-1}tx, t^2, z^{-1}y^{-1}zyw^{-1}z^{-1}, w^{-1}y^{-1}wyt^{-1}z^{-1},\\ & t^{-1}y^{-1}ty, w^{-1}z^{-1}wzt^{-1}, z^2t^{-1}, t^{-1}z^{-1}tz, t^{-1}w^{-1}tw, w^2t^{-1}\rangle.\end{aligned}$$

Also G_1 has 8 conjugacy classes as shown in the table below.

rep	1	x	y	z	t	xz	yw	xyz
order of rep	1	4	3	4	2	8	6	8

where rep means representative of conjugacy class. From the above discussion, clearly the exponent of G_1 is 24. Also $G_1' \cong SL(2,3)$. Next, we give the unit group of $\mathbb{F}_qG_1$ when $p > 3$.

Theorem 3.1. *The unit group of $\mathbb{F}_qG_1$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

(1) *for k even or $p^k \in \{1, 7, 17, 23\} \bmod 24$ with k odd*

$$U(\mathbb{F}_qG_1) \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q)^3 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q),$$

(2) *for $p^k \in \{5, 11, 13, 19\} \bmod 24$ with k odd*

$$U(\mathbb{F}_qG_1) \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q) \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^2}).$$

P r o o f. Since $\mathbb{F}_qG_1$ is semisimple, we have

$$\mathbb{F}_qG_1 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{3.1}$$

First assume that k is even which means for any prime $p > 3$, we have $p^k \equiv 1 \bmod 24$. This means $|S(\gamma_g)| = 1$ for each $g \in G_1$ as $I_{\mathbb{F}} = \{1\}$. Hence, (3.1), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_qG_1 \cong \mathbb{F}_q \bigoplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q). \tag{3.2}$$

Using Theorem 2.4 with $G_1' \cong SL(2,3)$ in (3.2), we reach

$$\mathbb{F}_qG_1 \cong \mathbb{F}_q^2 \bigoplus_{r=1}^{6} M_{n_r}(\mathbb{F}_q), \quad \text{where } n_r \geqslant 2 \text{ with } 46 = \sum_{r=1}^{6} n_r^2. \tag{3.3}$$

The above gives the only possibility $(2,2,2,3,3,4)$ for the possible values of n_r's and therefore, (3.3) implies that

$$\mathbb{F}_qG_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q). \tag{3.4}$$

It is straight-forward to deduce the unit group from WD. Now we consider that k is odd. We shall discuss this case in two parts:

(1) $p^k \in \{1,7,17,23\} \bmod 24$,

(2) $p^k \in \{5,11,13,19\} \bmod 24$.

Case (1): For $p^k \in \{1,7,17,23\} \bmod 24$, it can be verified that $|S(\gamma_g)| = 1$ for each $g \in G_1$. This means WD is given by (3.4).

Case (2): For $p^k \in \{5,11,13,19\} \bmod 24$, we can verify that $S(\gamma_g) = \{\gamma_g\}$ for each representative of the conjugacy classes except xz, for which $S(\gamma_{xz}) = \{S(\gamma_{xz}), S(\gamma_{xyz})\}$. Therefore, (3.1) and Theorems 2.1 and 2.2 imply that $\mathbb{F}_qG_1 \cong \mathbb{F}_q \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \oplus M_{n_6}(\mathbb{F}_{q^2})$. Using Theorem 2.4 in this to obtain (after suitable rearrangement of indexes)

$$\mathbb{F}_qG_1 \cong \mathbb{F}_q^2 \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus M_{n_5}(\mathbb{F}_{q^2}), \quad n_r \geqslant 2 \text{ with } 46 = \sum_{r=1}^{4} n_r^2 + 2n_5^2. \tag{3.5}$$

The above gives us two possibilities, namely $(2,2,2,4,3)$ and $(2,3,3,4,2)$ for the possible values of n_r's. However, we need to discard one of these possibilities. For that, consider the normal subgroup $H_1 = \langle t\rangle \cong C_2$ of G_1 with $G_1/H_1 \cong S_4$. From [9], we know that

$$\mathbb{F}_q S_4 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q) \oplus M_3(\mathbb{F}_q)^2. \tag{3.6}$$

Therefore, (3.5), (3.6) and Theorem 2.5 imply that $(2,3,3,4,2)$ is the only possibility for n_r's. □

3.2. The group $G_2 = GL(2,3)$. Group G_2 has the following presentation:

$$\begin{aligned}\langle x,y,z,w,t \mid x^2, z^{-1}x^{-1}zxt^{-1}w^{-1}z^{-1}, y^{-1}x^{-1}yxy^{-1}, w^{-1}x^{-1}wxw^{-1}z^{-1}, y^3,\\ t^{-1}x^{-1}tx, t^{-1}z^{-1}tz, z^{-1}y^{-1}zyw^{-1}z^{-1}, w^{-1}y^{-1}wyt^{-1}z^{-1},\\ t^{-1}y^{-1}ty, w^{-1}z^{-1}wzt^{-1}, w^2t^{-1}, t^{-1}w^{-1}tw, w^2t^{-1}, t^2\rangle.\end{aligned}$$

Further, G_2 has 8 conjugacy classes, as shown in the table below.

rep	1	x	y	z	t	xz	yw	xyz
order of rep	1	2	3	4	2	8	6	8

From the above discussion, clearly the exponent of G_2 is 24. Also $G_2' \cong SL(2,3)$.

Theorem 3.2. *The unit group of $\mathbb{F}_qG_2$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

(1) *for k even or $p^k \in \{1, 11, 17, 19\} \bmod 24$ with k odd*

$$U(\mathbb{F}_qG_2) \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q)^3 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q),$$

(2) *for $p^k \in \{5, 7, 13, 23\} \bmod 24$ with k odd*

$$U(\mathbb{F}_qG_2) \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q) \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_4(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^2}).$$

Proof. See Theorem 3.2 in [10]. □

3.3. The group $G_3 = A_4 \rtimes C_4$. Group G_3 has the following presentation:

$$\begin{aligned}\langle x,y,z,w,t \mid x^2y^{-1}, z^{-1}x^{-1}zxz^{-1}, y^{-1}x^{-1}yx, w^{-1}x^{-1}wxt^{-1}w^{-1}, y^2,\\ t^{-1}x^{-1}txt^{-1}w^{-1}, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, t^{-1}y^{-1}ty, z^3,\\ w^{-1}z^{-1}wzt^{-1}w^{-1}, t^{-1}z^{-1}tzw^{-1}, w^2, t^{-1}w^{-1}tw, t^2\rangle.\end{aligned}$$

Further, G_3 has 10 conjugacy classes, as shown in the table below.

rep	1	x	y	z	w	xy	xw	yz	yw	xyw
order of rep	1	4	2	3	2	4	4	6	2	4

From the above discussion, clearly the exponent of G_3 is 12. Also $G_3' \cong A_4$ and $G_3/G_3' \cong C_4$.

Theorem 3.3. *The unit group $U(\mathbb{F}_q G_3)$ of $\mathbb{F}_q G_3$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

(1) *for any p and k even or $p^k \in \{1, 5\} \bmod 12$ with k odd*

$$U(\mathbb{F}_q G_3) \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_q)^4,$$

(2) *for $p^k \in \{7, 11\} \bmod 12$ with k odd*

$$U(\mathbb{F}_q G_3) \cong (\mathbb{F}_q^*)^2 \oplus \mathbb{F}_{q^2}^* \oplus GL_2(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_{q^2}).$$

Proof. Since $\mathbb{F}_q G_3$ is semisimple, we have

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{3.7}$$

First assume that k is even, which means that for any prime $p > 3$, we have $p^k \equiv 1 \bmod 12$. This means $|S(\gamma_g)| = 1$ for each $g \in G_3$ as $I_{\mathbb{F}} = \{1\}$. Hence, (3.7), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q \bigoplus_{r=1}^{9} M_{n_r}(\mathbb{F}_q). \tag{3.8}$$

Using Theorem 2.4 with $G_3' \cong A_4$ and $G_3/G_3' \cong C_4$ in (3.8), we reach

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q^4 \bigoplus_{r=1}^{6} M_{n_r}(\mathbb{F}_q), \quad \text{where } n_r \geqslant 2 \text{ with } 44 = \sum_{r=1}^{6} n_r^2. \tag{3.9}$$

The above gives us the only possibility $(2, 2, 3, 3, 3, 3)$ for the possible values of n_r's. Therefore, we have

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4. \tag{3.10}$$

Now we consider that k is odd. We shall discuss this case into three parts:

(1) $p^k \equiv 1 \bmod 12$.

(2) $p^k \equiv \pm 1 \bmod 3$ and $p^k \equiv -1 \bmod 4$.

(3) $p^k \equiv -1 \bmod 3$ and $p^k \equiv 1 \bmod 4$.

Case (1): $p^k \equiv 1 \bmod 12$. In this case WD is given by (3.10).

Case (2): $p^k \equiv \pm 1 \bmod 3$ and $p^k \equiv -1 \bmod 4$ which means $p^k \in \{7, 11\} \bmod 12$. This means $I_{\mathbb{F}} = \{1, 7\}$ or $\{1, 11\}$ and accordingly we can verify that for both the cases, $|S(\gamma_g)| = 1$ for each representative g of conjugacy classes except the one's having order 4. For representatives of order 4, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xy}\}$, $S(\gamma_{xw}) = \{\gamma_{xw}, \gamma_{xyw}\}$. Therefore, (3.7) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=6}^{7} M_{n_r}(\mathbb{F}_{q^2}). \tag{3.11}$$

Since $G_3' \cong A_4$ with $G_3/G_3' \cong C_4$, we have $\mathbb{F}_q C_4 \cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{F}_{q^2}$. This with (3.11) and Theorem 2.5 implies that $\mathbb{F}_q G_3 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus M_{n_5}(\mathbb{F}_{q^2})$, $n_r \geqslant 2$ with $44 = \sum_{r=1}^{4} n_r^2 + 2n_5^2$, which further implies that the possible choices of n_r's are $(3, 3, 3, 3, 2)$, $(2, 2, 3, 3, 3)$. For uniqueness, consider the normal subgroup $H_3 = \langle y \rangle$ of G_3 having order 2 with $G_3/H_3 \cong S_4$. Using (3.6) and Theorem 2.5, we conclude that $(2, 2, 3, 3, 3)$ is the required choice.

Case (3): $p^k \equiv -1 \bmod 3$ and $p^k \equiv 1 \bmod 4$ which means $p^k \equiv 5 \bmod 12$. This means $I_{\mathbb{F}} = \{1, 5\}$ and accordingly we can verify that WD in this case is given by (3.10). □

3.4. The group $G_4 = C_2 \times SL(2, 3)$. Group G_4 has the following presentation:

$$\begin{aligned}\langle x, y, z, w, t \mid\ & x^2, z^{-1}x^{-1}zx, y^{-1}x^{-1}yx, w^{-1}x^{-1}wx, t^{-1}x^{-1}tx,\\ & z^{-1}y^{-1}zyt^{-1}w^{-1}z^{-1}, y^3, w^{-1}y^{-1}wyt^{-1}z^{-1}, t^{-1}y^{-1}ty,\\ & z^2t^{-1}, w^{-1}z^{-1}wzt^{-1}, t^{-1}z^{-1}tz, w^2t^{-1}, t^{-1}w^{-1}tw, t^2\rangle.\end{aligned}$$

Further, G_4 has 14 conjugacy classes, as shown in the table below.

rep	1	x	y	z	t	xy	xz	xt	y^2	yt	xy^2	xyt	y^2z	xy^2z
order of rep	1	2	3	4	2	6	4	2	3	6	6	6	6	6

From the above discussion, clearly the exponent of G_4 is 12. Also $G_4' \cong Q_8$ and $G_4/G_4' \cong C_6$.

Theorem 3.4. *The unit group $U(\mathbb{F}_qG_4)$ of $\mathbb{F}_qG_4$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

(1) *for any p and k even or $p^k \in \{1,7\} \bmod 12$ with k odd*

$$U(\mathbb{F}_qG_4) \cong (\mathbb{F}_q^*)^6 \oplus GL_2(\mathbb{F}_q)^6 \oplus GL_3(\mathbb{F}_q)^2,$$

(2) *for $p^k \in \{5,11\} \bmod 12$ with k odd*

$$U(\mathbb{F}_qG_4) \cong (\mathbb{F}_q^*)^2 \oplus (\mathbb{F}_{q^2}^*)^2 \oplus GL_2(\mathbb{F}_q)^4 \oplus GL_2(\mathbb{F}_{q^2}) \oplus GL_3(\mathbb{F}_{q^2}).$$

Proof. Since $\mathbb{F}_qG_4$ is semisimple, we have

$$\mathbb{F}_qG_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{3.12}$$

First assume that k is even, which means for any prime $p > 3$, we have $p^k \equiv 1 \bmod 12$. This means $|S(\gamma_g)| = 1$ for each $g \in G_4$. Hence, (3.12), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_qG_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{13} M_{n_r}(\mathbb{F}_q). \tag{3.13}$$

Using Theorem 2.4 with $G_4' \cong Q_8$ and $G_4/G_4' \cong C_6$ in (3.13), we reach to $\mathbb{F}_qG_4 \cong \mathbb{F}_q^6 \bigoplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$, with $42 = \sum_{r=1}^{8} n_r^2$. This gives the only possibility $(2,2,2,2,2,2,3,3)$ for the possible values of n_r's. Therefore, we have

$$\mathbb{F}_qG_4 \cong \mathbb{F}_q^6 \oplus M_2(\mathbb{F}_q)^6 \oplus M_3(\mathbb{F}_q)^2. \tag{3.14}$$

Now we consider that k is odd.

Case (1): $p^k \equiv 1 \bmod 3$ and $p^k \equiv 1 \bmod 4$ or $p^k \equiv 1 \bmod 3$ and $p^k \equiv -1 \bmod 4$ which means $p^k \equiv 1,7 \bmod 12$. It can be seen that for these possibilities, $|S(\gamma_g)| = 1$ for each $g \in G_4$. Therefore, WD is given by (3.14).

Case (2): $p^k \equiv -1 \bmod 3$ and $p^k \equiv \pm 1 \bmod 4$ which means $p^k \in \{5,11\} \bmod 12$. This means that $I_{\mathbb{F}} = \{1,5\}$ or $\{1,11\}$ and accordingly we can verify that for both the cases

$$S(\gamma_y) = \{\gamma_y, \gamma_{y^2}\}, \qquad S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xy^2}\},$$
$$S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{y^2z}\}, \quad S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xy^2z}\},$$

and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of the conjugacy classes. Therefore, (3.12) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=6}^{9} M_{n_r}(\mathbb{F}_{q^2}). \tag{3.15}$$

Since $G_4' \cong Q_8$ with $G_4/G_4' \cong C_6$, we have $\mathbb{F}_q C_6 \cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{F}_{q^2}^2$. This with (3.15) and Theorem 2.5 leads to $\mathbb{F}_q G_4 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=5}^{6} M_{n_r}(\mathbb{F}_{q^2})$, $n_r \geqslant 2$ with $42 = \sum_{r=1}^{4} n_r^2 + 2\sum_{r=5}^{6} n_r^2$, which further implies that the possible choices of n_r's are $(2,2,3,3,2,2)$, $(2,2,2,2,2,3)$. For uniqueness, consider the normal subgroup $H_4 = \langle xt \rangle$ of G_4 having order 2 with $G_4/H_4 \cong SL(2,3)$. Using Theorem 3.1 from [12] and Theorem 2.5, we conclude that $(2,2,2,2,2,3)$ is the required choice. □

3.5. The group $G_5 = ((C_4 \times C_2) \rtimes C_2) \rtimes C_3$. Group G_5 has the following presentation:

$$\begin{aligned}\langle x, y, z, w, t \mid\ & x^2t^{-1}, z^{-1}x^{-1}zx, y^{-1}x^{-1}yx, w^{-1}x^{-1}wx, t^{-1}x^{-1}tx,\\ & z^{-1}y^{-1}zyt^{-1}w^{-1}z^{-1}, y^3, w^{-1}y^{-1}wyt^{-1}z^{-1}, t^{-1}y^{-1}ty,\\ & z^2t^{-1}, w^{-1}z^{-1}wzt^{-1}, t^{-1}z^{-1}tz, w^2t^{-1}, t^{-1}w^{-1}tw, t^2\rangle.\end{aligned}$$

Further, G_5 has 14 conjugacy classes, as shown in the table below.

rep	1	x	y	z	t	xy	xz	xt	y^2	yt	xy^2	xyt	y^2z	xy^2z
order of rep	1	4	3	4	2	12	2	4	3	6	12	12	6	12

From the above discussion, clearly the exponent of G_5 is 12. Also, $G_5' \cong Q_8$ and $G_5/G_5' \cong C_6$.

Theorem 3.5. *The unit group $U(\mathbb{F}_q G_5)$ of $\mathbb{F}_q G_5$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

(1) *for any p and k even or $p^k \equiv 1 \bmod 12$ with k odd*

$$U(\mathbb{F}_q G_5) \cong (\mathbb{F}_q^*)^6 \oplus GL_2(\mathbb{F}_q)^6 \oplus GL_3(\mathbb{F}_q)^2,$$

(2) *for $p^k \equiv 7 \bmod 12$ with k odd*

$$U(\mathbb{F}_q G_5) \cong (\mathbb{F}_q^*)^6 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_2(\mathbb{F}_{q^2})^3,$$

(3) *for* $p^k \equiv 5 \bmod 12$ *with* k *odd*

$$U(\mathbb{F}_qG_5) \cong (\mathbb{F}_q^*)^2 \oplus (\mathbb{F}_{q^2}^*)^2 \oplus GL_2(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_2(\mathbb{F}_{q^2})^2,$$

(4) *for* $p^k \equiv 11 \bmod 12$ *with* k *odd*

$$U(\mathbb{F}_qG_5) \cong (\mathbb{F}_q^*)^2 \oplus (\mathbb{F}_{q^2}^*)^2 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_2(\mathbb{F}_{q^2})^3.$$

Proof. Since $\mathbb{F}_qG_5$ is semisimple, we have

$$\mathbb{F}_qG_5 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{3.16}$$

First assume that k is even, which means that for any prime $p > 3$, we have $p^k \equiv 1 \bmod 12$. This means $|S(\gamma_g)| = 1$ for each $g \in G_5$. Hence, (3.16), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_qG_5 \cong \mathbb{F}_q \bigoplus_{r=1}^{13} M_{n_r}(\mathbb{F}_q). \tag{3.17}$$

Proceeding similarly as in Theorem 3.4, we get the WD exactly similar to (3.14). Now we consider that k is odd.

Case (1): $p^k \equiv 1 \bmod 3$ and $p^k \equiv 1 \bmod 4$. In this case WD is given by (3.14).

Case (2): $p^k \equiv 1 \bmod 3$ and $p^k \equiv -1 \bmod 4$ which means that $p^k \equiv 7 \bmod 12$. This means that $I_{\mathbb{F}} = \{1, 7\}$ and accordingly we can verify that for this case

$$S(\gamma_x) = \{\gamma_x, \gamma_{xt}\}, \qquad S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xyt}\},$$
$$S(\gamma_{xy^2}) = \{\gamma_{xy^2}, \gamma_{xy^2z}\}, \qquad S(\gamma_g) = \{\gamma_g\}$$

for the remaining representatives g of the conjugacy classes. Therefore, (3.17) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_qG_5 \cong \mathbb{F}_q \bigoplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=8}^{10} M_{n_r}(\mathbb{F}_{q^2}). \tag{3.18}$$

Since $G_5' \cong Q_8$ with $G_5/G_5' \cong C_6$, we have $\mathbb{F}_qC_6 \cong \mathbb{F}_q^6$. This with (3.18) and Theorem 2.5 implies that $\mathbb{F}_qG_5 \cong \mathbb{F}_q^6 \bigoplus\limits_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \bigoplus\limits_{r=3}^{5} M_{n_r}(\mathbb{F}_{q^2})$, $n_r \geqslant 2$ with $42 = \sum\limits_{r=1}^{2} n_r^2 + 2\sum\limits_{r=3}^{5} n_r^2$, which further implies that the possible choices of n_r's are $(3, 3, 2, 2, 2)$,

$(2,2,2,2,3)$. For uniqueness, consider the normal subgroup $H_5 = \langle x, t\rangle$ of G_5 having order 4 with $G_5/H_5 \cong A_4$. From [19] and Theorem 2.5, we conclude that $(3,3,2,2,2)$ is the required choice.

Case (3): $p^k \equiv -1 \bmod 3$ and $p^k \equiv 1 \bmod 4$ which means $p^k \equiv 5 \bmod 12$. This means $I_{\mathbb{F}} = \{1,5\}$ and accordingly we can verify that

$$S(\gamma_y) = \{\gamma_y, \gamma_{y^2}\}, \qquad S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xy^2}\},$$
$$S(\gamma_{yz}) = \{\gamma_{yz}, \gamma_{y^2z}\}, \quad S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xy^2z}\},$$

and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of the conjugacy classes. Therefore, (3.17) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_qG_5 \cong \mathbb{F}_q \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=6}^{9} M_{n_r}(\mathbb{F}_{q^2}). \tag{3.19}$$

Since $G_5' \cong Q_8$ with $G_5/G_5' \cong C_6$, we have $\mathbb{F}_qC_6 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2$. This with Theorem 2.5 and (3.19) implies that $\mathbb{F}_qG_5 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=5}^{6} M_{n_r}(\mathbb{F}_{q^2})$, $n_r \geqslant 2$ with $42 = \sum_{r=1}^{4} n_r^2 + 2\sum_{r=5}^{6} n_r^2$, which further implies that the possible choices of n_r's are $(2,2,3,3,2,2)$, $(2,2,2,2,2,3)$. For uniqueness, again consider the normal subgroup $H_5 = \langle x, t\rangle$ of G_5. With the same approach used in Case (2), we conclude that $(2,2,3,3,2,2)$ is the required choice.

Case (4): $p^k \equiv -1 \bmod 3$ and $p^k \equiv -1 \bmod 4$, which means $p^k \equiv 11 \bmod 12$. This means $I_{\mathbb{F}} = \{1,11\}$ and accordingly we can verify that

$$S(\gamma_y) = \{\gamma_y, \gamma_{y^2}\}, \qquad S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xy^2z}\}, \quad S(\gamma_{yz}) = \{\gamma_{yz}, \gamma_{y^2z}\},$$
$$S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xy^2}\}, \quad S(\gamma_x) = \{\gamma_x, \gamma_{xt}\}, \qquad S(\gamma_g) = \{\gamma_g\}$$

for the remaining representatives g of the conjugacy classes. Therefore, (3.17) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_qG_5 \cong \mathbb{F}_q \bigoplus_{r=1}^{3} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=4}^{9} M_{n_r}(\mathbb{F}_{q^2}). \tag{3.20}$$

Since $G_5' \cong Q_8$ with $G_5/G_5' \cong C_6$, we have $\mathbb{F}_qC_6 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2$. This with Theorem 2.5 and (3.20) implies that $\mathbb{F}_qG_5 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=3}^{5} M_{n_r}(\mathbb{F}_{q^2})$, $n_r \geqslant 2$ with $42 = \sum_{r=1}^{2} n_r^2 + 2\sum_{r=3}^{5} n_r^2$, which further implies that the possible choices of n_r's are $(3,3,2,2,2)$, $(2,2,2,2,3)$. For uniqueness, again consider the normal subgroup $H_5 = \langle x, t\rangle$ of G_5. With the same approach used in Case (2), we conclude that $(3,3,2,2,2)$ is the required choice. $\square$

3.6. The group $G_6 = C_2 \times S_4$. Group G_6 has the following presentation:

$$\begin{aligned}\langle x, y, z, w, t \mid\ & x^2, z^{-1}x^{-1}zxz^{-1}, y^{-1}x^{-1}yx, w^{-1}x^{-1}wxt^{-1}w^{-1}, y^2,\\ & t^{-1}x^{-1}txt^{-1}w^{-1}, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, t^{-1}y^{-1}ty,\\ & z^3, w^{-1}z^{-1}wzt^{-1}w^{-1}, t^{-1}z^{-1}tzw^{-1}, w^2, t^{-1}w^{-1}tw, t^2\rangle.\end{aligned}$$

Group G_6 has 10 conjugacy classes as shown in the table below.

rep	1	x	y	z	w	xy	xw	yz	yw	xyw
order of rep	1	2	2	3	2	2	4	6	2	4

From the above discussion, clearly the exponent of G_6 is 12. Also, $G_6' \cong A_4$ and $G_6/G_6' \cong C_2 \times C_2$.

Theorem 3.6. *The unit group $U(\mathbb{F}_qG_6)$ of $\mathbb{F}_qG_6$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is isomorphic to $(\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_q)^4$.*

Proof. Since $\mathbb{F}_qG_6$ is semisimple, we have $\mathbb{F}_qG_6 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r)$. First assume that k is even, which means that for any prime $p > 3$, we have $p^k \equiv 1 \bmod 12$. This means that $|S(\gamma_g)| = 1$ for each $g \in G_6$ as $I_{\mathbb{F}} = \{1\}$. As $G_6' \cong A_4$ and $G_6/G_6' \cong C_2 \times C_2$, WD in this case follows on similar lines to Theorem 3.3, i.e. it is given by (3.10). Now we consider that k is odd, which means $p^k \in \{1, 5, 7, 11\} \bmod 12$. Here, we can verify that for all of these possibilities, $|S(\gamma_g)| = 1$ for each representative g of conjugacy classes. Therefore, WD is given by (3.10). □

4. Unit group of $\mathbb{F}_qG$ for non-metabelian group of order 54

In this section, we discuss the WD of $\mathbb{F}_qG$, where G is a non-metabelian group of order 54. There are 15 groups of order 54 up to isomorphism, but among these the only non-metabelian group is $G = ((C_3 \times C_3) \rtimes C_3) \rtimes C_2$ and it can be represented via four generators x, y, z, w as

$$\begin{aligned}\langle & x^2, z^{-1}x^{-1}zxz^{-1}, y^{-1}x^{-1}yxy^{-1}, w^{-1}x^{-1}wx, y^3,\\ & z^{-1}y^{-1}zyw^{-1}, w^{-1}y^{-1}wy, z^3, w^{-1}z^{-1}wz, w^3\rangle.\end{aligned}$$

Further, it can be seen that G has 10 conjugacy classes shown in the table below.

rep	e	x	y	z	w	xw	yz	w^2	xw^2	y^2z
order of rep	1	2	3	3	3	6	3	3	6	3

Theorem 4.1. *The unit group $U(\mathbb{F}_qG)$ of $\mathbb{F}_qG$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

(1) *for* $p \equiv 1 \bmod 6$ *and* k *is any positive integer or* $p \equiv 5 \bmod 6$ *and* k *is odd*

$$U(\mathbb{F}_q G) \cong (\mathbb{F}_q^*)^2 \times GL_2(\mathbb{F}_q)^4 \times GL_3(\mathbb{F}_q)^4,$$

(2) *for* $p \equiv 5 \bmod 6$ *and* k *is odd:*

$$U(\mathbb{F}_q G) \cong (\mathbb{F}_q^*)^2 \times GL_2(\mathbb{F}_q)^4 \times GL_3(\mathbb{F}_{q^2})^2.$$

P r o o f. As the group algebra $\mathbb{F}_q G$ is semisimple, we have

$$\mathbb{F}_q G \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{4.1}$$

Since p is an odd prime, we have the following two cases:

Case (1): $p \equiv 1 \bmod 6$ and k is any positive integer or $p \equiv 5 \bmod 6$ and k is an even integer. Then, clearly $q = p^k \equiv 1 \bmod 6$. This means $|S(\gamma_g)| = 1$ for each $g \in G$ as $I_{\mathbb{F}} = \{1\}$. Hence, (4.1), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_q G \cong \mathbb{F}_q \bigoplus_{r=1}^{9} M_{n_r}(\mathbb{F}_q) \quad \text{with } 53 = \sum_{r=1}^{9} n_r^2. \tag{4.2}$$

Further, it can be verified that G' is isomorphic to $(C_3 \times C_3) \rtimes C_3$. This means $\mathbb{F}_q(G/G') \cong \mathbb{F}_q \oplus \mathbb{F}_q$. Hence, Theorem 2.4 and (4.2) imply that the only possible values of n_r's satisfying (4.2) are $(1, 2, 2, 2, 2, 3, 3, 3, 3, 3)$.

Case (2): $p \equiv 5 \bmod 6$ and k is an odd positive integer. Then, clearly $q = p^k \equiv -1 \equiv 5 \bmod 6$. This means $I_{\mathbb{F}} = \{-1, 1\}$ and accordingly $S(\gamma_g) = \{\gamma_g\}$ for each representative g except when $g = w, w^2, xw, xw^2$. For these cases, we have $S(\gamma_w) = \{\gamma_w, \gamma_{w^2}\}$, $S(\gamma_{xw}) = \{\gamma_{xw}, \gamma_{xw^2}\}$. Therefore, this with (4.1), Theorems 2.1 and 2.2 implies that $\mathbb{F}_q G \cong \mathbb{F}_q \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=6}^{7} M_{n_r}(\mathbb{F}_{q^2})$. Incorporating Theorem 2.4 as in Case (1) to obtain $\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=5}^{6} M_{n_r}(\mathbb{F}_{q^2})$, where $n_r \geqslant 2$ with $52 = \sum_{r=1}^{4} n_r^2 + 2(n_5^2 + n_6^2)$. This gives the 3 choices $(3, 3, 3, 3, 2, 2)$, $(2, 2, 3, 3, 2, 3)$, $(2, 2, 2, 2, 3, 3)$ for n_r's and for uniqueness we need to discard 2 choices. Consider the normal subgroup $H = \langle w \rangle$ of G having order 3. It can be verified that $K = G/H \cong (C_3 \times C_3) \rtimes C_2$. To obtain the WD of $\mathbb{F}_q G$, we need to find the WD of $\mathbb{F}_q K$. Representation of K is $\langle x, y, z \mid x^2, z^{-1}x^{-1}zxz^{-1}, y^{-1}x^{-1}yxy^{-1}, y^3, z^3, z^{-1}y^{-1}zy, w^3 \rangle$. Further, K has 6 conjugacy classes shown in the table below.

rep	e	x	y	z	yz	y^2z
order of rep	1	2	3	3	3	3

For $p \equiv 5 \bmod 6$, it can be verified that $S(\gamma_k) = \{\gamma_k\}$ for each representative k of the conjugacy classes of K. Therefore, employ the fact that K' is isomorphic to $C_3 \times C_3$, we have $\mathbb{F}_q K \cong \mathbb{F}_q^2 \bigoplus\limits_{r=1}^{4} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$ with $16 = \sum\limits_{r=1}^{4} n_r^2$. This means that $F_q K \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4$. Finally, Theorem 2.5 implies that we remain with the only choice $(2, 2, 2, 2, 3, 3)$. $\square$

5. $U(\mathbb{F}_q G)$ for non-metabelian groups of order 72

The main objective of this section is to characterize the unit group of $\mathbb{F}_q G$, where G is a non-metabelian group of order 72. Up to isomorphism, there are 7 non-metabelian groups of order 72 from which 5, namely $G_1 = (C_3 \times A_4) \rtimes C_2$, $G_2 = C_3 \times S_4$, $G_3 = (S_3 \times S_3) \rtimes C_2$, $G_4 = C_3 \times SL(2, 3)$, $G_5 = (C_3 \times C_3) \rtimes Q_8$ have exponent 12 and rest 2, namely $G_6 = Q_8 \rtimes C_9$ and $G_7 = ((C_2 \times C_2) \rtimes C_9) \rtimes C_2$ have exponent 36.

5.1. The group $G_1 = (C_3 \times A_4) \rtimes C_2$. Group G_1 has the following presentation:

$$\begin{aligned}\langle x, y, z, w, t \mid\ & x^2, z^{-1}x^{-1}zxz^{-1}, y^{-1}x^{-1}yxy^{-1}, w^{-1}x^{-1}wxt^{-1}w^{-1},\\ & t^{-1}x^{-1}txt^{-1}w^{-1}, y^3, z^3, z^{-1}y^{-1}zy, w^{-1}y^{-1}wyt^{-1}w^{-1},\\ & t^{-1}y^{-1}tyw^{-1}, w^{-1}z^{-1}wz, t^{-1}z^{-1}tz, w^2, t^{-1}w^{-1}tw, t^2\rangle.\end{aligned}$$

Further, G_1 has 9 conjugacy classes, as shown in the table below.

rep	1	x	y	z	w	xw	yz	zw	y^2z
order of rep	1	2	3	3	2	4	3	6	3

From the above discussion, clearly the exponent of G_1 is 12. Also, $G_1' \cong C_3 \times A_4$.

Theorem 5.1. *The unit group $U(\mathbb{F}_q G_1)$ of $\mathbb{F}_q G_1$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is isomorphic to*

$$(\mathbb{F}_q^*)^2 \times GL_2(\mathbb{F}_q)^4 \times GL_3(\mathbb{F}_q)^2 \times GL_6(\mathbb{F}_q).$$

Proof. Since $\mathbb{F}_q G_1$ is semisimple, we have

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{5.1}$$

First assume that k is even, which means that for an odd prime p, we have $p^k \equiv 1 \bmod 12$. This means $|S(\gamma_g)| = 1$ for each $g \in G_1$. Hence, (5.1), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q \bigoplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q). \tag{5.2}$$

Using Theorem 2.4 with $G_1' \cong C_3 \times A_4$ in (5.2), we see that

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \bigoplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q), \quad \text{where } n_r \geqslant 2 \text{ with } 70 = \sum_{r=1}^{7} n_r^2. \tag{5.3}$$

The above gives us three possibilities $(2, 2, 2, 2, 2, 5, 5)$, $(2, 2, 2, 2, 3, 3, 6)$, and $(3, 3, 3, 3, 3, 3, 4)$ for the possible values of n_r's and for uniqueness we need to discard 2 choices. Consider the normal subgroup $H_1 = \langle z \rangle$ of G_1 having order 3. It can be verified that $K = G_1/H_1 \cong S_4$. Therefore, (3.6), (5.3) and Theorem 2.5 imply that

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus M_2(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_q)^2 \oplus M_6(\mathbb{F}_q). \tag{5.4}$$

Now we consider that k is odd. We shall discuss this case in three parts:
(1) $p^k \equiv 1 \bmod 12$,
(2) $p^k \equiv 1 \bmod 3$ and $p^k \equiv -1 \bmod 4$,
(3) $p^k \equiv -1 \bmod 3$ and $p^k \equiv \pm 1 \bmod 4$.

Case (1): $p^k \equiv 1 \bmod 12$. In this case, WD is given by (5.4).

Case (2): $p^k \equiv 1 \bmod 3$ and $p^k \equiv -1 \bmod 4$ which means $p^k \equiv 7 \bmod 12$. This means $I_{\mathbb{F}} = \{1, 7\}$ and accordingly $|S(\gamma_g)| = 1$ for each $g \in G_1$. Therefore, WD is given by (5.4).

Case (3): $p^k \equiv -1 \bmod 3$ and $p^k \equiv \pm 1 \bmod 4$ which means $p^k \equiv 5 \bmod 12$ or $p^k \equiv 11 \bmod 12$. This means $I_{\mathbb{F}} = \{1, 5\}$ or $I_{\mathbb{F}} = \{1, 11\}$ and accordingly $|S(\gamma_g)| = 1$ for each $g \in G_1$. Therefore, WD is given by (5.4). □

5.2. The group $G_2 = C_3 \times S_4$. Group G_2 has the following presentation:

$$\begin{aligned}\langle x, y, z, w, t \mid{}& x^2, z^{-1}x^{-1}zxz^{-1}, y^{-1}x^{-1}yx, w^{-1}x^{-1}wxt^{-1}w^{-1},\\ & t^{-1}x^{-1}txt^{-1}w^{-1}, y^3, z^3, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, t^{-1}y^{-1}ty,\\ & w^{-1}z^{-1}wzt^{-1}w^{-1}, t^{-1}z^{-1}tzw^{-1}, w^2, t^{-1}w^{-1}tw, t^2\rangle.\end{aligned}$$

Further, G_2 has 15 conjugacy classes, as shown in the table below.

rep	1	x	y	z	w	xy	xw	y^2	yz	yw	xy^2	xyw	y^2z	y^2w	xy^2w
order of rep	1	2	3	3	2	6	4	3	3	6	6	12	3	6	12

From the above discussion, clearly the exponent of G_2 is 12. Also, $G_2' \cong A_4$ and $G_2/G_2' \cong C_6$.

Theorem 5.2. *The unit group $U(\mathbb{F}_q G_2)$ of $\mathbb{F}_q G_2$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

(1) *for any p and k even or k odd with $p \equiv 1 \bmod 3$ and $p \equiv \pm 1 \bmod 4$,*

$$U(\mathbb{F}_q G_2) \cong (\mathbb{F}_q^*)^6 \oplus GL_2(\mathbb{F}_q)^3 \oplus GL_3(\mathbb{F}_q)^6,$$

(2) *for k odd and $p \equiv -1 \bmod 3$ and $p \equiv \pm 1 \bmod 4$,*

$$U(\mathbb{F}_q G_2) \cong (\mathbb{F}_q^*)^2 \oplus (\mathbb{F}_{q^2}^*)^2 \oplus GL_2(\mathbb{F}_q) \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_2(\mathbb{F}_{q^2}) \oplus GL_3(\mathbb{F}_{q^2})^2.$$

P r o o f. Since $\mathbb{F}_q G_2$ is semisimple, we have

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{5.5}$$

Now as in Theorem 5.1 for k even, we have $p^k \equiv 1 \bmod 12$ which means $|S(\gamma_g)| = 1$ for each $g \in G_2$. Hence, (5.5), Theorems 2.1 and 2.2 imply that $\mathbb{F}_q G_2 \cong \mathbb{F}_q \bigoplus_{r=1}^{14} M_{n_r}(\mathbb{F}_q)$. Using Theorem 2.4 with $G_2' \cong A_4$ in this to obtain $\mathbb{F}_q G_2 \cong \mathbb{F}_q^6 \bigoplus_{r=1}^{9} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$ with $66 = \sum_{r=1}^{9} n_r^2$. This gives us the only possibility $(2,2,2,3,3,3,3,3,3)$. Therefore, we have

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^6 \oplus M_2(\mathbb{F}_q)^3 \oplus M_3(\mathbb{F}_q)^6. \tag{5.6}$$

Now we consider that k is odd. We shall discuss this in same manner as in Theorem 5.1.

Case (1): $p^k \equiv 1 \bmod 12$. In this case WD is given by (5.6).

Case (2): $p^k \equiv 1 \bmod 3$ and $p^k \equiv -1 \bmod 4$, which means $p^k \equiv 7 \bmod 12$. This means $I_{\mathbb{F}} = \{1, 7\}$ and accordingly we can verify that $|S(\gamma_g)| = 1$ for each $g \in G_2$. Therefore, WD is given by (5.6).

Case (3): $p^k \equiv -1 \bmod 3$ and $p^k \equiv \pm 1 \bmod 4$ which means $p^k \equiv 5 \bmod 12$ or $p^k \equiv 11 \bmod 12$. This means $I_{\mathbb{F}} = \{1, 5\}$ or $I_{\mathbb{F}} = \{1, 11\}$ and accordingly we have

$$\begin{aligned} S(\gamma_y) &= \{\gamma_y, \gamma_{y^2}\}, & S(\gamma_{xy}) &= \{\gamma_{xy}, \gamma_{xy^2}\}, & S(\gamma_{yw}) &= \{\gamma_{yw}, \gamma_{y^2w}\}, \\ S(\gamma_{xyw}) &= \{\gamma_{xyw}, \gamma_{xy^2w}\}, & S(\gamma_{yz}) &= \{\gamma_{yz}, \gamma_{y^2z}\} & S(\gamma_g) &= \{\gamma_g\} \end{aligned}$$

for the remaining representatives g of conjugacy classes. Therefore, (5.6), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_qG_2 \cong \mathbb{F}_q \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=5}^{9} M_{n_r}(\mathbb{F}_{q^2}). \tag{5.7}$$

For $I_{\mathbb{F}} = \{1, 5\}$ or $I_{\mathbb{F}} = \{1, 11\}$, it is easy to see that $\mathbb{F}_qC_6 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2$. This, with (5.7) and Theorem 2.4, implies that

$$\mathbb{F}_qG_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \bigoplus_{r=1}^{3} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=4}^{6} M_{n_r}(\mathbb{F}_{q^2}), \tag{5.8}$$

$$\text{where } n_r \geqslant 2 \text{ with } 66 = \sum_{r=1}^{3} n_r^2 + 2\sum_{r=4}^{6} n_r^2.$$

The above gives us two possibilities $(2, 3, 3, 2, 3, 3)$ and $(2, 2, 2, 3, 3, 3)$, but we need to discard one of these. For that, consider the normal subgroup $H_2 = \langle y\rangle$ of G_2 having order 3. Observe that $G_2/H_2 \cong S_4$. Therefore, (3.6), (5.8) and Theorem 2.5 imply that $(2, 3, 3, 2, 3, 3)$ is the only choice. $\square$

5.3. The group $G_3 = (S_3 \times S_3) \rtimes C_2$. Group G_3 has the following presentation:

$$\begin{aligned}\langle x, y, z, w, t \mid\, & x^2, z^{-1}x^{-1}zx, y^{-1}x^{-1}yxz^{-1}, w^{-1}x^{-1}wxw^{-1}, t^{-1}x^{-1}tx,\\ & z^{-1}y^{-1}zy, y^2, z^2, w^{-1}y^{-1}wyt^{-1}w^{-2}, t^{-1}y^{-1}tyt^{-2}w^{-1},\\ & w^{-1}z^{-1}wzw^{-1}, t^{-1}z^{-1}tzt^{-1}, w^3, t^{-1}w^{-1}tw, t^3\rangle.\end{aligned}$$

Further, G_3 has 9 conjugacy classes, as shown in the table below.

rep	1	x	y	z	w	xy	xt	yw	wt
order of rep	1	2	2	2	3	4	6	6	3

Clearly the exponent of G_3 is 12 and we can verify that $G_3' \cong (C_3 \times C_3) \rtimes C_2$ with $G_3/G_3' \cong C_2 \times C_2$.

Theorem 5.3. *The unit group $U(\mathbb{F}_qG_3)$ of $\mathbb{F}_qG_3$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is isomorphic to $(\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q) \oplus GL_4(\mathbb{F}_q)^4$.*

Proof. Since $\mathbb{F}_qG_3$ is semisimple, we have

$$\mathbb{F}_qG_3 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{5.9}$$

Now as in Theorem 5.1 for k even, we have $p^k \equiv 1 \bmod 12$ which means $|S(\gamma_g)| = 1$ for each $g \in G_3$. Hence, (5.9), Theorems 2.1, 2.2 imply that $\mathbb{F}_q G_3 \cong \mathbb{F}_q^4 \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$ with $68 = \sum_{r=1}^{5} n_r^2$. This gives us two possibilities, namely $(2,4,4,4,4)$ and $(3,3,3,4,5)$, but we need one. For that, consider the normal subgroup $H_3 = \langle w, t\rangle$ of G_3 having order 9. Observe that $H_3 \cong C_3 \times C_3$ and $G_3/H_3 \cong D_8$. It can be clearly seen that WD of $\mathbb{F}_q D_8$ has no term of the form $M_3(\mathbb{F}_q)$ because of the dimension constraint. Therefore, Theorem 2.5 implies that $(2,4,4,4,4)$ is the only choice we have and hence

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q) \oplus M_4(\mathbb{F}_q)^4. \tag{5.10}$$

Now we consider that k is odd which means that $p^k \in \{1,5,7,11\} \bmod 12$. For all of these possibilities, we can easily see that $|S(\gamma_g)| = 1$ for all $g \in G_3$ and therefore, WD is given by (5.10). □

5.4. The group $G_4 = C_3 \times SL(2,3)$. Group G_4 has the following presentation:

$$\langle x, y, z, w, t \mid x^3, z^{-1}x^{-1}zxt^{-1}w^{-1}z^{-1}, y^{-1}x^{-1}yx, w^{-1}x^{-1}wxt^{-1}z^{-1}, t^{-1}x^{-1}tx, y^3, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, t^{-1}y^{-1}ty, z^2w^{-1}, w^{-1}z^{-1}wzt^{-1}, t^{-1}z^{-1}tz, w^2t^{-1}, t^{-1}w^{-1}tw, t^2\rangle.$$

Further, G_4 has 21 conjugacy classes, as shown in the two tables below.

rep	1	x	y	z	t	x^2	xy	xt	y^2	yz	yt	x^2y	x^2z
order of rep	1	3	3	4	2	3	3	6	3	12	6	3	6

rep	xy^2	xyt	y^2z	y^2t	x^2y^2	x^2yz	xy^2t	x^2y^2z
order of rep	3	6	12	6	3	6	6	6

From the above discussion, the exponent of G_4 is 12. Also, verify that $G_4' \cong Q_8$ and $G_4/G_4' \cong C_3 \times C_3$.

Theorem 5.4. *The unit group $U(\mathbb{F}_q G_4)$ of $\mathbb{F}_q G_4$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

(1) *for any p and k even or k odd with $p \equiv 1 \bmod 3$ and $p \equiv \pm 1 \bmod 4$,*

$$U(\mathbb{F}_q G_4) \cong (\mathbb{F}_q^*)^9 \oplus GL_2(\mathbb{F}_q)^9 \oplus GL_3(\mathbb{F}_q)^3,$$

(2) *for k odd and $p \equiv -1 \bmod 3$ and $p \equiv \pm 1 \bmod 4$,*

$$U(\mathbb{F}_q G_4) \cong \mathbb{F}_q^* \oplus (\mathbb{F}_{q^2}^*)^4 \oplus GL_2(\mathbb{F}_q) \oplus GL_3(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^2})^4 \oplus GL_3(\mathbb{F}_{q^2}).$$

P r o o f. Since $\mathbb{F}_q G_4$ is semisimple, we have

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{5.11}$$

Now as in Theorem 5.1 for k even, we get $\mathbb{F}_q G_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{20} M_{n_r}(\mathbb{F}_q)$. Using Theorem 2.4 with $G_4' \cong Q_8$ in above to obtain $\mathbb{F}_q G_4 \cong \mathbb{F}_q^9 \bigoplus_{r=1}^{12} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$ with $63 = \sum_{r=1}^{12} n_r^2$. This gives us the only possibility $(2,2,2,2,2,2,2,2,2,3,3,3)$. Therefore, we have

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q^9 \oplus M_2(\mathbb{F}_q)^9 \oplus M_3(\mathbb{F}_q)^3. \tag{5.12}$$

Now we consider that k is odd.

Case (1): $p^k \equiv 1 \bmod 12$. In this case WD is given by (5.12).

Case (2): $p^k \equiv 1 \bmod 3$ and $p^k \equiv -1 \bmod 4$ which means $p^k \equiv 7 \bmod 12$. This means that $I_{\mathbb{F}} = \{1,7\}$ and accordingly $|S(\gamma_g)| = 1$ for each $g \in G_4$. Therefore, WD is given by (5.12).

Case (3): $p^k \equiv -1 \bmod 3$ and $p^k \equiv \pm 1 \bmod 4$ which means $p^k \equiv 5 \bmod 12$ or $p^k \equiv 11 \bmod 12$. This means that $I_{\mathbb{F}} = \{1,5\}$ or $I_{\mathbb{F}} = \{1,11\}$ and accordingly we have

$$\begin{aligned}
S(\gamma_x) &= \{\gamma_x, \gamma_{x^2}\}, & S(\gamma_y) &= \{\gamma_y, \gamma_{y^2}\},\\
S(\gamma_{xy}) &= \{\gamma_{xy}, \gamma_{x^2y^2}\}, & S(\gamma_{xt}) &= \{\gamma_{xt}, \gamma_{x^2z}\},\\
S(\gamma_{yz}) &= \{\gamma_{yz}, \gamma_{y^2z}\}, & S(\gamma_{yt}) &= \{\gamma_{yt}, \gamma_{y^2t}\},\\
S(\gamma_{x^2y}) &= \{\gamma_{x^2y}, \gamma_{xy^2}\}, & S(\gamma_{xyt}) &= \{\gamma_{xyt}, \gamma_{x^2y^2z}\},\\
S(\gamma_{x^2yz}) &= \{\gamma_{x^2yz}, \gamma_{xy^2t}\}, & S(\gamma_g) &= \{\gamma_g\}
\end{aligned}$$

for the remaining representatives g of the conjugacy classes. Therefore, (5.11), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=3}^{11} M_{n_r}(\mathbb{F}_{q^2}). \tag{5.13}$$

For $I_{\mathbb{F}} = \{1,5\}$ or $I_{\mathbb{F}} = \{1,11\}$, it is easy to see that $\mathbb{F}_q(C_3 \times C_3) \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2}^4$. This, with (5.13) and Theorem 2.4, implies that $\mathbb{F}_q G_4 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2}^4 \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=3}^{7} M_{n_r}(\mathbb{F}_{q^2})$ with $63 = \sum_{r=1}^{2} n_r^2 + 2\sum_{r=3}^{7} n_r^2$, $n_r \geqslant 2$, which gives us the only possibility $(2,3,2,2,2,2,3)$.

□

5.5. The group $G_5 = (C_3 \times C_3) \rtimes Q_8$. Group G_5 has the following presentation:

$$\langle x,y,z,w,t \mid x^2z^{-1}, z^{-1}x^{-1}zx, y^{-1}x^{-1}yxz^{-1}, w^{-1}x^{-1}wxt^{-2}, t^{-1}x^{-1}txt^{-1}w^{-2}, z^{-1}y^{-1}zy, y^2z^{-1}, w^{-1}y^{-1}wyt^{-1}w^{-2}, t^{-1}y^{-1}tyt^{-2}w^{-2}, z^2, w^{-1}z^{-1}wzw^{-1}, t^{-1}z^{-1}tzt^{-1}, w^3, t^{-1}w^{-1}tw, t^3\rangle.$$

Further, G_5 has 6 conjugacy classes, as shown in the table below.

rep	1	x	y	z	w	xy
order of rep	1	4	4	2	3	4

Clearly the exponent of G_5 is 12 and we can verify that $G_5' \cong (C_3 \times C_3) \rtimes C_2$ with $G_5/G_5' \cong C_2 \times C_2$.

Theorem 5.5. *The unit group $U(\mathbb{F}_qG_5)$ of $\mathbb{F}_qG_5$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements and is isomorphic to $(\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q) \oplus GL_8(\mathbb{F}_q)$.*

Proof. Since $\mathbb{F}_qG_5$ is semisimple, we have

$$\mathbb{F}_qG_5 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{5.14}$$

Now as in Theorem 5.1 for k even, we have $p^k \equiv 1 \bmod 12$, which means $|S(\gamma_g)| = 1$ for each $g \in G_5$. Hence, (5.14), Theorems 2.1, and 2.2 imply that $\mathbb{F}_qG_5 \cong \mathbb{F}_q^4 \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q)$, where $n_r \geqslant 2$ with $68 = \sum_{r=1}^{2} n_r^2$. The above gives the only possibility namely (2,8) and therefore, the required WD is

$$\mathbb{F}_qG_5 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q) \oplus M_8(\mathbb{F}_q). \tag{5.15}$$

Now we consider that k is odd. We have $p^k \in \{1,5,7,11\} \bmod 12$. For all these possibilities, it can be verified that $|S(\gamma_g)| = 1$ for each $g \in G_5$. Therefore, WD is given by (5.15). $\square$

Now we characterize the unit group of $\mathbb{F}_qG$, where G is a non-metabelian group of order 72 and exponent 36.

5.6. The group $G_6 = Q_8 \rtimes C_9$. Group G_6 has the following presentation:

$$\langle x,y,z,w,t \mid x^3y^{-1}, z^{-1}x^{-1}zxt^{-1}w^{-1}z^{-1}, y^{-1}x^{-1}yx, w^{-1}x^{-1}wxt^{-1}z^{-1}, t^{-1}x^{-1}tx, y^3, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, t^{-1}y^{-1}ty, z^2t^{-1}, w^{-1}z^{-1}wzt^{-1}, t^{-1}z^{-1}tz, w^2t^{-1}, t^{-1}w^{-1}tw, t^2\rangle.$$

Further, G_6 has 21 conjugacy classes, as shown in the tables below.

rep	1	x	y	z	t	x^2	xy	xt	y^2	yz	yt	x^2y	x^2z
order of rep	1	9	3	4	2	9	9	18	3	12	6	9	18

rep	xy^2	xyt	y^2z	y^2t	x^2y^2	x^2yz	xy^2t	x^2y^2z
order of rep	9	18	12	6	9	18	18	18

Clearly the exponent of G_6 is 36. Also verify that $G_6' \cong Q_8$ and $G_6/G_6' \cong C_9$.

Theorem 5.6. *The unit group $U(\mathbb{F}_qG_6)$ of the group algebra $\mathbb{F}_qG_6$, for $q=p^k$, $p>3$ where $\mathbb{F}_q$ is a finite field having $q=p^k$ elements is as follows:*

(1) *for $p^k \equiv 1 \bmod 36$ or $p^k \equiv 19 \bmod 36$,*

$$U(\mathbb{F}_qG_6) \cong (\mathbb{F}_q^*)^9 \oplus GL_2(\mathbb{F}_q)^9 \oplus GL_3(\mathbb{F}_q)^3,$$

(2) *for $p^k \in \{5,11,23,29\} \bmod 36$,*

$$U(\mathbb{F}_qG_6) \cong \mathbb{F}_q^* \oplus \mathbb{F}_{q^2}^* \oplus \mathbb{F}_{q^6}^* \oplus GL_2(\mathbb{F}_q) \oplus GL_3(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^2}) \oplus GL_3(\mathbb{F}_{q^2}) \oplus GL_2(\mathbb{F}_{q^6}),$$

(3) *for $p^k \equiv 17 \bmod 36$ or $p^k \equiv 35 \bmod 36$,*

$$U(\mathbb{F}_qG_6) \cong \mathbb{F}_q^* \oplus (\mathbb{F}_{q^2}^*)^4 \oplus GL_2(\mathbb{F}_q) \oplus GL_3(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^2})^4 \oplus GL_3(\mathbb{F}_{q^2}),$$

(4) *for $p^k \in \{7,13,25,31\} \bmod 36$,*

$$U(\mathbb{F}_qG_6) \cong (\mathbb{F}_q^*)^3 \oplus (\mathbb{F}_{q^3}^*)^2 \oplus GL_2(\mathbb{F}_q)^3 \oplus GL_3(\mathbb{F}_q)^3 \oplus GL_2(\mathbb{F}_{q^3})^2.$$

Proof. As $\mathbb{F}_qG_6$ is semisimple, we have

$$\mathbb{F}_qG_6 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{5.16}$$

Since p is an odd prime, we have $p^k \in \{1,5,7,11,13,17,19,23,25,29,31,35\} \bmod 36$. We discuss each of the above mentioned possibilities one by one in the following cases.

Case (1): $p^k \equiv 1 \bmod 36$ or $p^k \equiv 19 \bmod 36$. In this case, we have $|S(\gamma_g)| = 1$ for each $g \in G_6$ as $I_{\mathbb{F}} = \{1\}$ or $I_{\mathbb{F}} = \{1, 19\}$. Hence, (5.16), Theorems 2.1 and 2.2 imply that $\mathbb{F}_q G_6 \cong \mathbb{F}_q \bigoplus_{r=1}^{20} M_{n_r}(\mathbb{F}_q)$. Using Theorem 2.4 with $G_6/G_6' \cong C_9$, we find

$$\mathbb{F}_q G_6 \cong \mathbb{F}_q^9 \bigoplus_{r=1}^{12} M_{n_r}(\mathbb{F}_q), \quad \text{where } n_r \geqslant 2 \text{ with } 63 = \sum_{r=1}^{12} n_r^2. \tag{5.17}$$

The above gives the only possibility $(2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3)$ for the possible values of n_r's. Therefore, (5.17) implies

$$\mathbb{F}_q G_6 \cong \mathbb{F}_q^9 \oplus M_2(\mathbb{F}_q)^9 \oplus M_3(\mathbb{F}_q)^3. \tag{5.18}$$

Case (2): $p^k \equiv 5 \bmod 36$ or $p^k \equiv 29 \bmod 36$. For both possibilities, we have $I_{\mathbb{F}} = \{1, 5, 13, 17, 25, 29\}$ and accordingly

$$\begin{aligned}
S(\gamma_x) &= \{\gamma_x, \gamma_{x^2y}, \gamma_{xy^2}, \gamma_{xy}, \gamma_{x^2}, \gamma_{x^2y^2}\}, \\
S(\gamma_{xt}) &= \{\gamma_{xt}, \gamma_{x^2yz}, \gamma_{xyt}, \gamma_{x^2y^2z}, \gamma_{xy^2t}, \gamma_{x^2z}\}, \\
S(\gamma_y) &= \{\gamma_y, \gamma_{y^2}\}, \\
S(\gamma_{yz}) &= \{\gamma_{yz}, \gamma_{y^2z}\}, \\
S(\gamma_{yt}) &= \{\gamma_{yt}, \gamma_{y^2t}\}, \\
S(\gamma_g) &= \{\gamma_g\}
\end{aligned}$$

for the remaining representatives g of conjugacy classes. Hence, (5.16), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_q G_6 \cong \mathbb{F}_q \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=3}^{5} M_{n_r}(\mathbb{F}_{q^2}) \bigoplus_{r=6}^{7} M_{n_r}(\mathbb{F}_{q^6}). \tag{5.19}$$

Since $G_6/G_6' \cong C_9$, it can be easily seen that $\mathbb{F}_q C_9 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus \mathbb{F}_{q^6}$. This, with (5.19) and Theorem 2.4, implies that $\mathbb{F}_q G_6 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2} \oplus \mathbb{F}_{q^6} \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=3}^{4} M_{n_r}(\mathbb{F}_{q^2}) \oplus M_{n_5}(\mathbb{F}_{q^6})$. Comparing dimensions on both the sides to obtain $63 = \sum_{r=1}^{2} n_r^2 + 2\sum_{r=3}^{4} n_r^2 + 6n_5^2$, $n_r \geqslant 2$, which gives the only possibility $(2, 3, 2, 3, 2)$.

Case (3): $p^k \equiv 11 \bmod 36$ or $p^k \equiv 23 \bmod 36$. For both possibilities, we have $I_{\mathbb{F}} = \{1, 11, 13, 23, 25, 35\}$. Further, we can verify that this case is exactly similar to Case (2).

Case (4): $p^k \equiv 17 \bmod 36$ or $p^k \equiv 35 \bmod 36$. For these possibilities, we have $I_{\mathbb{F}} = \{1, 17\}$ or $I_{\mathbb{F}} = \{1, 35\}$, respectively, and accordingly

$$\begin{aligned} S(\gamma_x) &= \{\gamma_x, \gamma_{x^2y^2}\}, & S(\gamma_y) &= \{\gamma_y, \gamma_{y^2}\}, \\ S(\gamma_{xy}) &= \{\gamma_{xy}, \gamma_{x^2y}\}, & S(\gamma_{xt}) &= \{\gamma_{xt}, \gamma_{x^2y^2z}\}, \\ S(\gamma_{x^2}) &= \{\gamma_{x^2}, \gamma_{xy^2}\}, & S(\gamma_{yz}) &= \{\gamma_{yz}, \gamma_{y^2z}\}, \\ S(\gamma_{yt}) &= \{\gamma_{yt}, \gamma_{y^2t}\}, & S(\gamma_{x^2z}) &= \{\gamma_{x^2z}, \gamma_{xy^2t}\}, \\ S(\gamma_{xyt}) &= \{\gamma_{xyt}, \gamma_{x^2yz}\}, & S(\gamma_g) &= \{\gamma_g\} \end{aligned}$$

for remaining g. Therefore, (5.16), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_q G_6 \cong \mathbb{F}_q \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=3}^{11} M_{n_r}(\mathbb{F}_{q^2}). \tag{5.20}$$

Also $G_6/G_6' \cong C_9$, which means $\mathbb{F}_q C_9 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2}^4$. This, with (5.20) and Theorem 2.4, implies that $\mathbb{F}_q G_6 \cong \mathbb{F}_q \oplus \mathbb{F}_{q^2}^4 \bigoplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=3}^{7} M_{n_r}(\mathbb{F}_{q^2})$. Applying the dimension formula to this to obtain $63 = \sum_{r=1}^{2} n_r^2 + 2 \sum_{r=3}^{7} n_r^2$, $n_r \geqslant 2$, which gives the only possibility $(2, 3, 2, 2, 2, 2, 3)$.

Case (5): $p^k \equiv 7 \bmod 36$ or $p^k \equiv 31 \bmod 36$. For both possibilities, we have $I_{\mathbb{F}} = \{1, 7, 13, 19, 25, 31\}$ and accordingly

$$\begin{aligned} &S(\gamma_x) = \{\gamma_x, \gamma_{xy^2}, \gamma_{xy}\}, \quad S(\gamma_{x^2}) = \{\gamma_{x^2}, \gamma_{x^2y}, \gamma_{x^2y^2}\}, \\ &S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyt}, \gamma_{xy^2t}\}, \quad S(\gamma_{x^2z}) = \{\gamma_{x^2z}, \gamma_{x^2yz}, \gamma_{x^2y^2z}\}, \quad \text{and} \quad S(\gamma_g) = \{\gamma_g\} \end{aligned}$$

for the remaining representatives g of conjugacy classes. Hence, (5.16), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_q G_6 \cong \mathbb{F}_q \bigoplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=9}^{12} M_{n_r}(\mathbb{F}_{q^3}). \tag{5.21}$$

Also $G_6/G_6' \cong C_9$, which means $\mathbb{F}_q C_9 \cong \mathbb{F}_q^3 \oplus \mathbb{F}_{q^3}^2$. This, with (5.21) and Theorem 2.4, implies that $\mathbb{F}_q G_6 \cong \mathbb{F}_q^3 \oplus \mathbb{F}_{q^3}^2 \bigoplus_{r=1}^{6} M_{n_r}(\mathbb{F}_q) \bigoplus_{r=7}^{8} M_{n_r}(\mathbb{F}_{q^3})$. Applying the dimension formula in the above to obtain $63 = \sum_{r=1}^{6} n_r^2 + 3 \sum_{r=7}^{8} n_r^2$, $n_r \geqslant 2$, which gives two possibilities, namely $(2, 2, 2, 3, 3, 3, 2, 2)$ and $(2, 2, 2, 2, 2, 2, 2, 3)$ but we need to discard one of these. For that, consider the normal subgroup $H_6 = \langle y \rangle$ of G_6 having order 3. Observe that $G_6/H_6 \cong SL(2,3)$, and from [11], we know that the WD of $\mathbb{F}_qG_6/H_6$ contains $M_2(\mathbb{F}_q)$ as well as $M_3(\mathbb{F}_q)$. Therefore, $(2,2,2,3,3,3,2,2)$ is the only possibility for n_r's.

Case (6): $p^k \equiv 13 \bmod 36$ or $p^k \equiv 25 \bmod 36$. For both the possibilities, we have $I_{\mathbb{F}} = \{1, 13, 25\}$ and one can verify that this case is similar to Case (5). □

5.7. The group $G_7 = ((C_2 \times C_2) \rtimes C_9) \rtimes C_2$. Group G_7 has the following presentation:

$$\langle x, y, z, w, t \mid x^2, z^{-1}x^{-1}zxz^{-1}, y^{-1}x^{-1}yxz^{-1}y^{-1}, w^{-1}x^{-1}wxt^{-1}w^{-1},\\ t^{-1}x^{-1}txt^{-1}w^{-1}, z^3, y^3z^{-2}, z^{-1}y^{-1}zy, w^{-1}y^{-1}wyt^{-1}w^{-1},\\ t^{-1}y^{-1}tyw^{-1}, w^{-1}z^{-1}wz, t^{-1}z^{-1}tz, w^2, t^{-1}w^{-1}tw, t^2\rangle.$$

Further, G_7 has 9 conjugacy classes, as shown in the table below.

rep	1	x	y	z	w	xw	y^2	zw	yz^2
order of rep	1	2	9	3	2	4	9	6	9

Clearly the exponent of G_7 is 36, and we can verify that $G_7' \cong (C_2 \times C_2) \rtimes C_9$ with $G_7/G_7' \cong C_2$.

Theorem 5.7. *The unit group $U(\mathbb{F}_qG_7)$ of $\mathbb{F}_qG_7$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

(1) *for* $p^k \in \{1, 17, 19, 35\} \bmod 36$

$$U(\mathbb{F}_qG_7) \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q)^4 \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_6(\mathbb{F}_q),$$

(2) $p^k \in \{5, 7, 11, 13, 23, 25, 29, 31\} \bmod 36$,

$$U(\mathbb{F}_qG_7) \cong (\mathbb{F}_q^*)^2 \oplus GL_2(\mathbb{F}_q) \oplus GL_3(\mathbb{F}_q)^2 \oplus GL_6(\mathbb{F}_q) \oplus GL_2(\mathbb{F}_{q^3}).$$

P r o o f. Since $\mathbb{F}_qG_7$ is semisimple, we have

$$\mathbb{F}_qG_7 \cong \mathbb{F}_q \bigoplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r). \tag{5.22}$$

Now, we proceed in a similar manner as in Theorem 5.6.

Case (1): $p^k \equiv 1 \bmod 36$ or $p^k \equiv 19 \bmod 36$. In this case, we have $|S(\gamma_g)| = 1$ for each $g \in G_7$ as $I_{\mathbb{F}} = \{1\}$ or $I_{\mathbb{F}} = \{1, 19\}$. Hence, (5.22), Theorems 2.1 and 2.2 imply that $\mathbb{F}_qG_7 \cong \mathbb{F}_q \bigoplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q)$. Using Theorem 2.4 with $G_7/G_7' \cong C_2$ in this to obtain $\mathbb{F}_qG_7 \cong \mathbb{F}_q \oplus \mathbb{F}_q \bigoplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q)$, with $70 = \sum_{r=1}^{7} n_r^2$, $n_r \geqslant 2$. This gives 3 possibilities $(2,2,2,2,2,5,5)$, $(2,2,2,2,3,3,6)$ and $(3,3,3,3,3,3,4)$ for the possible values of n_r's and we need to discard two. For that, consider the normal subgroup $H_7 = \langle z \rangle$ of G_7 having order 3. Observe that $G_7/H_7 \cong S_4$ and therefore, using (3.6) and Theorem 2.5, we conclude that $(2,2,2,2,3,3,6)$ is the only possibility.

Case (2): $p^k \equiv 5 \bmod 36$ or $p^k \equiv 29 \bmod 36$. For both possibilities, we have $I_{\mathbb{F}} = \{1,5,13,17,25,29\}$ and accordingly $S(\gamma_y) = \{\gamma_y, \gamma_{y^2}, \gamma_{yz^2}\}$, $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Hence, (5.22), Theorems 2.1 and 2.2 imply that $\mathbb{F}_q G_7 \cong \mathbb{F}_q \bigoplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \oplus M_6(\mathbb{F}_{q^3})$. Since $G_7/G_7' \cong C_2$, Theorem 2.4 further leads to

$$(5.23) \qquad \mathbb{F}_q G_7 \cong \mathbb{F}_q^2 \bigoplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus M_5(\mathbb{F}_{q^3}), \quad \text{with } 70 = \sum_{r=1}^{4} n_r^2 + 3n_5^2,\ n_r \geqslant 2.$$

The above gives us three possibilities, namely $(2,2,5,5,2)$, $(2,3,3,6,2)$, $(3,3,3,4,3)$. Now, again consider the normal subgroup H_7 of G_7. Therefore, (5.23) and Theorem 2.5 imply that $(2,3,3,6,2)$ is the only choice we have.

Case (3): $p^k \equiv 11 \bmod 36$ or $p^k \equiv 23 \bmod 36$. For both possibilities, we have $I_{\mathbb{F}} = \{1,11,13,23,25,35\}$. Further, we can verify that this case is exactly similar to Case (2).

Case (4): $p^k \equiv 17 \bmod 36$ or $p^k \equiv 35 \bmod 36$. For both possibilities, we have $I_{\mathbb{F}} = \{1,17\}$ or $I_{\mathbb{F}} = \{1,35\}$, respectively, and accordingly this case is exactly similar to Case (1).

Case (5): $p^k \equiv 7 \bmod 36$ or $p^k \equiv 31 \bmod 36$. For both possibilities, we have $I_{\mathbb{F}} = \{1,7,13,19,25,31\}$ and accordingly we can verify that this case is similar to Case (2).

Case (6): $p^k \equiv 13 \bmod 36$ or $p^k \equiv 25 \bmod 36$. For both possibilities, we have $I_{\mathbb{F}} = \{1,13,25\}$ and one can verify that this case is again similar to Case (2). □

6. Discussion

We have discussed the unit groups of semisimple group algebras of 14 non-metabelian groups. All the results are verified using GAP. It can be clearly seen that with the increase in the order of group, complexity in the determination of unique Wedderburn decomposition upsurges. This completes the study of the unit group of semisimple group algebras up to groups of order 72.

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