ON THE UNIT GROUP OF A SEMISIMPLE GROUP ALGEBRA $\mathbb{F}_qSL(2,\mathbb{Z}_5)$

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Abstract. We give the characterization of the unit group of $\mathbb{F}_qSL(2,\mathbb{Z}_5)$, where $\mathbb{F}_q$ is a finite field with $q=p^k$ elements for prime $p>5$, and $SL(2,\mathbb{Z}_5)$ denotes the special linear group of 2×2 matrices having determinant 1 over the cyclic group $\mathbb{Z}_5$.

Keywords: unit group; finite field; Wedderburn decomposition

MSC 2020: 16U60, 20C05

1. Introduction

Let $U(\mathbb{F}G)$ denote the unit group of the group algebra $\mathbb{F}G$ over the finite field $\mathbb{F}$ and the finite group G. For elementary definitions and results related to group rings, we refer to [17]. The units of the group rings are very important from an application point of view. As an application of the units of group rings, Hurley suggested the construction of convolutional codes from units in group rings (see [6]). The correspondence between a group ring and a ring of matrices was proposed in [5]. For the other applications of units, see [8], [10], [12], [20].

A great deal of research has been carried out in the direction of deducing the structure of the unit group of group algebras $\mathbb{F}G$. For instance, in [16], the case study is done when G is an abelian group of finite order. Articles [1], [4], [11], [9], and [14] provide the structure of the unit groups $U(\mathbb{F}G)$ for some of the dihedral groups G. For the alternating group A_4, the structure of $U(\mathbb{F}A_4)$ for the finite field $\mathbb{F}$ has been discussed in [3], [18]. For some nonabelian groups of small order, a characterization of the unit groups of their group algebras has been given in [13], [19], and [21]. Further, for the circulant matrices, unit groups of their group algebras have been discussed in [10], [20]. Maheshwari and Sharma (see [8]) gave the characterization of the unit

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group of the group algebra $\mathbb{F}_q SL(2,\mathbb{Z}_3)$, where $q = p^k$, $p \geqslant 5$ and $SL(2,\mathbb{Z}_3)$ is the group of 2×2 matrices having determinant 1 over the field $\mathbb{Z}_3$. Further, the unit groups of the semisimple group algebras of non-metabelian groups up to order 72 have been discussed in [15]. The paper [15] completes the study of the unit groups of semisimple group algebras of all the groups up to order 72.

The main motive of this paper is to characterize the unit group $U(\mathbb{F}_q SL(2,\mathbb{Z}_5))$ of $\mathbb{F}_q SL(2,\mathbb{Z}_5)$, where $q = p^k$ and $p > 5$. To characterize the unit group $U(\mathbb{F}_q SL(2,\mathbb{Z}_5))$, we utilize the results of [2] and [12] to obtain all the simple components (or Wedderburn decomposition) of $\mathbb{F}_q SL(2,\mathbb{Z}_5)$. Then it is straightforward to deduce the unit group from the knowledge of Wedderburn decomposition.

The rest of the paper is organized as follows. Section 2 sheds light on the basic definitions and known results required in our work. We present our main result in Section 3. Section 4 concludes the paper.

2. Preliminaries

Let e denote the exponent of G, ζ be the primitive eth root of unity, $\mathbb{F}$ be a finite field and $J(\mathbb{F}G)$ be the Jacobson radical of the group algebra $\mathbb{F}G$. On the lines of [2], let us denote

$$I_F = \{n\colon \ \zeta \mapsto \zeta^n \text{ is an automorphism of } \ \mathbb{F}(\zeta) \text{ over } \mathbb{F}\}.$$

Since the Galois group $\mathrm{Gal}(\mathbb{F}(\zeta),\mathbb{F})$ is a cyclic group, for any $\tau \in \mathrm{Gal}(\mathbb{F}(\zeta),\mathbb{F})$, there exists some s which is invertible modulo e such that $\tau(\zeta) = \zeta^s$. In other words, I_F is a subgroup of the multiplicative group $\mathbb{Z}_e^*$ (the group of integers which are invertible modulo e). For any p-regular element $g \in G$, i.e. an element whose order is not divisible by p, let the sum of all the conjugates of g be denoted by γ_g, and the cyclotomic $\mathbb{F}$ class of γ_g be denoted by

$$S(\gamma_g) = \{\gamma_{g^n}\colon \ n \in I_F\}.$$

Now let us recall two results from [2]. The first one relates the number of cyclotomic $\mathbb{F}$ classes with the number of simple components of $\mathbb{F}G/J(\mathbb{F}G)$ and the second one is about the cardinality of any cyclotomic $\mathbb{F}$ class in G.

Theorem 2.1. *The number of simple components of $\mathbb{F}G/J(\mathbb{F}G)$ and the number of cyclotomic $\mathbb{F}$ classes in G are equal.*

Theorem 2.2. *Let ζ be defined as above and j be the number of cyclotomic $\mathbb{F}$ classes in G. If K_i, $1 \leqslant i \leqslant j$, are the simple components of the center of $\mathbb{F}G/J(\mathbb{F}G)$*

and S_i, $1 \leqslant i \leqslant j$, are the cyclotomic $\mathbb{F}$ classes in G, then $|S_i| = [K_i : \mathbb{F}]$ for each i, after the suitable ordering of the indexes.

In order to determine the unit group of the group algebra $\mathbb{F}_q SL(2, \mathbb{Z}_5)$, we need to determine its Wedderburn decomposition, i.e. the simple components of $\mathbb{F}_q SL(2, \mathbb{Z}_5)$. From prior knowledge, we can always guarantee that $\mathbb{F}_q$ is one of its simple components.

Lemma 2.1 ([17], Corollary 2.5.4)**.** *Let M be a semisimple module and $M = \bigoplus_{i \in I} M_i$ be its decomposition as a direct sum of simple modules. If N is a submodule of M, then there exists a subset J of I such that $N \cong \bigoplus_{i \in J} M_i$.*

In the view of the above lemma, it is easy to deduce that $\mathbb{F}_q$ is one of the simple components of $\mathbb{F}_q SL(2, \mathbb{Z}_5)$. The next theorem tells us about the elements of the multiplicative group I_F.

Theorem 2.3 ([7], Theorem 2.21)**.** *Let $\mathbb{F}$ be a finite field with prime power order q. If e is such that $\gcd(e, q) = 1$, ζ is the primitive eth root of unity and z is the order of q modulo e, then we have*

$$I_F = \{1, q, q^2, \ldots, q^{z-1}\} \bmod e.$$

To this end, let us now recall a result which will be helpful in the determination of the commutative simple components of the group algebra $\mathbb{F}_q G$.

Theorem 2.4 ([17], Proposition 3.6.11)**.** *If RG is a semisimple group algebra, then*

$$RG \cong R(G/G') \oplus \Delta(G, G'),$$

where G' is the commutator subgroup of G, $R(G/G')$ is the sum of all commutative simple components of RG, and $\Delta(G, G')$ is the sum of all others.

Let us end this section by recalling a generalized version of the above Theorem 2.4. This result would be very crucial in obtaining the Wedderburn decomposition of the group algebra $\mathbb{F}_q SL(2, \mathbb{Z}_5)$.

Theorem 2.5 ([17], Proposition 3.6.7)**.** *Let RG be a semisimple group algebra and H be a normal subgroup of G. Then*

$$RG \cong R(G/H) \oplus \Delta(G, H),$$

where $\Delta(G, H)$ is a left ideal of RG generated by the set $\{h - 1 \colon h \in H\}$.

3. Unit group of $\mathbb{F}_q SL(2, \mathbb{Z}_5)$

In this section, we give the characterization of the unit group of group algebra $\mathbb{F}_q SL(2, \mathbb{Z}_5)$ for $p > 5$. If $\mathbb{F}_q$ is a finite field of order $q = p^k$, then the order of $SL(n, \mathbb{F}_q)$ is given by

$$\frac{1}{q-1}(q^n - 1)(q^n - q)\dots(q^n - q^{n-1}).$$

In our case, $G = SL(2, \mathbb{Z}_5)$ and hence $|G| = 120$. Since $p > 5$ and it does not divide the order of $|G|$, $J(\mathbb{F}_q G)$ is 0 by the well known Maschke's theorem (see [17]). This means that the group algebra $\mathbb{F}_q SL(2, \mathbb{Z}_5)$ is semisimple for $p > 5$.

In order to deduce the unit group of the group algebra $\mathbb{F}_q SL(2, \mathbb{Z}_5)$, let us first discuss the structure of the conjugacy classes of G. It can be verified that G has 9 conjugacy classes. Let us denote these classes by $[g_i]$, $1 \leqslant i \leqslant 9$, where for each i, g_i (defined below) represent the representative of the ith conjugacy class. To be more precise, we have:

$$g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}. \tag{1}$$

Moreover, g_1, and g_2 are the only elements in their conjugacy classes. Also, $|g_1| = 1$, and $|g_2| = 2$.

$$g_3 = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}, \quad g_4 = \begin{bmatrix} 0 & 3 \\ 3 & 3 \end{bmatrix}, \tag{2}$$

and both have 12 elements in their conjugacy classes. Also, $|g_3| = |g_4| = 10$.

$$g_5 = \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix}, \quad g_6 = \begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix}, \tag{3}$$

and both have 12 elements in their conjugacy classes. Also, $|g_5| = |g_6| = 5$.

$$g_7 = \begin{bmatrix} 0 & 4 \\ 1 & 4 \end{bmatrix}, \quad g_8 = \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}, \tag{4}$$

and both have 20 elements in their conjugacy classes. Also, $|g_7| = 3$ and $|g_8| = 6$.

$$g_9 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \tag{5}$$

and it has 30 elements in its conjugacy class. Moreover, $|g_9| = 4$.

From the above description, it is clear that the exponent of G is 60. Also, note that the derived subgroup G' of G is G. To this end, let us now discuss our main result on the Wedderburn decomposition of the group algebra $\mathbb{F}_q SL(2,\mathbb{Z}_5)$ for $p > 5$.

Theorem 3.1. *The Wedderburn decomposition of $\mathbb{F}_q SL(2,\mathbb{Z}_5)$ for $p > 5$, where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements, is as follows:*

Conditions on k and p	*Wedderburn decomposition*
k is even and $p > 5$ arbitrary or *k is odd and $q \equiv \pm 1 \bmod 5$*	$\mathbb{F}_q \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q)^2 \oplus M_5(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q)$
k is odd $q \equiv \pm 2 \bmod 5$	$\mathbb{F}_q \oplus M_4(\mathbb{F}_q)^2 \oplus M_5(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2})$

Proof. Since we know that $\mathbb{F}_q G$ is semisimple, its Wedderburn decomposition is given by

$$\mathbb{F}_q G \cong \bigoplus_{t=1}^{j} M_{n_t}(\mathbb{F}_t),$$

where for each t, $\mathbb{F}_t$ is a finite extension of $\mathbb{F}$, $n_t \geqslant 1$ and $G = SL(2,\mathbb{Z}_5)$. From the above isomorphism, the Wedderburn decomposition of $\mathbb{F}_q G$ can be determined, provided n_t's and $\mathbb{F}_t'$s are known for each $1 \leqslant t \leqslant j$. So, for this, by utilizing Lemma 2.1 and re-ordering the indexes (if required), one can obtain

$$\mathbb{F}_q G \cong \mathbb{F}_q \bigoplus_{t=1}^{j-1} M_{n_t}(\mathbb{F}_t). \tag{3.1}$$

For any odd prime greater than 5, with the usage of the Chinese remainder theorem, it can be easily seen that $q = p^k \equiv 1 \bmod 60$, whenever k is a multiple of 4. Therefore, for this case, we can verify that $|S(\gamma_g)| = 1$ for each $g \in G$. Hence, (3.1), and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_q G \cong \mathbb{F}_q \bigoplus_{t=1}^{8} M_{n_t}(\mathbb{F}_q). \tag{3.2}$$

Further, apply the dimesion formula to (3.2) to obtain

$$119 = \sum_{t=1}^{8} n_t^2, \quad n_t \geqslant 1 \ \forall\ t.$$

Moreover, use the fact that $G' = SL(2, \mathbb{Z}_5)$ and Theorem 2.4 in (3.2) to deduce that

$$\mathbb{F}_q G \cong \mathbb{F}_q \bigoplus_{t=1}^{8} M_{n_t}(\mathbb{F}_q) \quad \text{with } 119 = \sum_{t=1}^{8} n_t^2,\ n_t \geqslant 2\ \forall\, t. \tag{3.3}$$

The above equation provides 7 possibilities for the possible choices of n_t's, namely

$$(2,2,2,2,2,3,3,9),\ (2,2,2,2,2,5,5,7),\ (2,2,2,2,3,3,6,7),\ (2,2,2,3,3,3,4,8),$$
$$(2,2,2,4,4,5,5,5),\ (2,2,3,3,4,4,5,6) \text{ and } (3,3,3,3,3,3,4,7).$$

In order to uniquely determine the Wedderburn decomposition of the group algebra $\mathbb{F}_q G$, let us consider the subgroup H of G generated by $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$. It can be observed that H is a normal subgroup of G with $G/H \cong A_5$, where A_5 is the group of all even permutations of degree 5. To this end, let us recall from [12], Theorem 4.1, that for $q \equiv \pm 1 \bmod 5$, we have

$$\mathbb{F}_q A_5 \cong \mathbb{F}_q \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q) \oplus M_5(\mathbb{F}_q). \tag{3.4}$$

Therefore, by combining equations (3.3), (3.4) and Theorem 2.5, we conclude that $(2,2,3,3,4,4,5,6)$ is the only possibility for n_t's which means that

$$\mathbb{F}_q G \cong \mathbb{F}_q \oplus M_2(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^2 \oplus M_4(\mathbb{F}_q)^2 \oplus M_5(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q). \tag{3.5}$$

Now we move on to the possibility that k is a multiple of 2 but not of 4. In this case, we have

$$p^k \equiv 1 \bmod 3, \quad p^k \equiv 1 \bmod 4, \quad p^k \equiv \pm 1 \bmod 5.$$

We discuss the above possibility in the following two cases:

(a) $p^k \equiv 1 \bmod 3$, $p^k \equiv 1 \bmod 4$, $p^k \equiv 1 \bmod 5$.
(b) $p^k \equiv 1 \bmod 3$, $p^k \equiv 1 \bmod 4$, $p^k \equiv -1 \bmod 5$.

For part (a), we have $p^k \equiv 1 \bmod 60$. Therefore, the Wedderburn decomposition in this case is exactly same as given in (3.5).

For part (b), employ the Chinese remainder theorem to obtain

$$q = p^k \equiv 49 \bmod 60.$$

In this case, for any $g \in G$, we have

$$I_F = \{1, 49\} \text{ which means } S(\gamma_g) = \{\gamma_g, \gamma_{g^{49}}\}.$$

It can be verified that in this case, $|S(\gamma_g)| = 1$ for each $g \in G$. Hence, the Wedderburn decomposition is exactly same as given in (3.5).

The next possibility is when k is odd and it is discussed in the following 8 cases.

Case 1: $p \equiv 1 \bmod 3$, $p \equiv 1 \bmod 4$, and $p \equiv \pm 1 \bmod 5$. This means $p^k \equiv 1 \bmod 60$ or $p^k \equiv 49 \bmod 60$. For both of these possibilities, it can be verified that $S(\gamma_g) = \{\gamma_g\}$ for each $g \in G$. Hence, the Wedderburn decomposition is given by (3.5).

Case 2: $p \equiv 1 \bmod 3$, $p \equiv 1 \bmod 4$, and $p \equiv \pm 2 \bmod 5$. For this case, we get $p^k \equiv 1 \bmod 12$, $p^k \equiv 2 \bmod 5$ or $p^k \equiv 1 \bmod 12$, $p^k \equiv 3 \bmod 5$ which further implies that

$$p^k \equiv 37 \bmod 60, \quad \text{or} \quad p^k \equiv 13 \bmod 60.$$

For both of these possibilities, we have $I_F = \{1, 13, 37, 49\}$ which means that

$$S(\gamma_{g_1}) = \{\gamma_{g_1}\}, \quad S(\gamma_{g_2}) = \{\gamma_{g_2}\}, \quad S(\gamma_{g_3}) = \{\gamma_{g_3}, \gamma_{g_4}\}, \quad S(\gamma_{g_5}) = \{\gamma_{g_5}, \gamma_{g_6}\},$$
$$S(\gamma_{g_7}) = \{\gamma_{g_7}\}, \quad S(\gamma_{g_8}) = \{\gamma_{g_8}\}, \quad S(\gamma_{g_9}) = \{\gamma_{g_9}\}.$$

Therefore, (3.1) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_q G \cong \mathbb{F}_q \bigoplus_{t=1}^{4} M_{n_t}(\mathbb{F}_q) \bigoplus_{t=5}^{6} M_{n_t}(\mathbb{F}_{q^2}).$$

To this end, let us now apply the dimension formula in the above to get

$$119 = \sum_{t=1}^{4} n_t^2 + 2n_5^2 + 2n_6^2, \quad n_t \geqslant 1 \; \forall\, t.$$

Now employ Theorem 2.4 and $G' = SL(2, \mathbb{Z}_5)$ to further obtain that

$$\text{(3.6)} \quad \mathbb{F}_q G \cong \mathbb{F}_q \bigoplus_{t=1}^{4} M_{n_t}(\mathbb{F}_q) \bigoplus_{t=5}^{6} M_{n_t}(\mathbb{F}_{q^2}) \text{ with } 119 = \sum_{t=1}^{4} n_t^2 + 2n_5^2 + 2n_6^2, \; n_t \geqslant 2 \; \forall\, t.$$

Again, from Theorem 4.1 in [12], we know that for $q \equiv \pm 2 \bmod 5$ we have

$$\text{(3.7)} \qquad \mathbb{F}_q A_5 \cong \mathbb{F}_q \oplus M_3(\mathbb{F}_{q^2}) \oplus M_4(\mathbb{F}_q) \oplus M_5(\mathbb{F}_q).$$

Combining (3.6), (3.7) and Theorem 2.5, we get

$$\mathbb{F}_q G \cong \mathbb{F}_q \oplus M_4(\mathbb{F}_q) \oplus M_5(\mathbb{F}_q) \bigoplus_{t=1}^{2} M_{n_t}(\mathbb{F}_q) \oplus M_3(\mathbb{F}_{q^2}) \oplus M_{n_3}(\mathbb{F}_{q^2})$$

with

$$60 = \sum_{t=1}^{2} n_t^2 + 2n_3^2, \quad n_t \geqslant 2 \; \forall\, t.$$

The above equation leaves us with the only choice $(4,6,2)$. Therefore, we have

$$\mathbb{F}_q G \cong \mathbb{F}_q \oplus M_4(\mathbb{F}_q)^2 \oplus M_5(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q) \oplus M_2(\mathbb{F}_{q^2}) \oplus M_3(\mathbb{F}_{q^2}). \tag{3.8}$$

Case 3: $p \equiv 1 \bmod 3$, $p \equiv -1 \bmod 4$, and $p \equiv \pm 1 \bmod 5$. For this case, we have $p^k \equiv 31 \bmod 60$ or $p^k \equiv 19 \bmod 60$. This means that

$$I_F = \{1, 31\}, \quad \text{or} \quad I_F = \{1, 19\}.$$

For both of these possibilities, it can be verified that $S(\gamma_g) = \{\gamma_g\}$ for each $g \in G$. Therefore, the required Wedderburn decomposition is given by (3.5).

Case 4: $p \equiv 1 \bmod 3$, $p \equiv -1 \bmod 4$, and $p \equiv \pm 2 \bmod 5$. For this case, we have $p^k \equiv 7 \bmod 60$ or $p^k \equiv 43 \bmod 60$, which means that

$$I_F = \{1, 7, 43, 49\}.$$

So, we can see that this case is similar to Case 2 which means the Wedderburn decomposition is given by (3.8).

Case 5: $p \equiv -1 \bmod 3$, $p \equiv 1 \bmod 4$, and $p \equiv \pm 1 \bmod 5$. Here, we have $p^k \equiv 41 \bmod 60$ or $p^k \equiv 29 \bmod 60$. This means that

$$I_F = \{1, 41\}, \quad \text{or} \quad I_F = \{1, 29\}.$$

For both of these possibilities, it can be verified that $S(\gamma_g) = \{\gamma_g\}$ for each $g \in G$. Therefore, the Wedderburn decomposition is given by (3.5).

Case 6: $p \equiv -1 \bmod 3$, $p \equiv 1 \bmod 4$, and $p \equiv \pm 2 \bmod 5$. For this case, we get $p^k \equiv 17 \bmod 60$ or $p^k \equiv 53 \bmod 60$, which means that

$$I_F = \{1, 17, 49, 53\},$$

and one can verify that for this case, the Wedderburn decomposition is given by (3.8).

Case 7: $p \equiv -1 \bmod 3$, $p \equiv -1 \bmod 4$, and $p \equiv \pm 1 \bmod 5$. Here, we have $p^k \equiv 11 \bmod 60$ or $p^k \equiv 59 \bmod 60$. This means that

$$I_F = \{1, 11\}, \quad \text{or} \quad I_F = \{1, 59\}.$$

For both of these possibilities, it can be verified that $S(\gamma_g) = \{\gamma_g\}$ for each $g \in G$. Therefore, the Wedderburn decomposition is given by (3.5).

Case 8: $p \equiv -1 \bmod 3$, $p \equiv -1 \bmod 4$, and $p \equiv \pm 2 \bmod 5$. For this case, we get $p^k \equiv 47 \bmod 60$ or $p^k \equiv 23 \bmod 60$. For both of these possibilities, we have

$$I_F = \{1, 23, 47, 49\},$$

and one can verify that the Wedderburn decomposition in this case is given by (3.8).

This completes the proof. □

Corollary 3.1. *The unit group of* $\mathbb{F}_qSL(2,\mathbb{Z}_5)$*, when* $p>5$*, where* $\mathbb{F}_q$ *is a finite field having* $q=p^k$ *elements, is isomorphic to:*

Conditions on k *and* p	*unit group*
k *is even and* $p>5$ *arbitrary* *or* k *is odd and* $q\equiv\pm1 \bmod 5$	$\mathbb{F}_q^*\oplus GL_2(\mathbb{F}_q)^2\oplus GL_3(\mathbb{F}_q)^2\oplus GL_4(\mathbb{F}_q)^2\oplus GL_5(\mathbb{F}_q)\oplus GL_6(\mathbb{F}_q)$
k *is odd* $q\equiv\pm2 \bmod 5$	$\mathbb{F}_q^*\oplus GL_4(\mathbb{F}_q)^2\oplus GL_5(\mathbb{F}_q)\oplus GL_6(\mathbb{F}_q)\oplus GL_2(\mathbb{F}_{q^2})\oplus GL_3(\mathbb{F}_{q^2})$

where $GL_n(\mathbb{F}_q)$ *denotes the group of all* $n\times n$ *invertible matrices over the field* $\mathbb{F}_q$*.*

4. Discussion

We have obtained the unit group of the semisimple group algebra $\mathbb{F}_qSL(2,\mathbb{Z}_5)$ for $p>5$. The approach used in this paper for obtaining the Wedderburn decomposition is suitable for almost all the groups up to order 120 (at least) having two or more nontrivial normal subgroups unlike the symmetric group S_5.

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