THE NIEMYTZKI PLANE IS $\varkappa$-METRIZABLE

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Abstract. We prove that the Niemytzki plane is $\varkappa$-metrizable and we try to explain the differences between the concepts of a stratifiable space and a $\varkappa$-metrizable space. Also, we give a characterisation of $\varkappa$-metrizable spaces which is modelled on the version described by Chigogidze.

Keywords: stratifiable space; $\varkappa$-metrizable space; Niemytzki plane; Sorgenfrey line

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1. Introduction

The aim of this paper is to present elementary or alternative proofs of some facts about the class of $\varkappa$-metrizable spaces. Our approach is focused on completely regular spaces, as it was intended by Shchepin, compare [8], page 164. The class of $\varkappa$-metrizable spaces was first considered by Shchepin, see [8]. This class contains all metric spaces and it is wide enough to include many important classes of spaces that are not metrizable, compare [8] and [9]. To emphasise our motivations, let us quote Sierpiński's book [10].

The theorems of any geometry (e.g. Euclidean) follow, as is well known, from a number of axioms, i.e. hypotheses about the space considered, and from accepted definitions. A given theorem may be a consequence of some of the axioms and may not require all of them.

As a by-product, we obtain a class of spaces which we call ro-stratifiable. We were not able to find a publication in which ro-stratifiable spaces are examined. As it is to be shown, the case of the Niemytzki plane, see the definition in [5], page 22, indicates that certain properties of the Euclidean metric are crucial in a non-metrizable setting.

Our notations are standard, following [5] or [11]. Let us recall that a subset U of a topological space X is *regular open* whenever it is the interior of a closed set;

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in other words, U is a regular open set whenever $U = \operatorname{int}_X \operatorname{cl}_X(U)$. We denote the family of all regular open subsets of X by $\mathrm{RO}(X)$. The complement of a regular open set is called a *regular closed* set. So, $F \subseteq X$ is a regular closed set whenever $F = \operatorname{cl} \operatorname{int} F$. A subset G of a topological space X is a *co-zero* set whenever there exists a continuous function $f\colon X \to [0,1]$ such that $G = f^{-1}((0,1])$.

A T_1-space X is *stratifiable* if, for the family $\mathcal{B}$ of all open subsets of X there exists a family of functions $\{f_U\colon U \in \mathcal{B}\}$ satisfying conditions (1)–(3):

(1) If $U \in \mathcal{B}$, then $U = f_U^{-1}((0,1])$.

(2) If $U, V \in \mathcal{B}$ and $U \subseteq V$, then $f_U(x) \leqslant f_V(x)$ for any $x \in X$.

(3) For any $U \in \mathcal{B}$, the function $f_U\colon X \to [0,1]$ is continuous.

(4) For any decreasing sequence (U^α) of regular open sets, if

$$W = \operatorname{int} \bigcap_\alpha U^\alpha,$$

then $f_W(x) = \inf_\alpha f_{U^\alpha}(x)$ for any $x \in X$.

A family $\{f_U\colon U \in \mathrm{RO}(X)\}$ of functions satisfying (1)–(4) is called a *$\varkappa$-metric.* Following Shchepin, see [8], page 164 and compare [9], page 407, a completely regular space is called *$\varkappa$-metrizable* whenever it has a $\varkappa$-metric.

The class of M_3-spaces, see [3], is thus the class of stratifiable spaces, compare [2]. Shchepin has introduced the concept of a $\varkappa$-metrizable space with the help of regular closed sets; conditions (K1)–(K4) for the notion of a $\varkappa$-metric, see [9], page 164, are direct translations, via de Morgan's laws, of conditions (1)–(4).

In the next part, we show why the Sorgenfrey line, see the definition in [5], page 21, is not stratifiable, even though it is $\varkappa$-metrizable. Also, we show that the double arrow space is ro-stratifiable, but not $\varkappa$-metrizable, see the ends of parts 2 and 3. This indicates that condition (4) is independent of conditions (1)–(3). A characterisation of a $\varkappa$-metrizable space is stated at Propositions 5 and 6. In the last part, we discuss the properties of the Niemytzki plane.

2. $\mathcal{B}$-approximations and ro-stratifiable spaces

If X is a T_0-space, $\mathcal{B}$ is a family of open subsets of X, then a family of functions $\{f_U\colon U \in \mathcal{B}\}$, where $f_U\colon X \to [0,1]$ for all U in $\mathcal{B}$, is called a *$\mathcal{B}$-stratification* if it fulfils conditions (1)–(3). If a space X has a $\mathcal{B}$-stratification, then the space X is said to be *$\mathcal{B}$-stratifiable.* If $\mathcal{A} \subseteq \mathcal{B}$ and the space X is $\mathcal{B}$-stratifiable, then it is also $\mathcal{A}$-stratifiable. If $\mathcal{B} = \mathrm{RO}(X)$, then we will say that X is ro-*stratifiable* instead of $\mathrm{RO}(X)$-stratifiable. If a space X is ro-stratifiable, then any regular open set of X is a co-zero set by conditions (1) and (3). Moreover, if the family $\mathcal{B} = \mathrm{RO}(X)$ fulfils conditions (1) and (3), then the space X is $\varkappa$-normal; recall that a completely regular space is $\varkappa$-*normal* whenever any pair of nonempty disjoint and regular closed sets can be separated by disjoint open sets, see [9], compare [1]. To see this, fix disjoint and regular closed subsets $F, G \subseteq X$. Conditions (1) and (3) imply that there exist continuous functions $f, g\colon X \to [0,1]$ such that $F = f^{-1}(0)$ and $G = g^{-1}(0)$. Then preimages of $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ via the continuous function $f/(f+g)$ separate F and G. Under the additional assumption that each regular closed subset of X is a G_δ-set, one can verify, using a modified proof of Urysohn's lemma, that if the space X is $\varkappa$-normal, then it is also ro-stratifiable. This additional assumption is necessary as shown below.

There are compact Hausdorff spaces, and hence $\varkappa$-normal spaces, which are not ro-stratifiable. For example, a compact Hausdorff space, containing a regular open subset which is not a co-zero set, cannot be ro-stratifiable. Any such space is $\varkappa$-normal, being a normal space. To see an example, set

$$Y = \{\alpha\colon \alpha \leqslant \omega_1\} \cup \Big\{\frac{1}{n}\colon n > 0\Big\}$$

and consider the linear order $(Y, <)$ which is the restriction of the well order of the ordinals on $\{\alpha\colon \alpha \leqslant \omega_1\}$ and inherits the order from the real line on $\{1/n\colon n > 0\}$ and if $\alpha \leqslant \omega_1$ and $n > 0$, then $\alpha < 1/n$. The linear topology on Y generated by $<$ is compact and Hausdorff. In this topology, there are regular open sets which are not co-zero sets, for example $\{\alpha\colon \alpha < \omega_1\}$.

In the above reasoning, we do not use condition (2). For more results concerning $\varkappa$-normal spaces, compare [6]. Also, there are many examples of completely regular spaces which are not $\varkappa$-normal, e.g. the ones which can be built using a technique called the Jones' machine, compare [7] or [1].

It was noted in [3], pages 106–107 that the Sorgenfrey line $\mathbb{S}$, i.e. the real line with a topology generated by the collection of all intervals of the form $[a, b)$, where $a, b \in \mathbb{R}$ and $a < b$, is not stratifiable, being a paracompact and perfectly normal space; in other words, if $\mathcal{B}$ is the family of all open subsets of $\mathbb{S}$, then no family $\{f_U\colon U \in \mathcal{B}\}$ of functions fulfils conditions (1)–(3) with respect to $\mathbb{S}$ and $\mathcal{B}$, see Proposition 1. Nonetheless, the family consisting of characteristic functions of closed-open sets of $\mathbb{S}$ fulfils conditions (1)–(3).

Proposition 1. *If* $\mathcal{A} = \{[x, y)\colon x < y\} \cup \{(x, y)\colon x, y \in \mathbb{Q}\}$, *then the Sorgenfrey line is not* $\mathcal{A}$*-stratifiable.*

P r o o f. Suppose that a family $\{f_U\colon U \in \mathcal{A}\}$ is an $\mathcal{A}$-stratification, i.e. it satisfies conditions (1)–(3) with respect to $\mathbb{S}$ and $\mathcal{A}$. For an interval $(a, a+2) \in \mathcal{A}$ and $n > 0$, put

$$R_n = (a, a+2) \cap \Big\{x\colon f_{[x,x+1)}(x) > \frac{1}{n}\Big\}.$$

Since $(a, a+1) \subseteq \bigcup\{R_n\colon\ n > 0\}$, using the Baire category theorem, choose n such that

$$(a, a+1) \cap \operatorname{int}\operatorname{cl} R_n \neq \emptyset,$$

where the interior and the closure are taken with respect to the Euclidean topology. Next, choose a rational number $x \in (a, a+1) \cap R_n$ and a decreasing sequence (x_k) converging to x such that $x_k \in (a, a+1) \cap R_n$. Thus, for all k we have $x_k + 1 \in (x, a+2)$, so by condition (2), we obtain

$$f_{(x,a+2)}(x_k) \geqslant f_{[x_k,x_k+1)}(x_k) > \frac{1}{n}.$$

Since $x \notin (x, a+2)$, by condition (1), we obtain $f_{(x,a+2)}(x) = 0$, which contradicts the continuity of $f_{(x,a+2)}$. □

It is known that the Sorgenfrey line $\mathbb{S}$ is a $\varkappa$-metrizable space, compare [12], page 507. Therefore the space $\mathbb{S}$ is ro-stratifiable. We present an alternative proof, using the sequential criterion for the continuity of a function. If U is a regular open subset of $\mathbb{S}$, then put

$$f_U(x) = \begin{cases} \sup\{(q-x)\colon\ [x,q) \subseteq U \cap [x, x+1)\} & \text{when } x \in U; \\ 0 & \text{when } x \notin U. \end{cases}$$

By the definition, the family $\{f_U\colon\ U \in \mathrm{RO}(\mathbb{S})\}$ fulfils conditions (1) and (2). To verify condition (3), we shall check that each function $f_U\colon\ \mathbb{S} \to [0,1]$ is continuous. Indeed, suppose that a sequence (x_n) is convergent to x. Since we consider convergence in $\mathbb{S}$, we can assume that for all n, $x \leqslant x_n$. If $x \in U$, then, by the definition of f_U, the sequence $(f_U(x_n))$ converges to $f_U(x)$. But if $x \notin U$, then the fact that $U \in \mathrm{RO}(\mathbb{S})$ implies that there is a decreasing sequence (y_n) converging to x such that $y_n \notin U$ and $x_n < y_n$. Then, again using the definition of f_U, we check that $f_U(x_n) \leqslant y_n - x$, which implies that the sequence $(f_U(x_n))$ converges to $0 = f_U(x)$.

Now, we will slightly modify this definition of a stratifiable space which was proposed in [2], page 1. Let $\mathbb{I} = (0,1) \cap \mathbb{Q}$ be the set of all rational numbers from the open unit interval. Fix a topological space X and its base $\mathcal{B}$. Let us assume that every $U \in \mathcal{B}$ is assigned a family $\{U_q\colon\ q \in \mathbb{I}\}$, consisting of open sets. We will call the collection $\{\{U_q\colon\ q \in \mathbb{I}\}\colon\ U \in \mathcal{B}\}$ a $\mathcal{B}$-*approximation* if it satisfies conditions (a)–(c).

(a) If $U \in \mathcal{B}$, then $U = \bigcup\{U_q\colon\ q \in \mathbb{I}\}$.
(b) If $U, V \in \mathcal{B}$, $q \in \mathbb{I}$ and $U \subseteq V$, then $U_q \subseteq V_q$.
(c) If $U \in \mathcal{B}$, $p, q \in \mathbb{I}$ and $p < q$, then $\mathrm{cl}(U_q) \subseteq U_p$.

Observe that if $\{\{U_q\colon\ q \in \mathbb{I}\}\colon\ U \in \mathrm{RO}(X)\}$ is an $\mathrm{RO}(X)$-approximation, then the family

$$\{\{\operatorname{int}\operatorname{cl}(U_q)\colon\ q \in \mathbb{I}\}\colon\ U \in \mathcal{B}\}$$

is also an RO(X)-approximation. The following propositions explain a connection between $\mathcal{B}$-approximations and $\mathcal{B}$-stratifications.

Proposition 2. *If a family $\{f_U\colon U \in \mathcal{B}\}$ is a $\mathcal{B}$-stratification, then the family*

$$\{\{f_U^{-1}((q,1])\colon q \in \mathbb{I}\}\colon U \in \mathcal{B}\}$$

is a $\mathcal{B}$-approximation.

Proof. The sets $U_q = f_U^{-1}((q,1])$ are open since each f_U is a continuous function. By the definition of f_U, conditions (1) and (a) are equivalent. For the same reasons, conditions (2) and (b) are equivalent. If $p < q$, then we have

$$\mathrm{cl}(U_q) \subseteq f_U^{-1}([q,1]) \subseteq U_p$$

since f_U is a continuous function. □

Proposition 3. *If a collection $\{\{U_q\colon q \in \mathbb{I}\}\colon U \in \mathcal{B}\}$ is a $\mathcal{B}$-approximation, then the family $\{f_U\colon U \in \mathcal{B}\}$, where*

$$f_U(x) = \begin{cases} \sup\{q \in \mathbb{I}\colon x \in U_q\} & \text{when } x \in U, \\ 0 & \text{when } x \notin U, \end{cases}$$

is a $\mathcal{B}$-stratification.

Proof. Clearly, condition (b) implies (2). For every $U \in \mathcal{B}$, the function f_U is upper semi-continuous since

$$f_U^{-1}([0,q)) = \bigcup\{X \setminus \mathrm{cl}(U_p)\colon p < q\}.$$

Indeed, if $f_U(x) < q$, then take $p_1, p_2 \in \mathbb{I}$ such that $f_U(x) < p_1 < p_2 < q$. Condition (c) implies that $x \notin U_{p_1} \supseteq \mathrm{cl}(U_{p_2})$. But when $p < q$ and $x \notin \mathrm{cl}(U_p)$, we have $x \notin U_p$. Again, by condition (c) and the definition of f_U, we check that $f_U(x) \leqslant p$. Each function f_U is lower semi-continuous since

$$f_U^{-1}((q,1]) = \bigcup\{U_p\colon p > q\}.$$

Indeed, if $f_U(x) > q$, then, by the definition of f_U, there exists $p > q$ such that $x \in U_p$. But when $x \in U_p$ and $p > q$, then $f_U(x) \geqslant p > q$. We have shown that each function f_U is continuous. Obviously, $U = \bigcup\{U_q\colon q \in \mathbb{I}\}$ implies $U = f_U^{-1}((0,1])$. □

Borges in Theorem 5.2 of [2] characterised a stratifiable space as a space with a $\mathcal{B}$-stratification, where $\mathcal{B}$ consists of all nonempty open sets. So, Propositions 2 and 3 bring us to another characterisation of stratifiable spaces.

Theorem 4. *Let $\mathcal{B}$ be the family of all open subsets of a topological space X. Then the space X is stratifiable if and only if it has a $\mathcal{B}$-approximation.*

Consider the lexicographic order on $\mathbb{D} = [0,1] \times \{0,1\}$. Note that $\mathbb{D}$ with the order topology is well known as the double arrow space or the two arrows space. Observe that regular open subsets of $\mathbb{D}$ are unions of pairwise disjoint closed-open intervals. Indeed, if $U \subseteq \mathbb{D}$ is an open set, then for every x in U let U_x be the union of all open intervals $I \subseteq U$ such that $x \in I$. If U is regular open, then the family $\{U_x\colon x \in U\}$ consists of closed and open (i.e. clopen) subsets of $\mathbb{D}$ and is a partition of U. Since for each $x \in U$ there exist $a, b \in \mathbb{R}$ such that $U_x = ((a,0),(b,1)) = [(a,1),(b,0)]$, we put

$$f_U(x) = \begin{cases} 0 & \text{when } x \notin U; \\ b-a & \text{when } x \in U_x = [(a,1),(b,0)]; \\ 1 & \text{when } x \in U \cap \{(0,0),(1,1)\}. \end{cases}$$

Then, check that the family $\{f_U\colon U \in \mathrm{RO}(\mathbb{D})\}$ is an $\mathrm{RO}(\mathbb{D})$-stratification.

3. On $\varkappa$-metrizable spaces

The notion of a $\varkappa$-metrizable space (a $\varkappa$-metric space) has been introduced by Shchepin, see [8], compare [9]. In [4], Chigogidze gave a characterisation of $\varkappa$-metrizable spaces. However, in Mathematical Reviews H. H. Wicke, the reviewer of [4], noted: *This article is an announcement of results; proofs are not included.* So, we propose a slight modification of the characterisation from [4]. Assume that a space X is completely regular and ro-stratifiable. Fix an $\mathrm{RO}(X)$-stratification $\{f_U\colon U \in \mathrm{RO}(X)\}$ and let $\{\{U_q\colon q \in \mathbb{I}\}\colon U \in \mathrm{RO}(X)\}$ be its corresponding $\mathrm{RO}(X)$-approximation, obtained by the formula $U_q = f_U^{-1}((q,1])$. Then consider the following condition, where the sequence (U^α) may be transfinite.

(d) If (U^α) is a decreasing sequence of regular open sets, $p, q \in \mathbb{I}$ and $p < q$, then

$$\bigcap_\alpha \mathrm{cl}(U^\alpha_q) \subseteq \Big(\mathrm{int}\bigcap_\alpha U^\alpha\Big)_p.$$

Because of [9], Theorem 18 the double arrow space $\mathbb{D}$, being compact, first-countable and of weight continuum, is not $\varkappa$-metrizable. Thus, the class of all ro-stratifiable spaces is wider than the class of all $\varkappa$-metrizable spaces.

Proposition 5. *If an* RO(X)*-approximation fulfils condition* (d), *then its corresponding* RO(X)*-stratification fulfils condition* (4).

Proof. Fix an RO(X)-approximation $\{\{U_q\colon q\in\mathbb{Q}\}\colon U\in \mathrm{RO}(X)\}$ and its corresponding RO(X)-stratification $\{f_U\colon U\in\mathrm{RO}(X)\}$. Let (U^α) be a decreasing sequence of regular open sets and let

$$W=\operatorname{int}\bigcap_\alpha U^\alpha.$$

Suppose that there exists x in X such that $f_W(x)\neq\inf_\alpha f_{U^\alpha}(x)$. By condition (2), we have $f_W(x)<\inf_\alpha f_{U^\alpha}(x)$. Choose rationals p, q such that

$$f_W(x)<p<q<\inf_\alpha f_{U^\alpha}(x).$$

Since $f_W(x)<p$ implies $x\in X\setminus\mathrm{cl}(W_p)$, by condition (d), we get

$$x\in X\setminus W_p\subseteq\bigcup_\alpha X\setminus\mathrm{cl}(U_q^\alpha).$$

So, there exists β such that $x\in X\setminus\mathrm{cl}(U_q^\beta)$, which implies that $f_{U^\beta}(x)\leqslant q$; a contradiction. □

Proposition 6. *If an* RO(X)*-stratification fulfils condition* (4), *then its corresponding* RO(X)*-approximation fulfils condition* (d).

Proof. Fix an RO(X)-stratification $\{f_U\colon U\in\mathrm{RO}(X)\}$ and its corresponding RO(X)-approximation $\{\{U_q\colon q\in\mathbb{Q}\}\colon U\in\mathrm{RO}(X)\}$. Let (U^α) be a decreasing sequence of regular open sets and let

$$W=\operatorname{int}\bigcap_\alpha U^\alpha.$$

Fix rationals $p<q$. Suppose that there exists $x\in\bigcap_\alpha\mathrm{cl}(U_q^\alpha)\setminus W_p$. Thus $f_W(x)\leqslant p<q$. By condition (4) there exists β such that $f_{U^\beta}(x)<q$. But the function f_{U^β} is continuous, hence there exists an open $V\ni x$ such that $V\subseteq f_{U^\beta}^{-1}([0,q))$. Therefore $V\cap U_q^\beta\neq\emptyset$. If $b\in V\cap U_q^\beta$, then $q\leqslant f_{U^\beta}(b)<q$; a contradiction. □

Assume that a family $\{f_U\colon U\in\mathrm{RO}(X)\}$ witnesses that a space X is ro-stratifiable. This family fulfils condition (4) if and only if it yields the RO(X)-approximation which fulfils condition (d). Thus, we obtain the following theorem, which is a characterisation of $\varkappa$-metrizable spaces, resembling those given in [4].

Theorem 7. *A T_0-space is $\varkappa$-metrizable if and only if it is* ro*-stratifiable.*

Now, we will show why the double arrow space $\mathbb{D}$ does not satisfy condition (4). This gives an alternative proof that this space is not $\varkappa$-metrizable, compare [9], Theorem 18. Suppose that the space $\mathbb{D}$ is $\varkappa$-metrizable. Then there exists an $\mathrm{RO}(\mathbb{D})$-approximation $\{\{U_q\colon q \in \mathbb{I}\}\colon U \in \mathrm{RO}(\mathbb{D})\}$. For every $U = [(a,1),(b,0)] \subseteq \mathbb{D}$ let

$$t(U) = \sup\{p \in (0,1) \cap \mathbb{Q}\colon U = U_p\}.$$

Since each U is a compact subset, by condition (a), numbers $t(U)$ are well defined. Put

$$R_p = \Big\{x \in \Big[0, \frac{1}{10}\Big]\colon t\Big(\Big((x,0),\Big(\frac{1}{5},1\Big)\Big)\Big) > p\Big\},$$

where $p \in (0,1)\cap\mathbb{Q}$. Note that $[0,\frac{1}{10}] \subseteq \bigcup\{R_p\colon p \in (0,1)\cap\mathbb{Q}\}$. By the Baire category theorem there is $p \in (0,1) \cap \mathbb{Q}$ such that $\operatorname{int}\operatorname{cl} R_p \neq \emptyset$. Thus, there exists x and an increasing sequence (x_k) converging to x such that for all k, $x_k \in R_p$. Then

$$\mathrm{cl}\Big(\Big((x_k,0),\Big(\frac{1}{5},1\Big)\Big)_p\Big) = \mathrm{cl}\Big(\Big((x_k,0),\Big(\frac{1}{5},1\Big)\Big)\Big) = \Big[(x_k,1),\Big(\frac{1}{5},0\Big)\Big]$$

and

$$\bigcap_k \Big[(x_k,1),\Big(\frac{1}{5},0\Big)\Big] = \Big[(x,0),\Big(\frac{1}{5},0\Big)\Big] \nsubseteq \Big((x,0),\Big(\frac{1}{5},1\Big)\Big),$$

which contradicts condition (d).

4. A $\varkappa$-metric for the Niemytzki plane

In [9], page 827, it has been noted that Zaitsev showed that the Niemytzki plane is $\varkappa$-normal. A proof of this fact can be found in [1]. The Niemytzki plane L is the closed upper half-plane $L = \mathbb{R} \times [0,\infty)$ endowed with the topology generated by the family of all open discs disjoint with the real axis $L_1 = \{(x,0)\colon x \in \mathbb{R}\}$ and all sets of the form $\{a\} \cup D$, where $D \subseteq L$ is an open disc tangent to L_1 at the point $a \in L_1$. For our purposes here, we use the following notations. Let $B((x,y),r)$ denote the open disc with centre (x,y) and radius r, and let $B^*(x,r) = B((x,r),r) \cup \{(x,0)\}$. Put

$$\mathcal{B} = \{B((x,y),r)\colon (x,y) \in L \setminus L_1 \text{ and } r \leqslant y,\ 0 < r \leqslant 1\}$$

and $\mathcal{B}^* = \{B^*(x,r)\colon 0 < r \leqslant 1 \text{ and } x \in \mathbb{R}\}$. Thus, the family $\mathfrak{B} = \mathcal{B}^* \cup \mathcal{B}$ is a base for L.

Fact 8. *The family $\mathfrak{B}$ is closed with respect to increasing unions.*

Proof. Let $U_n = B((x_n, y_n), r_n) \in \mathcal{B}$ and let (U_n) be an increasing sequence. Thus the sequence of reals (r_n), being bounded and increasing, is convergent, i.e. there exists r such that $r_n \to r$. Also, if (x_{k_n}, y_{k_n}), (x_{m_n}, y_{m_n}) are convergent subsequences to (x, y), (x', y'), respectively, then $(x, y) = (x', y')$. Indeed, if $(x, y) \neq (x', y')$, then the union $\bigcup\{U_n\colon\ n \geqslant 0\}$ is a disc with radius r and with two different centres, which is impossible in the Euclidean metric. Thus, the sequence (x_n, y_n) is convergent and let (x, y) be its limit. We have $\bigcup\{U_n\colon\ n \geqslant 0\} = B((x, y), r)$. If $U_n = B^*(x_n, r_n) \in \mathcal{B}^*$ for infinitely many n, then $(x_n, y_n) \to (x, r)$ and $\bigcup\{U_n\colon\ n \geqslant 0\} = B^*(x, r)$. □

The above proposition is surely folklore. We include it to make elementary methods, that we use below, more understandable. So, we think the reader will have no trouble verifying that if a sequence $(B^*(x_n, r_n))$ is decreasing, then the sequence (x_n) is constant, hence the set $\operatorname{int}_L \bigcap\{U_n\colon\ n \geqslant 0\}$ is empty or belongs to $\mathcal{B}^*$. We are in a position to define an $\mathrm{RO}(L)$-stratification. If $U = B((a, b), r) \in \mathcal{B}$, then put

$$f_U(x, y) = \begin{cases} r - \sqrt{(x-a)^2 + (y-b)^2} & \text{when } (x, y) \in U; \\ 0 & \text{for other cases.} \end{cases}$$

Thus, $f_U(x, y)$ is the distance between the point (x, y) and the complement of the open disc $B((a, b), r) = U$.

If $U = B((a, r), r) \cup \{(a, 0)\} \in \mathcal{B}^*$, then put

$$f_U(x, y) = \begin{cases} r - \sqrt{(x-a)^2 + (y-r)^2} & \text{when } (x, y) \in U \text{ and } r \leqslant y; \\ r & \text{when } (x, y) = (a, 0); \\ r - \dfrac{r|x-a|}{\sqrt{2yr - y^2}} & \text{when } (x, y) \in U \text{ and } y < r; \\ 0 & \text{for other cases.} \end{cases}$$

For every $U \in \mathcal{B}^*$ the function f_U is continuous in $L \setminus L_1$ with respect to the Euclidean topology, and hence it is continuous in $L \setminus L_1$ with respect to the Niemytzki plane. Suppose that $\lim\limits_{n\to\infty} (x_n, y_n) = (a, 0)$ with respect to the Niemytzki plane. Without loss of generality, we can assume that $(x_n, y_n) \in B((a, 1/n), 1/n)$ and $2/n < r$. Since for every natural number n the inequality $|x_n - a| < \sqrt{2y_n/n - y_n^2}$ holds and $y_n \to 0$, we obtain

$$r \geqslant f_U(x_n, y_n) = r - \frac{r|x_n - a|}{\sqrt{2y_n r - y_n^2}} \geqslant r - \frac{r\sqrt{2/n - y_n}}{\sqrt{2r - y_n}} \underset{n\to\infty}{\longrightarrow} r.$$

Thus, we have checked that for every $U \in \mathfrak{B}$ the function $f_U\colon L \to [0, 1]$ is continuous.

For every $V \in \mathrm{RO}(L)$ put

$$f_V(x,y) = \sup\{f_U(x,y)\colon\ U \in \mathfrak{B} \text{ and } U \subseteq V\}.$$

If $V \in \mathfrak{B}$, then both definitions of f_V coincide. Also, if $(x,y) \in L \setminus V$, then $f_V(x,y) = 0$.

Lemma 9. *If $(x,y) \in V \in \mathrm{RO}(L)$, then there exists $U \in \mathfrak{B}$ such that $U \subseteq V$ and $f_V(x,y) = f_U(x,y) > 0$.*

Proof. Suppose $0 < f_V(x,y) = \lim_{n\to\infty} f_{U_n}(x,y)$, where $U_n \in \mathfrak{B}$ and $U_n \subseteq V$. If $U_n \in \mathcal{B}$ for infinitely many n, then we can assume that there are sequences $((x_n,y_n))$, (r_n) and a, b, r such that $U_n = B((x_n,y_n),r_n)$, $x_n \to a$, $y_n \to b$ and $r_n \to r > 0$. We will show that $B((a,b),r) \subseteq V$. Indeed, fix $(c,e) \in B((a,b),r)$. Let $\varepsilon > 0$ be such that $d((c,e),(a,b)) = r - \varepsilon$, where d is the Euclidean distance. Choose n such that

$$r_n > r - \frac{\varepsilon}{2} \quad \text{and} \quad d((a,b),(x_n,y_n)) < \frac{\varepsilon}{2}.$$

We have

$$d((c,e),(x_n,y_n)) \leqslant d((c,e),(a,b)) + d((a,b),(x_n,y_n)) < r - \frac{\varepsilon}{2} < r_n.$$

Therefore $(c,e) \in U_n \subseteq V$. Moreover,

$$\begin{aligned} f_V(x,y) &= \lim_{n\to\infty} f_{U_n}(x,y) = \lim_{n\to\infty} \max\left\{0, r_n - \sqrt{(x-x_n)^2+(y-y_n)^2}\right\} \\ &= \max\left\{0, r - \sqrt{(x-a)^2+(y-b)^2}\right\} = f_{B((a,b),r)}(x,y). \end{aligned}$$

If $U_n \in \mathcal{B}^*$ for almost all n, then we can assume that there exist sequences (a_n), (r_n) and a, r such that $U_n = B^*(a_n,r_n)$, $a_n \to a$, $r_n \to r$ and $0 < y < r_n$, since the case when $y \geqslant r_n$ for infinitely many n one can reduce to the previous reasoning. Similarly to the above argument, we check that $B((a,r),r) \subseteq V$. Moreover, $B^*(a,r) \subseteq V$ since $V \in \mathrm{RO}(L)$. Therefore

$$\begin{aligned} f_V(x,y) &= \lim_{n\to\infty} f_{U_n}(x,y) = \lim_{n\to\infty} \max\left\{0, r_n - \frac{r_n|x-a_n|}{\sqrt{2yr_n - y^2}}\right\} \\ &= \max\left\{0, r - \frac{r|x-a|}{\sqrt{2yr-y^2}}\right\} = f_{B^*(a,r)}(x,y). \end{aligned}$$

The family $\{U \in \mathfrak{B}\colon\ (x,0) \in U\} = \{U \in \mathfrak{B}\colon\ (x,0) \in U \in \mathcal{B}^*\}$ is linearly ordered by inclusion, hence if $y = 0$, then, by Fact 8, the union $\bigcup_n U_n$ belongs to $\mathcal{B}^*$ and is contained in V. Thus, this union is the desired set. □

Proposition 10. *If $V \in \mathrm{RO}(L)$, then the function $f_V\colon\ L \to [0,1]$ is continuous.*

P r o o f. Assume that $\lim_{n\to\infty}(x_n, y_n) = (x, y)$ with respect to the topology of the Niemytzki plane. Suppose that $\lim_{n\to\infty} f_V(x_n, y_n) > p > f_V(x, y)$. For every n there exists $U_n \in \mathfrak{B}$ such that $(x_n, y_n) \in U_n \subseteq V$ and $f_{U_n}(x_n, y_n) > p$. Then there exist a_n, b_n, r_n such that either $U_n = B((a_n, b_n), r_n)$ or $U_n = B^*(a_n, r_n)$. We can assume that there are a, b, r such that $a_n \to a$ and $b_n \to b$ and $r_n \to r > 0$. If for infinitely many n, $U_n = B((a_n, b_n), r_n)$, then

$$p \leqslant \lim_{n\to\infty} f_{U_n}(x_n, y_n) = \lim_{n\to\infty} \max\bigl\{0, r_n - \sqrt{(x_n - a_n)^2 + (y_n - b_n)^2}\bigr\}$$
$$= \max\bigl\{0, r - \sqrt{(x - a)^2 + (y - b)^2}\bigr\} = f_U(x, y) \leqslant f_V(x, y) < p;$$

a contradiction. We can use the same argument if for infinitely many n, $r_n \leqslant y_n$. Thus, we can assume that for all n, $U_n = B^*(a_n, r_n)$ and $r_n \geqslant y_n$. If $y > 0$, then

$$p \leqslant \lim_{n\to\infty} f_{U_n}(x, y) = \lim_{n\to\infty} \max\Bigl\{0, r_n - \frac{r_n|x - a_n|}{\sqrt{2yr_n - y^2}}\Bigr\}$$
$$= \max\Bigl\{0, r - \frac{r|x - a|}{\sqrt{2yr - y^2}}\Bigr\} = f_{B^*(a,r)}(x, y) \leqslant f_V(x, y) < p;$$

again we have a contradiction.

If $y = 0$, then $a_n \to x$ and $B^*(x, r) \subseteq V$. So,

$$p > f_V(x, 0) \geqslant f_{B^*(x,r)}(x, 0) = r = \lim_{n\to\infty} r_n \geqslant \lim_{n\to\infty} f_{U_n}(x, y) \geqslant p;$$

a contradiction, which finishes the proof. □

Proposition 10 gives an alternative proof that the Niemytzki plane is $\varkappa$-normal since the family $\{f_V\colon V \in \mathrm{RO}(L)\}$ fulfils conditions (1) and (3).

Corollary 11. *The Niemytzki plane is* ro*-stratifiable.*

P r o o f. If $U_1, U_2 \in \mathcal{B}$ and $U_1 \subseteq U_2$, then $f_{U_1}(x, y) \leqslant f_{U_2}(x, y)$ since $f_{U_1}(x, y)$ equals to the distance between (x, y) and the complement of U_1, which is smaller than the distance $f_{U_2}(x, y)$ between (x, y) and the complement of U_2.

If $U = B((a, r), r)$ and $U^* = B^*(a, r)$, then $y \geqslant r$ implies $f_U(x, y) = f_{U^*}(x, y)$. But if $0 < y < r$, under the assumption $|x - a| \leqslant \sqrt{2yr - y^2}$, we get

$$\sqrt{(x - a)^2 + (y - r)^2} \geqslant \frac{r|x - a|}{\sqrt{2yr - y^2}}.$$

Therefore $f_U(x, y) \leqslant f_{U^*}(x, y)$ for every $(x, y) \in L$.

If $r_1 < r_2$ and $U_1 = B^*(a, r_1)$ and $U_2 = B^*(a, r_2)$, then we verify that $f_{U_1}(x, y) < f_{U_2}(x, y)$ for every $(x, y) \in U_2$. We have obtained that the family $\{f_U\colon U \in \mathcal{B} \cup \mathcal{B}^*\}$ fulfils conditions (1)–(3). Accordingly, the family $\{f_U\colon U \in \mathrm{RO}(L)\}$ fulfils conditions (1)–(3). □

Now, it seems natural to verify that the Niemytzki plane is $\varkappa$-metrizable.

Theorem 12. *The Niemytzki plane is $\varkappa$-metrizable.*

P r o o f. We have shown that the family $\{f_V\colon V \in \mathrm{RO}(L)\}$ is an $\mathrm{RO}(L)$-stratification of the Niemytzki plane L. So, it remains to show that it satisfies condition (4). Fix a decreasing chain $\{U_n\colon n>0\}$ consisting of regular open sets of the Niemytzki plane and put $W = \operatorname{int}\bigcap\{U_n\colon n>0\}$. Since for all n, $W \subseteq U_n$, we have $f_W(x) \leqslant \inf\{f_{U_n}(x)\colon n>0\}$ for any $x \in L$. Fix $x \in L$. For every n, by Lemma 9, there exists $V_n \in \mathfrak{B}$ such that $f_{U_n}(x) = f_{V_n}(x)$. If for infinitely many n, $V_n \in \mathcal{B}$, then we can assume that there exist sequences $((x_n,y_n))$, (r_n) and a, b, r such that $B((x_n,y_n),r_n) = V_n$, $x_n \to a$, $y_n \to b$ and $r_n \to r$. Then $B((a,b),r) \subseteq W$ and $f_{B((a,b),r)}(x) = \lim\limits_{n\to\infty} f_{V_n}(x)$. But if for all n, $V_n = B^*(x_n,r_n)$ and $x_n \to a$ and $r_n \to r > 0$, then we get $B^*(a,r) \subseteq W$ and $f_{B^*(a,r)}(x) = \lim\limits_{n\to\infty} f_{V_n}(x)$. Therefore $f_W(x) = \lim\limits_{n\to\infty} f_{U_n}(x)$. □

Proposition 13. *The Niemytzki plane is not stratifiable.*

P r o o f. Suppose that there exists a family of functions

$$\{f_U\colon U \text{ is an open subset of } L\}$$

which fulfils conditions (1), (2) and (3). Put

$$P_{m,n} = \Big\{x \in \mathbb{R}\colon\ f_{B^*(x,1)}(x,y) > \frac{1}{n} \text{ whenever } 0 \leqslant y < \frac{1}{m}\Big\}.$$

Since $\mathbb{R} = \bigcup\{P_{n,m}\colon m>0 \text{ and } n>0\}$, by the Baire category theorem, there exist a set $P_{n,m}$ and an interval (a,b) such that the intersection $P_{n,m}\cap(a,b)$ is dense in (a,b). Choose $(x_k,c_k) \in B^*(a,1/k)$ such that $x_k \in P_{n,m}\cap(a,b)$ and $c_k < 1/m$. Thus, the sequence $((x_k,c_k))$ is convergent to the point $(a,0)$ with respect to the Niemytzki plane. By condition (2) we get $f_{L\setminus\{(a,0)\}}(x_k,c_k) \geqslant f_{B^*(x_k,1)}(x_k,c_k) > 1/n$; a contradiction with $f_{L\setminus\{(a,0)\}}(a,0) = 0$. □

Put

$$g_{B^*(a,r)}(x,y) = \begin{cases} r - \sqrt{(a-x)^2+(r-y)^2} & \text{if } (x,y)\in B^*(a,r),\ r \leqslant y;\\ \Big(r - \dfrac{r|x-a|}{\sqrt{2yr-y^2}}\Big)\dfrac{(r-1)y+r}{r^2} & \text{if } (x,y)\in B^*(a,r),\ 0<y<r;\\ 1 & \text{if } (x,y) = (a,0);\\ 0 & \text{for other cases.} \end{cases}$$

Mimicking the proof of Corollary 11, we check that the family

$$\mathcal{G} = \{g_{B^*(a,r)}\colon\ B^*(a,r) \in \mathcal{B}^*\}$$

is a $\mathcal{B}^*$-stratification. But this family cannot be extended to an RO(L)-stratification. Indeed, the set $V = \{(x,y) \in L\colon\ x > 0\}$ is a regular open subset of the Niemytzki plane and $(0,0) \notin V$. Suppose that the family $\mathcal{G} \cup \{g_V\}$ fulfils conditions (1)–(3). Observe that $(1/(3n), 1/(6n)) \in B^*(0,1/n) \cap B^*(1/(3n), 1/(3n))$. Since $B^*(1/(3n), 1/(3n)) \subseteq V$, we get

$$g_V\Big(\frac{1}{3n}, \frac{1}{6n}\Big) \geqslant g_{B^*(1/(3n),1/(3n))}\Big(\frac{1}{3n}, \frac{1}{6n}\Big) > \frac{1}{2};$$

this is a contradiction with continuity of g_V and the equality $g_V(0,0) = 0$.

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